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On Elliptical Trigonometry

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Abstract

The ordinary trigonometric functions are defined by means of angles and lengths on the unit circle. There are several derivatives of classical trigonometric functions such as hyperbolic, polar, spherical, Fourier, inverse, log, complex, q-versions etc. In this paper, we add this list a new version of trigonometry which will be consistently called as elliptic trigonometry by working on an ellipse instead of a circle. Although the definitions of elliptic trigonometric functions remind us the classical trigonometry, there are several interesting differences between the two whence the relations and expansions are considered. Also as ellipses has applications in engineering, medicine, astronomy, architecture, etc., it is expected that this new notion will have new applications or improve the existing applications in these and other areas.

1 Introduction

In analytic geometry, a sub branch of geometry, a conic is defined as a plane algebraic curve of degree 2. That is, as the set of points whose coordinates satisfy a quadratic equation in two variables. This equation may be written in matrix form, and some geometric properties of it can be studied as algebraic conditions. The three main types of conics are parabola, hyperbola and ellipse. Although circle is a special ellipse, in many sources, it is counted as the fourth type of conics due to its importance as the result of its applications.

The word trigonometry coming from Greek trigonon meaning triangle and metron meaning measure, is a branch of mathematics that deals with relationships involving lengths and angles in triangles. Trigonometry emerged in the Anatolia in the Hellenistic period between 3^{rd} century BC from applications of geometry to astronomical studies. Today, nearly all branches of science make

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use of trigonometry.

There are many applications of trigonometric functions as they help us to state real life situations including angles and lengths. They occur in the solutions of many linear differential equations such as the equation defining a catenary, of some cubic equations, in calculations of angles and distances in hyperbolic geometry, and of Laplace's equation in Cartesian coordinates. Laplace's equations are important in many areas of physics, including electromagnetic theory, heat transfer, fluid dynamics, and special relativity.

There are a number of ways of defining the ordinary trigonometric functions on real numbers the most frequently used ones being right-angled triangle definitions and unit-circle definitions. Like different versions of geometries, trigonometric functions can also be defined in other ways besides the geometrical definitions above, using tools from calculus such as infinite series, differential equations, functional equations, etc. Several such definitions are resulted in hyperbolic trigonometry, polar trigonometry, spherical trigonometry, Fourier trigonometry, inverse trigonometry, log trigonometry, complex trigonometry, q-trigonometry, etc. For example, the hyperbolic trigonometric functions are defined just by taking the points $(cosht, sinht)$ form the right half of the equilateral hyperbola instead of taking the points $(cost, sint)$ from the unit circle. The hyperbolic functions take a real argument called a hyperbolic angle. The size of a hyperbolic angle is twice the area of its hyperbolic sector. The hyperbolic functions can be defined in terms of the legs of a right triangle covering this sector. For example

$$sinhx = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad coshx = \frac{e^x + e^{-x}}{2}.$$

Similarly the trigonometric functions can be defined by means of complex exponential function $e^{ix} = cosx + isinx$, where i is the imaginary unit satisfying $i^2 = -1$, indeed

$$sinx = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad cosx = \frac{e^{ix} + e^{-ix}}{2}.$$

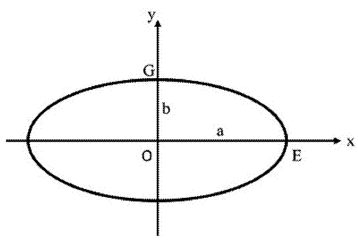

Figure 1: Ellipse

For the relation with elliptic functions and elliptic integrals, see [3, 4, 5].

But, interestingly enough, although some researchers mentioned the elliptical trigonometry, no clear description of it have been given, see [1], [7]. As well-known, an ellipse in the plane is a curve surrounding two focal points such that the sum of the distances to the two focal points is constant for every point on the curve. The points on the ellipse satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

where $a, b \in \mathbb{R}^+$, see Figure 1. When $a = b$, we have a circle.

