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A NOTE ON SUMS OF FINITE PRODUCTS OF LUCAS-BALANCING POLYNOMIALS

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Abstract. Behera and Panda introduced balancing numbers about twenty years ago. Since then, these numbers have been intensively studied by many researchers and lots of interesting properties of them have been unveiled. Lucas-balancing numbers have close connection with balancing numbers and their natural extensions are Lucas-balancing polynomials. In this paper, we will study sums of finite products of Lucas-balancing polynomials and represent them in terms of nine orthogonal polynomials in two different ways each. In particular, we obtain an expression of such sums of finite products in terms of Lucas-balancing polynomials. Our proof is based on a fundamental relation between Lucas-balancing polynomials and Chevbyshev polynomials of the first kind observed by Frontczak.

1. Introduction

Behera and Panda [3] introduced the balancing numbers B_n about twenty years ago. Since then, these numbers have been intensively studied by many researchers and lots of interesting properties of them have been unveiled [5, 6, 7, 8, 14, 15, 16, 20, 23]. On the other hand, as we will see, the Lucas-balancing numbers C_n arise naturally from the balancing numbers. Natural extensions of balancing numbers and those of Lucas-balancing numbers are respectively the balancing polynomials $B_n(x)$, and the Lucas-balancing polynomials $C_n(x)$. Then $B_n = B_n(1)$, $C_n = C_n(1)$, $(n \geq 0)$.

Recently, Frontczak [6] observed that there are simple relations between balancing polynomials and Chebyshev polynomials of the second kind, and also between Lucas-balancing polynomials and Chebyshev polynomials of the first kind. It is very strange that these fundamental relations had not been observed much earlier. From these observations and some well-known properties of Chebyshev polynomials of the first and second kinds, we will be able to derive several immediate results for Lucas-balancing polynomials and hence also for Lucas-balancing numbers. Then, from these observations and by utilizing these

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results, we will represent sums of finite products of Lucas-balancing polynomials in terms of nine orthogonal polynomials in two different ways each. In particular, this gives a representation of sums of products of Lucas-balancing polynomials in terms of Lucas-balancing polynomials.

These may be viewed as a generalization of the classical linearization problem. We will recall here that the classical linearization problem in general consists in determining the coefficients in the expansion of the product of two polynomials in terms of an arbitrary polynomial sequence. The problem of representing sums of products of some special polynomials has drawn the attention of many mathematicians in recent years (see [11, 12] and the references therein).

The rest of this section is devoted to recalling some preliminary facts and results on the balancing and Lucas-balancing polynomials and numbers, and Chebyshev polynomials of the first kind and second kinds. For more details on the balancing and Lucas-balancing polynomials and numbers, we recommend the Ph. D thesis of Ray [20] and the survey paper [14]. As to a general reference for Chebyshev polynomials, we let the reader refer to [13].

The Lucas-balancing polynomials $C_n(x)$ are given by the recurrence relation:

$$\begin{aligned} &C_{n+1}(x) = 6xC_n(x) - C_{n-1}(x), \ \ (n \geq 1), \\ &C_0(x) = 1, \ \ C_1(x) = 3x. \end{aligned} \tag{1.1}$$

They are also given by the generating function

$$\frac{1-3xt}{1-6xt+t^2} = \sum_{n=0}^{\infty} C_n(x)t^n. \tag{1.2}$$

The first few terms of Lucas-balancing polynomials are given as follows:

$$\begin{aligned} &C_2(x) = 18x^2 - 1, \ \ C_3(x) = 108x^3 - 9x, \\ &C_4(x) = 648x^4 - 72x^2 + 1, \ \ C_5(x) = 3888x^5 - 540x^3 + 15x, \ \ \cdots \end{aligned}$$

The $C_n = C_n(1), \ (n \geq 0)$, are called the Lucas-balancing numbers, so that they are given either by

$$C_{n+1} = 6C_n - C_{n-1}, \ \ (n \geq 1), C_0 = 1, C_1 = 3, \tag{1.3}$$

or by

$$\frac{1-3t}{1-6t+t^2} = \sum_{n=0}^{\infty} C_n t^n. \tag{1.4}$$

The first few terms of Lucas-Balancing numbers are :

$$C_2 = 17, C_3 = 99, C_4 = 577, C_5 = 3363, \cdots .$$

The balancing polynomials $B_n(x)$ are given by

$$\begin{aligned} &B_{n+1}(x) = 6xB_n(x) - B_{n-1}(x), \quad (n \geq 1), \\ &B_0(x) = 1, \quad B_1(x) = 6x. \end{aligned} \tag{1.5}$$

Alternatively, they are given by the generating function

$$\frac{1}{1 - 6xt + t^2} = \sum_{n=0}^{\infty} B_n(x)t^n. \tag{1.6}$$

The $B_n = B_n(1)$, $(n \geq 0)$, are called the balancing numbers. Then they are given either by

$$B_{n+1} = 6B_n - B_{n-1}, \quad (n \geq 1), B_0 = 1, B_1 = 6, \tag{1.7}$$

or by

$$\frac{1}{1 - 6t + t^2} = \sum_{n=0}^{\infty} B_n t^n. \tag{1.8}$$

We note here that, following the original convention in [3], we defined B_0 and $B_0(x)$ as 1, not as 0.

Actually, the balancing numbers B_n are originally defined as follows: A positive integer n is called a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

holds for some positive integer r, in which case r is the balancer corresponding to the balancing number n.