There are many applications of ellipses. In Physics and Astronomy, Kepler's first law of planetary motion states that the path of each planet is an ellipse with the sun at one focus, see [2]. In Architecture, there are buildings all over the world having what is called whispering galleries. These galleries are rooms where the ceilings are elliptical and a sound made at one focus can be heard very clearly at the other focus. One of the important properties of an ellipse is its reflective property. This property is used in Medicine for light rays and also for other forms of energy, including shockwaves. Shockwaves generated at one focus will reflect off the ellipse and pass through the second focus by the reflective characteristic, a property, unique to the ellipse, and therefore inspired useful medical applications. Medical specialists used an ellipse to create a device that effectively treats kidney stones and gallstones. A device called lithotripter uses shockwaves to successfully shatter a troubling kidney stone or gallstone into tiny pieces that can be easily passed by the body without open surgery. This process is known as lithotripsy. There are too many satellites in the space for several purposes. If the orbit of a satellite has zero angular acceleration, this means that in suchan elliptical orbit, the satellite remains stationery in space which reduces the maintanance costs. Also, this enables getting better view of the world. In mechanics, in the area of IC engine design, there is an offset of circumference for piston and cylinder arrangement making the circular cross section an ellipse, and the difference between minor and major axes of this ellipse is about 1/2 mm. This helps to ease the flow of engine oil, improve clearance, reduce friction, hence reduces the sound and increases engine's life. In heat transfer, the fins have elliptical shape for effective heat transfer. The hulls of the ships are formed in elliptical shape in order to increase buoyant force by changing the meta centric height. We think that the approach, notions and formulae given in this paper will help to ease the calculations that were done by means of classical trigonometric functions.

In the rest of this paper, we shall introduce trigonometric functions on an ellipse and study their properties. There are some literature mentioning elliptic sine and cosine functions, but these are respectively just the odd and even solutions of the second order Mathieu differential equation

$$\frac{d^2V}{dv^2} + [a - 2qcos(2v)]V = 0.,$$

and should not be mixed with the ones we define here.

2 Elliptic trigonometric functions

We now define two new functions denoted by $esin\alpha$ and $ecos\alpha$ on an ellipse and we then define the remaining elliptic trigonometric functions. Also we calculate some values of these functions.

2.1 Definitions of fundamental elliptic trigonometric functions

We begin with the definitions of the two main elliptic trigonometric functions:

Definition 2.1. *Let B be a point on the ellipse (1) as shown in Fig. 2. The ordinate value $y(B)$ of the point B is called the elliptic sine value of the angle α and denoted by $esin\alpha$. Similarly the apsis $x(B)$ of the point B is called the elliptic cosine value of the angle α and denoted by $ecos\alpha$.*

Consequently, we have $B = (ecos\alpha, esin\alpha)$.

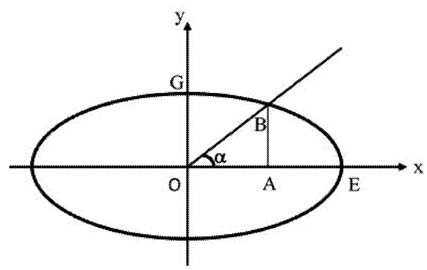

Figure 2: $esin\alpha$ and $ecos\alpha$

Now we will define the next two elliptic trigonometric functions $etan\alpha$, $ecot\alpha$ as follows. One expects the similar definitions to classical trigonometry such as $tan\alpha = sin\alpha/cos\alpha$, but there are slight differences making the elliptic trigonometry rather exciting.

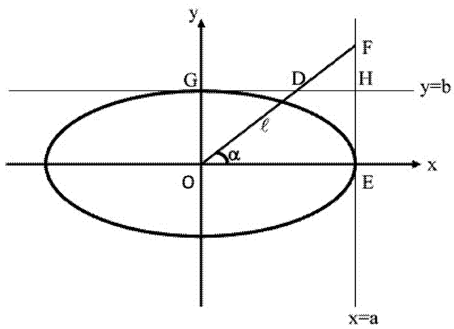

Figure 3: Tangent and cotangent axes

Definition 2.2. *Let ℓ be the line passing through origin and making an angle α with x-axis, see Fig. 3. The vertical line $x = a$ is called as elliptic tangent axis, denoted by $etan$ and the ordinate of the intersection point F of that line with the line ℓ will be called the etangent of α denoted by $etan\alpha$. Similarly, the horizontal line $y = b$ will be called as elliptic cotangent axis briefly denoted by $ecot$ and the apsis of the intersection point D of the line $y = b$ with the line ℓ will be called ecotangent of α and denoted by $ecot\alpha$.*

Therefore $etan\alpha = y(F)$ and $ecot\alpha = x(D)$.

2.2 Some fundamental relations

In this subsection, we shall try to obtain some basic properties of the elliptic trigonometric functions. These relations will be parallel to the ordinary ones with slight differences. First we have the following obvious result giving the relation between $\sin\alpha$, $\cos\alpha$, $esin\alpha$ and $ecos\alpha$:

Theorem 2.1. *For any $\alpha \in \mathbb{R}$, we have*

$$\frac{ecos\alpha}{\cos\alpha} = \frac{esin\alpha}{\sin\alpha}. \tag{2}$$

Proof. In Fig. 2, note that $\mid OA \mid = z \cdot \cos\alpha = ecos\alpha$ and $\mid AB \mid = z \cdot \sin\alpha = esin\alpha$ implying the result. □

Theorem 2.2. *For any $\alpha \in \mathbb{R}$, we have*

$$b^2 ecos^2\alpha + a^2 esin^2\alpha = a^2 b^2. \tag{3}$$

Proof. It follows from the definitions of $ecos$ and $esin$ functions. □

Theorem 2.3. *The connection between elliptic tangent, elliptic sine and elliptic cosine functions is given by*

$$etan\alpha = a\frac{esin\alpha}{ecos\alpha}.$$

Proof. The similarity $FOE \sim BOA$ in Fig. 4 implies that

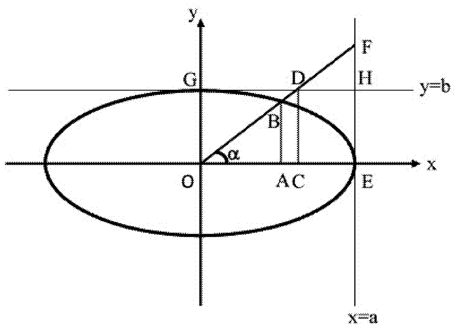

Figure 4

$$\frac{|FE|}{|BA|} = \frac{|OE|}{|OA|}.$$

This is equivalent to

$$\frac{etan\alpha}{esin\alpha} = \frac{a}{ecos\alpha}$$

and hence we have the required result. □

A similar relation can easily be given for the function $ecot$:

Theorem 2.4. *The connection between elliptic cotangent, elliptic sine and elliptic cosine functions is given by*

$$ecot\alpha = b\frac{ecos\alpha}{esin\alpha}.$$

Proof. See Fig. 4. Because of the similarity $DGO \sim OAB$, we have

$$\frac{|DG|}{|OA|} = \frac{|GO|}{|AB|}.$$

This is equivalent to

$$\frac{ecot\alpha}{ecos\alpha} = \frac{b}{esin\alpha}$$

and hence we get the result. □

By means of Theorems 2.3 and 2.4, we reach to the following surprising identity.

Corollary 2.1. *We have*

$$etan\alpha \cdot ecot\alpha = ab.$$

The distance $\mid OB \mid$ in Fig. 4 is unfixed unlike the classical trigonometry case and frequently used in many calculations. It can be stated in terms of trigonometric and elliptic trigonometric functions in many ways:

Lemma 2.1. *In the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *given in Fig. 4, we have*

$$\begin{aligned} \mid OB \mid &= \frac{ab}{\sqrt{b^2 + (a^2 - b^2)\sin^2\alpha}} \\ &= \frac{ab}{\sqrt{a^2 + (b^2 - a^2)\cos^2\alpha}} \\ &= \frac{ab}{\sqrt{b^2\cos^2\alpha + a^2\sin^2\alpha}} \\ &= ab\sqrt{\frac{esin^2\alpha + ecos^2\alpha}{a^2esin^2\alpha + b^2ecos^2\alpha}}. \end{aligned}$$

Proof. We shall just prove the first one as the others easily follow using the definitions. Let's assume that $\mid BA \mid = y$, $\mid OA \mid = x$ and $\mid OB \mid = z$ so that $x = ecos\alpha = z \cdot cos\alpha$ and $y = esin\alpha = z \cdot sin\alpha$. Also we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then substituting the above values in this ellipse equation, we have

$$\frac{z^2 \cdot \cos^2\alpha}{a^2} + \frac{z^2 \cdot \sin^2\alpha}{b^2} = 1,$$

$$z^2b^2 \cdot \cos^2\alpha + z^2a^2 \cdot \sin^2\alpha = a^2b^2,$$

and hence

$$z = \mid OB \mid = \frac{ab}{\sqrt{b^2 + (a^2 - b^2)\sin^2\alpha}}.$$

□

Now we define and study the elliptic versions of the two relatively less used trigonometric functions. Consider Fig. 5.