Then it is easy to see that a positive integer n is a balancing number if and only if $8n^2 + 1$ is a perfect square. In fact, $C_{n+1} = \sqrt{8B_n^2 + 1}$, $(n \geq 0)$, are the Lucas-balancing numbers.

If we denote the balancer corresponding to B_n by R_n, then it can be shown that $C_{n+1} = 2B_n + 2R_n + 1$. For example, for $B_2 = 35$, $C_3 = 99$ and $R_2 = 14$, so that

$$1 + 2 + \cdots + 34 = 595 = 36 + \cdots + 49.$$

Further, we observe that $C_{n+1} = 3B_n - B_{n-1}$, and hence that $R_n = \frac{1}{2}(B_n - B_{n-1} - 1)$. This follows from the trivial identity between the Chebyshev polynomials of the first kind $T_n(x)$ (see (1.9)–(1.11)) and of the second kind $U_n(x)$ (see (1.12)–(1.14)), given by $T_{n+1}(x) = xU_n(x) - U_{n-1}(x)$, and the equations in (1.15).

Next, we recall a few facts on the Chebyshev polynomials of the first and second kinds. The Chebyshev polynomials of the first kind $T_n(x)$ are given either by the generating function

$$\frac{1-xt}{1-2xt+t^2}=\sum_{n=0}^{\infty}T_n(x)t^n. \tag{1.9}$$

or by the recurrence relation

$$\begin{aligned}&T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\ \ (n\geq 1),\\ &T_0(x)=1,\ \ T_1(x)=x.\end{aligned} \tag{1.10}$$

Also, they are explicitly given by

$$\begin{aligned}T_n(x)&={}_2F_1(-n,n;\tfrac{1}{2};\tfrac{1-x}{2})\\ &=\frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l},\ \ (n\geq 1),\end{aligned} \tag{1.11}$$

where ${}_2F_1(a,b;c;x)$ is the Gaussian hypergeometric function, (see (2.9)).

The Chebyshev polynomials of the second $U_n(x)$ are given either by the generating function

$$\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n. \tag{1.12}$$

or by the recurrence relation

$$\begin{aligned}&U_{n+1}(x)=2xU_n(x)-U_{n-1}(x),\ \ (n\geq 1),\\ &U_0(x)=1,\ \ U_1(x)=2x.\end{aligned} \tag{1.13}$$

In explicit terms, they are also given by

$$\begin{aligned}U_n(x)&=(n+1){}_2F_1(-n,n+2;\tfrac{3}{2};\tfrac{1-x}{2})\\ &=\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2x)^{n-2l},\ \ (n\geq 0).\end{aligned} \tag{1.14}$$

The first one in (1.15) follows either from (1.1) and (1.10) or (1.2) and (1.9), while the second one does either from (1.5) and (1.13) or (1.6) and (1.12). This fundamental result is first observed in [6].

Lemma 1.1. *For any integer $n\geq 0$, we have*

$$C_n(x)=T_n(3x),\ \ B_n(x)=U_n(3x). \tag{1.15}$$

Remark 1. *From Lemma 1.1, Lucas-balancing polynomials fall in the class of modified or shifted Chebyshev polynomials. This class of orthogonal polynomials can be found, for instance, in the works* [4, 9, 10, 25, 26, 27].

As are known, the orthogonalities of $T_n(x)$ and $U_n(x)$ are given by

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_n(x)T_m(x)\,dx=\frac{\pi}{\mathcal{E}_n}\delta_{n,m}, \tag{1.16}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_n(x)U_m(x)\,dx=\frac{\pi}{2}\delta_{n,m}, \tag{1.17}$$

where

$$\mathcal{E}_n=\begin{cases}1, \text{if } n=0,\\ 2, \text{if } n\geq 1.\end{cases}$$

From (1.16), (1.17) and (1.15), we deduce the following orthogonality relations for $C_n(x)$ and $B_n(x)$:

$$\int_{-\frac{1}{3}}^{\frac{1}{3}}(1-9x^2)^{-\frac{1}{2}}C_n(x)C_m(x)dx=\frac{\pi}{3\mathcal{E}_n}\delta_{n,m}, \tag{1.18}$$

$$\int_{-\frac{1}{3}}^{\frac{1}{3}}(1-9x^2)^{\frac{1}{2}}B_n(x)B_m(x)dx=\frac{\pi}{6}\delta_{n,m}. \tag{1.19}$$

In addition, from the Rodrigues' formulas of $T_n(x)$ and $U_n(x)$, respectively given by

$$T_n(x)=\frac{(-1)^n2^nn!}{(2n)!}(1-x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{1.20}$$

$$U_n(x)=\frac{(-1)^n2^n(n+1)!}{(2n+1)!}(1-x^2)^{-\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}, \tag{1.21}$$

we obtain

$$C_n(x)=\frac{(-1)^n(\frac{2}{3})^nn!}{(2n)!}(1-9x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-9x^2)^{n-\frac{1}{2}}, \tag{1.22}$$

$$B_n(x)=\frac{(-1)^n(\frac{2}{3})^n(n+1)!}{(2n+1)!}(1-9x^2)^{-\frac{1}{2}}\frac{d^n}{dx^n}(1-9x^2)^{n+\frac{1}{2}}. \tag{1.23}$$

Now, the next Proposition follows from (1.18) – (1.23).