Definition 2.3. *The elliptic secant (cosecant) of the angle α is defined as the ordinate $y(A)$ (apsis $x(B)$) of the intersection point A (point B) and denoted by $esec\alpha$ ($ecsc\alpha$).*

That is

$$esec\alpha = x(B) \text{ and } ecsc\alpha = y(A).$$

Theorem 2.5. *We have*

$$esec\alpha = \frac{a^2}{ecos\alpha} \quad \textit{and} \quad ecsc\alpha = \frac{b^2}{esin\alpha}.$$

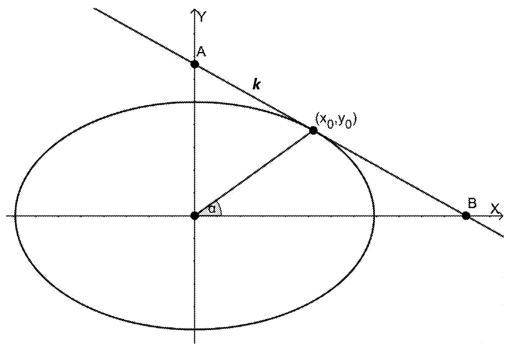

Figure 5: Tangent line of ellipse

The line ***k*** passing through the points A and B is tangent to the ellipse at the point (x_0, y_0). The parts of ellipse which lie in the first and second quadrants of the plane has explicit equation form

$$y = f(x) = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Therefore the equation of the tangent line ***k*** is

$$y = -b\frac{ecot\alpha}{a^2}(x - x_0) + y_0.$$

If we substitute $ecos\alpha$ instead of x_0 and $esin\alpha$ instead of y_0 remembering that $ecot\alpha = b\dfrac{ecos\alpha}{esin\alpha}$ we obtain

$$y = -b\frac{ecot\alpha}{a^2}(x - ecos\alpha) + esin\alpha.$$

Substituting $x = 0$, we find the ordinate $y(A)$ of the intersection point A with y-axis as

$$y = y(A) = -b\frac{ecot\alpha}{a^2}(0 - ecos\alpha) + esin\alpha$$

and finally

$$y(A) = \frac{b^2}{esin\alpha}.$$

Similarly, substituting $y = 0$ gives

$$x(B) = \frac{a^2}{ecos\alpha}.$$

If we consider the parts of the ellipse in third and fourth quadruples, we reach to the same tangent line.

2.3 Domain and range of elliptic trigonometric functions

Because of periodicity, we had to be careful when selecting the domains and range of trigonometric functions. The same is valid for elliptic trigonometric functions. Following, the domains and ranges of the elliptic trigonometric functions are listed:

For all $\alpha \in \mathbb{R}$, $\mid ecos\alpha \mid \leq a$ and $\mid esin\alpha \mid \leq b$,

for all $\alpha \in \mathbb{R}/\{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}$, $etan\alpha \in \mathbb{R}$,

for all $\alpha \in \mathbb{R}/\{k\pi : k \in \mathbb{Z}\}$, $ecot\alpha \in \mathbb{R}$.

3 Transformation formulae

In this section, we state elliptic trigonometric functions in terms of classical trigonometric functions and vice versa.

3.1 Elliptic trigonometric functions in terms of trigonometric functions

Extending some notion in mathematics, especially one like trigonometry having so many applications of our lives implies that we need to establish links with the existing notions. In that manner, we shall obtain the connections of elliptic trigonometric functions with the classical trigonometric functions. First we give the elliptic trigonometric functions in terms of classical trigonometric functions and in the next subsection, we shall state the classical trigonometric functions in terms of elliptic trigonometric functions. Let us consider the ellipse in Fig. 2:

Theorem 3.1.

$$esin\alpha = \frac{ab\sin\alpha}{\sqrt{b^2 + (a^2 - b^2)\sin^2\alpha}} \tag{4}$$

and

$$ecos\alpha = \frac{ab\cos\alpha}{\sqrt{a^2 + (b^2 - a^2)\cos^2\alpha}}. \tag{5}$$

Proof. We know that $\sin\alpha = \dfrac{\mid BA \mid}{\mid OB \mid}$, and hence we get $\mid BA \mid = \mid OB \mid \sin\alpha$. Then by Lemma 2.1, we obtain the required relation.

Similarly to $esin\alpha$, we have $\mid OA \mid = \mid OB \mid \cos\alpha$ implying the result.

It can easily be shown using the existing formulae that both denominators in $esin\alpha$ and $ecos\alpha$ are equal.