Proposition 1.2. *Let $q(x)\in\mathbb{R}[x]$ be a polynomial of degree n, and let $q(x)=\sum_{k=0}^{n}\beta_kC_k(x)=\sum_{k=0}^{n}\alpha_kB_k(x)$. Then we have*

$$(a)\quad \beta_k(x)=\frac{3\mathcal{E}_k(-\frac{2}{3})^kk!}{\pi(2k)!}\int_{-\frac{1}{3}}^{\frac{1}{3}}q(x)\frac{d^k}{dx^k}(1-9x^2)^{k-\frac{1}{2}}dx, \tag{1.24}$$

$$(b)\quad \alpha_k(x)=\frac{6(-\frac{2}{3})^k(k+1)!}{\pi(2k+1)!}\int_{-\frac{1}{3}}^{\frac{1}{3}}q(x)\frac{d^k}{dx^k}(1-9x^2)^{k+\frac{1}{2}}dx. \tag{1.25}$$

Before we move on to the next section, we would like to mention that balancing, Lucas-balancing, cobalancing, Lucas-cobalancing, Pell and Pell-Lucas numbers and polynomials have close connections with many interesting problems in number theory. Indeed, some divisibility or congruence properties of balancing and Lucas-balancing numbers were studied in [21]. A repdigit is a nonnegative integer whose digits are all equal. In [24], it was proved the necessary and sufficient conditions for two products of Pell or Pell-Lucas or balancing or Lucas-balancing numbers to be repdigits. Some Diophantine equations involving balancing and Lucas-balancing numbers were investigated in [22]. In [17], the authors obtained some explicit lower and upper bounds for reciprocal sums with terms from balancing and Lucas-balancing sequences. The periodicity of balancing numbers modulo terms of certain sequences and modulo primes were explored in [19]. Among other things, it was shown that the period of balancing sequence coincides with the modulus of congruence if and only if the modulus is any power of 2. It was observed [18] that the sequences of balancing and cobalancing numbers have very close connections with the Pell sequence, whereas the sequences of Lucas-balancing and Lucas-cobalancing numbers are closely related to the associated Pell sequence.

2. Preliminaries and statements of results

Here we are going to recall some elementary facts on orthogonal polynomials and to state our main results in this paper. For more details on orthogonal polynomials, we let the reader refer to the books [1, 2, 13] and to the papers [11, 12].

The falling factorial $(x)_n$ and the rising factorial $\langle x\rangle_n$ are respectively defined by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{2.1}$$

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ \langle x\rangle_0 = 1. \tag{2.2}$$

The two factorial are clearly related by

$$(-1)^n (x)_n = \langle -x\rangle_n,\ (-1)^n\langle x\rangle_n = (-x)_n. \tag{2.3}$$

$$\frac{(2n-2s)!}{(n-s)!} = \frac{2^{2n-2s}(-1)^s\langle\frac{1}{2}\rangle_n}{\langle\frac{1}{2}-n\rangle_s},\ (n \geq s \geq 0). \tag{2.4}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n,\ \frac{\Gamma(x+n)}{\Gamma(x)} = \langle x\rangle_n,\ (n \geq 0). \tag{2.5}$$

$$\Gamma(n+\frac{1}{2})=\frac{(2n)!\sqrt{\pi}}{2^{2n}n!},\ (n\geq 0). \tag{2.6}$$

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ (Re(x),Re(y)>0), \tag{2.7}$$

when $\Gamma(x)$ and $B(x,y)$ are the gamma and beta functions respectively.

The hypergeometric function ${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x)$ is given by

$$\begin{aligned}&{}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x)\\&=\sum_{n=0}^{\infty}\frac{\langle a_1\rangle_n\cdots\langle a_p\rangle_n}{\langle b_1\rangle_n\cdots\langle b_q\rangle_n}\frac{x^n}{n!},\ (p\leq q+1,\mid x\mid<1).\end{aligned} \tag{2.8}$$

Especially, the Gaussian hypergeometric function is given by

$$\begin{aligned}&{}_2F_1(a,b;c;x)\\&=\sum_{n=0}^{\infty}\frac{\langle a\rangle_n\langle b\rangle_n}{\langle c\rangle_n}\frac{x^n}{n!},\ (\mid x\mid<1).\end{aligned} \tag{2.9}$$

Next, we will recall Chebyshev polynomials of the third kind $V_n(x)$, those of the fourth kind $W_n(x)$, Hermite polynomials $H_n(x)$, generalized Laguerre polynomials $L_n^{\alpha}(x)$, Legendre polynomials $P_n(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and Jacobi polynomais $P_n^{(\alpha,\beta)}(x)$.
In explicit terms, they are given by

$$\begin{aligned}V_n(x)&={}_2F_1(-n,n+1;\tfrac{1}{2};\tfrac{1-x}{2})\\&=\sum_{l=0}^{n}\binom{2n-l}{l}2^{n-l}(x-1)^{n-l},\ (n\geq 0),\end{aligned} \tag{2.10}$$

$$\begin{aligned}W_n(x)&=(2n+1){}_2F_1(-n,n+1;\tfrac{3}{2};\tfrac{1-x}{2})\\&=(2n+1)\sum_{l=0}^{n}\frac{2^{n-l}}{2n-2l+1}\binom{2n-l}{l}(x-1)^{n-l},\ (n\geq 0),\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
H_n(x) &= n! \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l}{l!(n-2l)!}(2x)^{n-2l},\ (n \geq 0), && (2.12)\\
L_n^{\alpha}(x) &= \frac{\langle \alpha+1 \rangle_n}{n!} {}_1F_1(-n; \alpha+1; x)\\
&= \sum_{l=0}^{n} \frac{(-1)^l \binom{n+\alpha}{n-l}}{l!} x^l,\ (n \geq 0), && (2.13)\\
P_n(x) &= {}_2F_1(-n, n+1; 1; \tfrac{1-x}{2})\\
&= \frac{1}{2^n} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} x^{n-2l},\ (n \geq 0), && (2.14)\\
C_n^{(\lambda)}(x) &= \binom{n+2\lambda-1}{n} {}_2F_1(-n, n+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1-x}{2})\\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda) k! (n-2k)!} (2x)^{n-2k},\ (n \geq 0), && (2.15)\\
P_n^{(\alpha,\beta)}(x) &= \frac{\langle \alpha+1 \rangle_n}{n!} {}_2F_1(-n; 1+\alpha+\beta+n; \alpha+1; \tfrac{1-x}{2})\\
&= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (\frac{x-1}{2})^k (\frac{x+1}{2})^{n-k},\ (n \geq 0), && (2.16)\\
& && (2.17)
\end{aligned}
$$

where $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$.

In Theorem 1 of [11] and Theorem 1 of [12], the sums of finite products of Chebyshev polynomials of the first kind

$$
\sum_{i_1+i_2+\cdots+i_{r+1}=n} T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x)
$$

were expressed in terms of the orthogonal polynomials in (1.11), (1.14) and (2.10)–(2.16).

Now, the following theorem is obtained from those expressions by replacing x by $3x$ and making use of (1.15).

Theorem 2.1. *Let n, r be nonnegative integers. Then, we have the following representations.*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} C_{i_1}(x)C_{i_2}(x)\cdots C_{i_{r+1}}(x)$$

$$= \frac{1}{r!}\sum_{s=0}^{[\frac{n}{2}]}\sum_{l=0}^{s} \frac{\mathcal{E}_{n-2s}(-1)^l(n+r-l)!}{l!(n-s-l)!(s-l)!}\,{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})C_{n-2s}(x) \tag{2.18}$$

$$= \frac{1}{r!}\sum_{s=0}^{[\frac{n}{2}]}\sum_{l=0}^{s} \frac{(-1)^l(n-2s+1)(n+r-l)!}{l!(n-s+1-l)!(s-l)!}\,{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) \times U_{n-2s}(3x) \tag{2.19}$$

$$= \frac{1}{r!}\sum_{s=0}^{n}\sum_{l=0}^{[\frac{s}{2}]} \frac{(-1)^l(n+r-l)!}{l!(n-[\frac{s}{2}]-l)!([\frac{s}{2}]-l)!}\,{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})V_{n-s}(3x) \tag{2.20}$$

$$= \frac{1}{r!}\sum_{s=0}^{n}\sum_{l=0}^{[\frac{s}{2}]} \frac{(-1)^{s+l}(n+r-l)!}{l!(n-[\frac{s}{2}]-l)!([\frac{s}{2}]-l)!}\,{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})W_{n-s}(3x) \tag{2.21}$$

$$= \frac{1}{r!}\sum_{s=0}^{[\frac{n}{2}]}\frac{1}{(n-2s)!}\sum_{l=0}^{s} \frac{(-1)^l(n+r-l)!}{l!(s-l)!}\,{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) \times H_{n-2s}(3x) \tag{2.22}$$

$$= \frac{2^n}{r!}\sum_{k=0}^{n}\frac{(-1)^k}{\Gamma(\alpha+k+1)}\sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{4})^l(n+r-l)!\Gamma(n-2l+\alpha+1)}{l!(n-k-2l)!} \times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})L_n^{\lambda}(3x) \tag{2.23}$$

$$= \frac{4^n}{r!}\sum_{s=0}^{[\frac{n}{2}]}2^{1-2s}(2n-4s+1)\sum_{l=0}^{s} \frac{(-\frac{1}{4})^l(n+r-l)!(n-s-l+1)!}{l!(s-l)!(2n-2s-2l+2)!} \times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})P_{n-2s}(3x) \tag{2.24}$$

$$
\begin{aligned}
&= \frac{\Gamma(\lambda)}{r!} \sum_{s=0}^{[\frac{n}{2}]} (n-2s+\lambda) \sum_{l=0}^{s} \frac{(-1)^l (n+r-l)!}{l!(s-l)!\Gamma(n+\lambda-s-l+1)} \\
&\times {}_2F_1(2l-n, -r-1; l-n-r; \tfrac{1}{2}) C_{n-2s}^{(\lambda)}(3x) \qquad (2.25)\\
&= \frac{(-2)^n}{r!} \sum_{k=0}^{n} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{4})^l (n+r-l)!}{l!(n-k-2l)!} \\
&\times {}_2F_1(2l-n, -r-1; 1-n-r; \tfrac{1}{2}) {}_2F_1(2l+k-n, k+\beta+1; 2k+\alpha+\beta+2; 2) \\
&\times P_k^{(\alpha,\beta)}(3x). \qquad (2.26)
\end{aligned}
$$

We will show the following theorem in the next section. Note here that the polynomials on the right hand sides are in x, not in $3x$.