□

Corollary 3.1. *The remaining elliptic trigonometric functions are stated in terms of trigonometric functions as follows:*

i) $etan\alpha = a\tan\alpha$,

ii) $ecot\alpha = b\cot\alpha$,

iii) $esec\alpha = \dfrac{a\sqrt{a^2\sec^2\alpha - (a^2-b^2)}}{b}$,

iv) $ecsc\alpha = \dfrac{b\sqrt{b^2\csc^2\alpha + (a^2-b^2)}}{a}$.

Proof. i) Consider Fig. 4. Note that $\mid OA \mid = ecos\alpha$, $\mid AB \mid = esin\alpha$, $\mid OE \mid = a$, $\mid CD \mid = b$, $\mid EF \mid = etan\alpha$, $\mid OC \mid = ecot\alpha$. Because of the similarity $OAB \sim OCD \sim OEF$, we have

$$\frac{\mid BA \mid}{\mid OA \mid} = \frac{\mid DC \mid}{\mid OC \mid} = \frac{\mid FE \mid}{\mid OE \mid}.$$

This is equivalent to

$$\frac{esin\alpha}{ecos\alpha} = \frac{b}{ecot\alpha} = \frac{etan\alpha}{a}. \tag{6}$$

Therefore we have

$$etan\alpha = a\frac{esin\alpha}{ecos\alpha} = a\frac{\frac{ab\sin\alpha}{\sqrt{b^2+(a^2-b^2)\sin^2\alpha}}}{\frac{ab\cos\alpha}{\sqrt{a^2-(a^2-b^2)\cos^2\alpha}}} = a\tan\alpha.$$

ii) Similarly we have

$$b\cot\alpha = \frac{b}{\tan\alpha} = \frac{ab}{a\tan\alpha} = \frac{etan\alpha \cdot ecot\alpha}{etan\alpha} = ecot\alpha.$$

iii) -iv) For $esec$ and $ecsc$ functions, from the above equalities, we have the required results.

□

3.2 Trigonometric functions in terms of elliptic trigonometric functions

Now we give inverse relations. That is, we state the classical trigonometric functions by means of elliptic trigonometric functions.

Theorem 3.2. *The functions* sin *and* cos *can be stated in terms of* $esin$ *and* $ecos$ *as follows:*

$$\sin\alpha = \frac{b\cdot esin\alpha}{\sqrt{a^2b^2+(b^2-a^2)esin^2\alpha}} = \frac{esin\alpha}{\sqrt{esin^2\alpha+ecos^2\alpha}} \tag{7}$$

and

$$\cos\alpha = \frac{a\cdot ecos\alpha}{\sqrt{a^2b^2+(a^2-b^2)ecos^2\alpha}} = \frac{ecos\alpha}{\sqrt{esin^2\alpha+ecos^2\alpha}} \tag{8}$$

Proof. Both statements follows from Theorem 3.1. □

Similarly to these two functions, all the remaining four trigonometric functions can be stated in terms of their elliptic trigonometric counterparts.

4 Some elliptic trigonometric values

In this section, we calculate the values of the elliptic trigonometric functions at some special angles 0°, 90°, 180° and 270° which are multiples of a right angle and also at the angles 30°, 45° and 60° which are frequently needed in scientific calculations. We do not give the detailed calculations as they are obtained using the above definitions:

	0°	30°	45°	60°	90°	180°	270°
$esin$	0	$\frac{ab}{\sqrt{a^2+3b^2}}$	$\frac{ab}{\sqrt{a^2+b^2}}$	$\frac{\sqrt{3}ab}{\sqrt{3a^2+b^2}}$	b	0	$-b$
$ecos$	a	$\frac{\sqrt{3}ab}{\sqrt{a^2+3b^2}}$	$\frac{ab}{\sqrt{a^2+b^2}}$	$\frac{ab}{\sqrt{3a^2+b^2}}$	0	$-a$	0
$etan$	0	$\frac{a\sqrt{3}}{3}$	a	$a\sqrt{3}$	∞	0	∞
$ecot$	∞	$b\sqrt{3}$	b	$\frac{b\sqrt{3}}{3}$	0	∞	0
$esec$	a	$\frac{a\sqrt{a^2+3b^2}}{\sqrt{3}b}$	$\frac{a\sqrt{a^2+b^2}}{b}$	$\frac{a\sqrt{3a^2+b^2}}{b}$	∞	$-a$	∞
$ecsc$	∞	$\frac{b\sqrt{a^2+3b^2}}{a}$	$\frac{b\sqrt{a^2+b^2}}{a}$	$\frac{b\sqrt{3a^2+b^2}}{\sqrt{3}a}$	b	∞	$-b$

For the angles in other quadruples of the plane other than the first one, we can use the following identities in Theorem 4.1, Theorem 4.2, Corollary 4.2 and Corollary 4.3 to calculate the elliptic trigonometric values of these angles.