Theorem 2.2. *Let n, r be nonnegative integers. Then, we have the following expressions.*

$$
\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{r+1}=n} C_{i_1}(x) C_{i_2}(x) \cdots C_{i_{r+1}}(x) \\
&= \frac{3^n}{r!} \sum_{s=0}^{[\frac{n}{2}]} \sum_{l=0}^{s} \frac{\mathcal{E}_{n-2s} (-\frac{1}{9})^l (n+r-l)!}{l!(n-s-l)!(s-l)!} {}_2F_1(2l-n, -r-1; l-n-r; \tfrac{1}{2}) T_{n-2s}(x) \qquad (2.27)\\
&= \frac{3^n}{r!} \sum_{s=0}^{[\frac{n}{2}]} \sum_{l=0}^{s} \frac{(n-2s+1)(-\frac{1}{9})^l (n+r-l)!}{l!(n-s-l+1)!(s-l)!} \qquad (2.28)\\
&\times {}_2F_1(2l-n, -r-1; l-n-r; \tfrac{1}{2}) U_{n-2s}(x) \qquad (2.29)\\
&= \frac{3^n}{r!} \sum_{s=0}^{n} \sum_{l=0}^{[\frac{s}{2}]} \frac{(-\frac{1}{9})^l (n+r-l)!}{l!(n-[\frac{s}{2}]-l)!([\frac{s}{2}]-l)!} {}_2F_1(2l-n, -r-1; l-n-r; \tfrac{1}{2}) V_{n-s}(x) \qquad (2.30)
\end{aligned}
$$

$$= \frac{3^n}{r!} \sum_{s=0}^{n} \sum_{l=0}^{[\frac{s}{2}]} \frac{(-1)^s(-\frac{1}{9})^l(n+r-l)!}{l!(n-[\frac{s}{2}]-l)!([\frac{s}{2}]-l)!} {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) W_{n-s}(x) \tag{2.31}$$

$$= \frac{3^n}{r!} \sum_{s=0}^{[\frac{n}{2}]} \sum_{l=0}^{s} \frac{(-\frac{1}{9})^l(n+r-l)!}{l!(n-2s)!(s-l)!}$$
$$\times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) H_{n-2s}(x) \tag{2.32}$$

$$= \frac{6^n}{r!} \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{36})^l(n+r-l)!\Gamma(n-2l+\alpha+1)}{l!(n-k-2l)!}$$
$$\times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) L_n^{\alpha}(x) \tag{2.33}$$

$$= \frac{(12)^n}{r!} \sum_{s=0}^{[\frac{n}{2}]} 2^{1-2s}(2n-4s+1) \sum_{l=0}^{s} \frac{(-\frac{1}{36})^l(n+r-l)!(n-s-l+1)!}{l!(s-l)!(2n-2s-2l+2)!}$$
$$\times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) P_{n-2s}(x) \tag{2.34}$$

$$= \frac{3^n\Gamma(\lambda)}{r!} \sum_{s=0}^{[\frac{n}{2}]} (n-2s+\lambda) \sum_{l=0}^{s} \frac{(-\frac{1}{9})^l(n+r-l)!}{l!(s-l)!\Gamma(n-s-l+\lambda+1)}$$
$$\times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) C_{n-2s}^{(\lambda)}(x) \tag{2.35}$$

$$= \frac{(-6)^n}{r!} \sum_{k=0}^{n} \frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{36})^l(n+r-l)!}{l!(n-2l-k)!}$$
$$\times {}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2}) {}_2F_1(k+2l-n,k+\beta+1;2k+\alpha+\beta+2;2)$$
$$\times P_k^{(\alpha,\beta)}(x). \tag{2.36}$$

3. Proof of Theorem 2.2

In this section, we are going to show Theorem 2.2. The next two results are respectively from Proposition 1 in [12] and Proposition 1 in [11].

Proposition 3.1. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then we have the following.*

$$(a) \quad q(x) = \sum_{k=0}^{n} C_{k,1} T_k(x),$$

where

$$C_{k,1} = \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx.$$

$$(b) \quad q(x) = \sum_{k=0}^{n} C_{k,2} U_k(x),$$

where

$$C_{k,2} = \frac{(-1)^k 2^{k+1} (k+1)!}{(2k+1)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\frac{1}{2}} dx.$$

$$(c) \quad q(x) = \sum_{k=0}^{n} C_{k,3} V_k(x),$$

where

$$C_{k,3} = \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx.$$

$$(d) \quad q(x) = \sum_{k=0}^{n} C_{k,4} W_k(x),$$

where

$$C_{k,4} = \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx.$$

Proposition 3.2. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then the following hold true.*

$$(a) \quad q(x) = \sum_{k=0}^{n} C_{k,5} H_k(x),$$

where

$$C_{k,5} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} e^{-x^2} dx.$$

$$(b) \quad q(x)=\sum_{k=0}^{n} C_{k,6} L_k^{\alpha}(x),$$

where

$$C_{k,6}=\frac{1}{\Gamma(\alpha+k+1)}\int_0^{\infty} q(x)\frac{d^k}{dx^k}\left(e^{-x}x^{k+\alpha}\right)dx.$$

$$(c) \quad q(x)=\sum_{k=0}^{n} C_{k,7} P_k(x),$$

where

$$C_{k,7}=\frac{2k+1}{2^{k+1}k!}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(x^2-1)^k dx.$$

$$(d) \quad q(x)=\sum_{k=0}^{n} C_{k,8} C_k^{(\lambda)}(x),$$

where

$$C_{k,8}=\frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}dx.$$

$$(e) \quad q(x)=\sum_{k=0}^{n} C_{k,9} P_k^{(\alpha,\beta)}(x),$$

where

$$\begin{aligned} C_{k,9}=&\frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\ &\times\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx. \end{aligned}$$

We also need the following two results from Proposition 2 in [11] and Proposition 2 in [12], respectively.

Proposition 3.3. *Let m, k be nonnegative integers. Then following hold true.*

$$\begin{aligned} &(a) \int_{-\infty}^{\infty} x^m e^{-x^2} dx \\ &= \begin{cases} 0, & \text{if } m \equiv 1 \ (mod\ 2), \\ \frac{m!\sqrt{\pi}}{(\frac{m}{2})!2^m}, & \text{if } m \equiv 0 \ (mod\ 2), \end{cases} \end{aligned}$$

$$(b) \int_{-1}^{1} x^m (1-x^2)^k dx$$
$$= \begin{cases} 0, & if\ m \equiv 1\ (mod\ 2), \\ \dfrac{2^{2k+2} k! m! (k+\frac{m}{2}+1)!}{(\frac{m}{2})!(2k+m+2)!}, & if\ m \equiv 0\ (mod\ 2), \end{cases}$$

$$(c) \int_{-1}^{1} x^m (1-x^2)^{k+\lambda-\frac{1}{2}} dx$$
$$= \begin{cases} 0, & if\ m \equiv 1\ (mod\ 2), \\ \dfrac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m}{2}+1)}, & if\ m \equiv 0\ (mod\ 2), \end{cases}$$

$$(d) \int_{-1}^{1} x^m (1-x)^{k+\alpha}(1+x)^{k+\beta} dx$$
$$= 2^{2k+\alpha+\beta+1} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} 2^s$$
$$\times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)}.$$

Proposition 3.4. *For any nonnegative integers m, k, we have the following.*

$$(a) \int_{-1}^{1} (1-x^2)^{k-\frac{1}{2}} x^m dx$$
$$= \begin{cases} 0, & if\ m \equiv 1\ (mod\ 2), \\ \dfrac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & if\ m \equiv 0\ (mod\ 2), \end{cases}$$

$$(b) \int_{-1}^{1} (1-x^2)^{k+\frac{1}{2}} x^m dx$$
$$= \begin{cases} 0, & if\ m \equiv 1\ (mod\ 2), \\ \dfrac{m!(2k+2)!\pi}{2^{m+2k+2}(\frac{m}{2}+k+1)!(\frac{m}{2})!(k+1)!}, & if\ m \equiv 0\ (mod\ 2), \end{cases}$$

$$(c) \int_{-1}^{1} (1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} x^m dx$$
$$= \begin{cases} \dfrac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & if\ m \equiv 1\ (mod\ 2), \\ \dfrac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & if\ m \equiv 0\ (mod\ 2), \end{cases}$$

$$(d) \int_{-1}^{1} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} x^m dx$$
$$= \begin{cases} -\dfrac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m \equiv 1 \ (mod\ 2), \\ \dfrac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m \equiv 0 \ (mod\ 2). \end{cases}$$

All of the computations in Theorem 2.2 rely on the next important lemma.

Lemma 3.5. *Let n, r be nonnegative integers. Then, we have the following identity.*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} C_{i_1}(x)C_{i_2}(x)\cdots C_{i_{r+1}}(x)$$
$$= \frac{1}{6^r r!} \sum_{j=0}^{n} (-3)^j \binom{r+1}{j} x^j B_{n-j+r}^{(r)}(x).$$

Proof. Let

$$G(t,x) = \frac{1}{1-6xt+t^2} = \sum_{n=0}^{\infty} B_n(x)t^n. \tag{3.1}$$

By taking the rth derivative with respect to x on both sides of (3.1), we get

$$6^r r!(1-6xt+t^2)^{-(r+1)} = \sum_{m=0}^{\infty} B_{m+r}^{(r)}(x)t^m. \tag{3.2}$$

Then, from (3.2), we get

$$\begin{aligned} \left(\frac{1-3xt}{1-6xt+t^2}\right)^{r+1} &= \frac{1}{6^r r!}(1-3xt)^{(r+1)} \sum_{m=0}^{\infty} B_{m+r}^{(r)}(x)t^m \\ &= \frac{1}{6^r r!} \sum_{j=0}^{\infty} \binom{r+1}{j} (-3xt)^j \sum_{m=0}^{\infty} B_{m+r}^{(r)}(x)t^m \\ &= \frac{1}{6^r r!} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} (-3)^j \binom{r+1}{j} x^j B_{n-j+r}^{(r)}(x) \right) t^n, \end{aligned}$$

from which the result follows. □

Remark 2. *In Lemma 2 of* [12], *it was shown that*

$$\begin{aligned}&\sum_{i_1+i_2+\cdots+i_{r+1}=n} T_{i_1}(x)T_{i_2}(x)\cdots T_{i_{r+1}}(x)\\ =&\frac{1}{2^r r!}\sum_{j=0}^{n}(-1)^j\binom{r+1}{j}x^j U_{n-j+r}^{(r)}(x).\end{aligned} \tag{3.3}$$

We observe here that (3.1) also follows from (3.3) by replacing x by $3x$.

For convenience, we let

$$\alpha_{n,r}(x)=\sum_{i_1+i_2+\cdots+i_{r+1}=n} C_{i_1}(x)C_{i_2}(x)\cdots C_{i_{r+1}}(x). \tag{3.4}$$

From (1.11), (1.14) and (1.15), we obtain explicit expressions for $B_n(x)$ and $C_n(x)$:

$$B_n(x)=\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(6x)^{n-2l},\ \ (n\geq 0), \tag{3.5}$$

$$C_n(x)=\frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}(6x)^{n-2l},\ \ (n\geq 1). \tag{3.6}$$

From (3.5), we see that the rth derivative of $B_n(x)$ is given by

$$B_n^{(r)}(x)=\sum_{l=0}^{[\frac{n-r}{2}]}(-1)^l\binom{n-l}{l}(n-2l)_r 6^{n-2l}x^{n-2l-r}. \tag{3.7}$$

Now, by (3.1), (3.4) and (3.7), we obtain an expressions for $\alpha_{n,r}(x)$:

$$\alpha_{n,r}(x)=\frac{1}{6^r r!}\sum_{j=0}^{n}(-3)^j\binom{r+1}{j}\sum_{l=0}^{[\frac{n-j}{2}]}(-1)^l\binom{n+r-j-l}{l} \tag{3.8}$$

$$\times (n+r-j-2l)_r 6^{n+r-j-2l}x^{n-2l}. \tag{3.9}$$

Every expression in (2.27) - (2.36) in Theorem 2.2 can be shown by using Propositions 3.1– 3.4, (3.8) and the facts in (2.1) - (2.7). Here we are going to show (2.30) and (2.36), while leaving the others as exercises for the interested reader. Let

$$\alpha_{n,r}(x)=\sum_{k=0}^{n} C_{k,3}V_k(x). \tag{3.10}$$

Then, from (c) of Proposition 1, (3.8), and integration by parts k times, we have

$$\begin{aligned}
C_{k,3} &= \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} \alpha_{n,r}(x) \frac{d^k}{dx^k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} dx \\
&= \frac{(-1)^k k! 2^k}{(2k)!\pi 6^r r!} \sum_{j=0}^{n} (-3)^j \binom{r+1}{j} \sum_{l=0}^{[\frac{n-j}{2}]} (-1)^l \binom{n+r-j-l}{l} \\
&\times (n+r-j-2l)_r 6^{n+r-j-2l} \int_{-1}^{1} x^{n-2l} \frac{d^k}{dx^k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} dx \\
&= \frac{(-1)^k k! 2^k}{(2k)!\pi 6^r r!} \sum_{l=0}^{[\frac{n-k}{2}]} \sum_{j=0}^{n-2l} (-3)^j \binom{r+1}{j} (-1)^l \binom{n+r-j-l}{l} \\
&\times (n+r-j-2l)_r 6^{n+r-j-2l} (-1)^k (n-2l)_k \\
&\times \int_{-1}^{1} x^{n-2l-k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} dx.
\end{aligned} \tag{3.11}$$

From (c) of Proposition 4, we observe that

$$\begin{aligned}
&\int_{-1}^{1} x^{n-2l-k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} dx \\
&= \begin{cases} \dfrac{(n-2l-k+1)!(2k)!\pi}{2^{n+k-2l+1}(\frac{n+k+1}{2}-l)!(\frac{n-k+1}{2}-1)!k!}, & \text{if } n \not\equiv k \pmod 2, \\ \dfrac{(n-2l-k)!(2k)!\pi}{2^{n+k-2l}(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!k!}, & \text{if } n \equiv k \pmod 2, \end{cases}
\end{aligned} \tag{3.12}$$

Then, from (3.10)–(3.12), and after some simplifications, we have

$$\begin{aligned}
\alpha_{n,r}(x) &= \frac{3^n}{r!} \sum_{\substack{0 \le k \le n \\ n \not\equiv k \ (\mathrm{mod}\ 2)}} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l (\frac{1}{9})^l (n+r-l)!(n-2l-k+1)}{2l!(\frac{n+k+1}{2}-l)!(\frac{n-k+1}{2}-l)!} V_k(x) \\
&\times \sum_{j=0}^{n-2l} \frac{(-1)^l (\frac{1}{2})^j (n-2l)_j (r+1)_j}{j!(n+r-l)_j}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3^n}{r!}\sum_{\substack{0\le k\le n\\ n\equiv k \pmod 2}}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l(\frac{1}{9})^l(n+r-l)!}{l!(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!}V_k(x)\\
&\times\sum_{j=0}^{n-2l}\frac{(-1)^l(\frac{1}{2})^j(n-2l)_j(r+1)_j}{j!(n+r-l)_j}\\
&=\frac{3^n}{r!}\sum_{\substack{0\le k\le n\\ n\not\equiv k \pmod 2}}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l(\frac{1}{9})^l(n+r-l)!(n-2l-k+1)}{2l!(\frac{n+k+1}{2}-l)!(\frac{n-k+1}{2}-l)!}V_k(x)\\
&\times\sum_{j=0}^{n-2l}\frac{(\frac{1}{2})^j<2l-n>_j<-r-1>_j}{j!<l-n-r>_j}\\
&+\frac{3^n}{r!}\sum_{\substack{0\le k\le n\\ n\equiv k \pmod 2}}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l(\frac{1}{9})^l(n+r-l)!}{l!(\frac{n+k}{2}-l)!(\frac{n-k}{2}-l)!}V_k(x)\\
&\times\sum_{j=0}^{n-2l}\frac{(\frac{1}{2})^j<2l-n>_j<-r-1>_j}{j!<l-n-r>_j}.
\end{aligned}
\tag{3.13}
$$

Putting in (3.13) $n-k=2s+1$, for the first sum and $n-k=2s$, for the second, and after some simplifications, we obtain

$$
\begin{aligned}
\alpha_{n,r}(x)&=\frac{3^n}{r!}\sum_{s=0}^{[\frac{n-1}{2}]}\sum_{l=0}^{s}\frac{(-\frac{1}{9})^l(n+r-l)!}{l!(n-s-l)!(s-l)!}\\
&\times{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})V_{n-2s-1}(x)\\
&+\frac{3^n}{r!}\sum_{s=0}^{[\frac{n}{2}]}\sum_{l=0}^{s}\frac{(-\frac{1}{9})^l(n+r-l)!}{l!(n-s-l)!(s-l)!}\\
&\times{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})V_{n-2s}(x)\\
&=\frac{3^n}{r!}\sum_{s=0}^{n}\sum_{l=0}^{[\frac{s}{2}]}\frac{(-\frac{1}{9})^l(n+r-l)!}{l!(n-[\frac{s}{2}]-l)!([\frac{s}{2}]-l)!}\\
&\times{}_2F_1(2l-n,-r-1;l-n-r;\tfrac{1}{2})V_{n-s}(x).
\end{aligned}
\tag{3.14}
$$

This shows (2.20).

Next, we put

$$\alpha_{n,r}(x) = \sum_{k=0}^{n} C_{k,9} P_k^{(\alpha,\beta)}(x). \tag{3.15}$$

Then, by (e) of Proposition 3.2, (3.8), integration by parts k times, and proceeding just as in (3.11), we get

$$\begin{aligned} C_{k,9} = &\frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{k+\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)6^r r!} \\ &\times \sum_{j=0}^{n}(-3)^j\binom{r+1}{j}\sum_{l=0}^{[\frac{n-j}{2}]}(-1)^l\binom{n+r-j-l}{l}(n+r-j-2l)_r \\ &\times 6^{n+r-j-2l}(-1)^k(n-2l)_k\int_{-1}^{1}x^{n-2l-k}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx, \end{aligned} \tag{3.16}$$

where we note from (d) of Proposition 3.3 that

$$\begin{aligned} &\int_{-1}^{1}x^{n-2l-k}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\ &= 2^{2k+\alpha+\beta+1}\sum_{s=0}^{n-2l-k}\binom{n-2l-k}{s}(-1)^{n-2l-k-s}2^s \\ &\times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)} \end{aligned} \tag{3.17}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned} \alpha_{n,r}(x) = &\sum_{k=0}^{n}\sum_{l=0}^{[\frac{n-k}{2}]}\sum_{j=0}^{n-2l}\frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)P_k^{(\alpha,\beta)}(x)}{2^{k+\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)6^r r!} \\ &\times (-3)^j\binom{r+1}{j}(-1)^l\binom{n+r-j-l}{l}(n+r-j-2l)_r 6^{n+r-j-2l} \\ &\times (-1)^k(n-2l)_k 2^{2k+\alpha+\beta+1}\sum_{s=0}^{n-2l-k}\binom{n-2l-k}{s} \\ &\times (-1)^{n-2l-k-s}2^s\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-6)^n}{r!} \sum_{k=0}^{n} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} P_k^{(\alpha,\beta)}(x) \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{36})^l (n+r-l)!}{l!(n-2l-k)!} \\
&\times \sum_{j=0}^{n-2l} \frac{(\frac{1}{2})^j (-1)^j (n-2l)_j (r+1)_j}{j!(n+r-l)_j} \\
&\times \sum_{s=0}^{n-2l-k} \frac{2^s (-1)^s (n-2l-k)_s < k+\beta+1 >_s}{s! < 2k+\alpha+\beta+2 >_s} \qquad (3.18) \\
&= \frac{(-6)^n}{r!} \sum_{k=0}^{n} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{36})^l (n+r-l)!}{l!(n-2l-k)!} \\
&\times {}_2F_1(2l-n, -r-1; l-n-r; \tfrac{1}{2}) \\
&\times {}_2F_1(k+2l-n, k+\beta+1; 2k+\alpha+\beta+2; 2) P_k^{(\alpha,\beta)}(x).
\end{aligned}
$$

This completes the proof for (2.36).

4. Conclusions

The balancing numbers B_n were introduced about twenty years ago by Behera and Panda. Since their introduction, these numbers have been intensively studied and many interesting properties of them have been discovered. The Lucas-balancing numbers C_n have close connection with balancing numbers, namely $C_{n+1} = \sqrt{8B_n^2+1}$, $(n \geq 0)$. Natural extensions of Lucas-balancing numbers are Lucas-balancing polynomials $C_n(x)$.

The linearization problem in general consists in determining the coefficients $c_{nm}(k)$ in the expansion of the product of two polynomials $q_n(x)$ and $r_m(x)$ in terms of an arbitrary polynomial sequence $\{p_k(x)\}_{k\geq 0}$:

$$q_n(x) r_m(x) = \sum_{k=0}^{n+m} c_{nm}(k) p_k(x).$$

A special problem of this is the case when $p_n(x) = q_n(x) = r_n(x)$, which is called either the standard linearization or Clebsch-Gordan-type problem.

In this paper, we studied sums of finite products of Lucas-balancing polynomials given by

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} C_{i_1}(x) C_{i_2}(x) \cdots C_{i_{r+1}}(x),$$

and represented them in terms of nine orthogonal polynomials in two different ways each. In particular, we obtained an expression of such sums of finite products in terms of Lucas-balancing polynomials. These can be viewed as a generalization of the aforementioned classical linearization problem. Our proof was based on a fundamental relation between Lucas-balancing polynomials and Chevbyshev polynomials of the first kind observed by Frontczak.

As our immediate future projects, we would like to continue to study the problem of representing sums of finite products of some special polynomials in terms of various special polynomials.

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