Theorem 4.1. *Let* α *be an angle in the first quadruple. Then we have*

$$esin(-\alpha) = -esin\alpha \quad and \quad ecos(-\alpha) = ecos\alpha.$$

Proof. See Fig. 6.

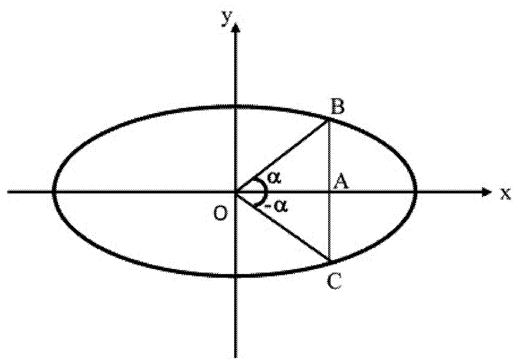

Figure 6: $esin$ is an odd function

By the similarity $BOA \cong COA$, we can easily obtain the required properties. □

By means of Theorem 4.1, we obtain the following obvious result:

Corollary 4.1. *We see that $esin\alpha$, $etan\alpha$, $ecot\alpha$ are all odd functions and $ecos\alpha$ is an even function.*

Now we consider the angles in the second quadruple:

Theorem 4.2. *Let α be an angle in the first quadruple. Then we have*

$$esin(180^\circ - \alpha) = esin\alpha \quad \textit{and} \quad ecos(180^\circ - \alpha) = -ecos(\alpha).$$

By means of Theorem 4.2, we obtain

Corollary 4.2. *We have*

$$etan(180^\circ - \alpha) = -etan\alpha \quad \textit{and} \quad ecot(180^\circ - \alpha) = -ecot\alpha.$$

The following identities can be obtained similarly:

Corollary 4.3. *For the angles in third and fourth quadruples, we have*
$esin(180^\circ + \alpha) = -esin\alpha$, $ecos(180^\circ + \alpha) = -ecos\alpha$,
$etan(180^\circ + \alpha) = etan\alpha$, $ecot(180^\circ + \alpha) = ecot\alpha$,
$esin(360^\circ + \alpha) = esin\alpha$, $ecos(360^\circ + \alpha) = ecos\alpha$,
$esin(360^\circ - \alpha) = -esin\alpha$, $ecos(360^\circ - \alpha) = ecos\alpha$.

The above identities give us the periods of the elliptic trigonometric functions which are exactly the same with the classical trigonometric functions.

Corollary 4.4. *The period of $esin\alpha$ and $ecos\alpha$ is 2π and the period of $etan\alpha$ and $ecotan\alpha$ is π.*

5 Graphics of elliptic trigonometric functions

In this section, we will draw the graphics of elliptic trigonometric functions. Note that these graphics will be changing depending on a, b. These graphics can also be found in Geogebra website, see https://www.geogebra.org/m/jjwwrdy3, https://www.geogebra.org/m/hjrjmc5n, https://www.geogebra.org/m/cxjrzgpq, https://www.geogebra.org/m/rthj3smb, https://www.geogebra.org/m/henarnha and https://www.geogebra.org/m/hqmepuaq.

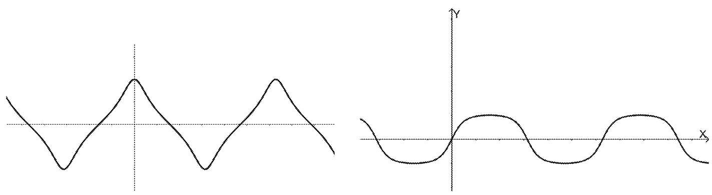

Figure 7: Graphics of $ecos\alpha$ and $esin\alpha$

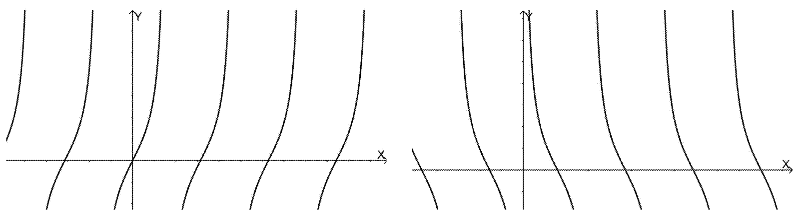

Figure 8: Graphics of $etan\alpha$ and $ecot\alpha$

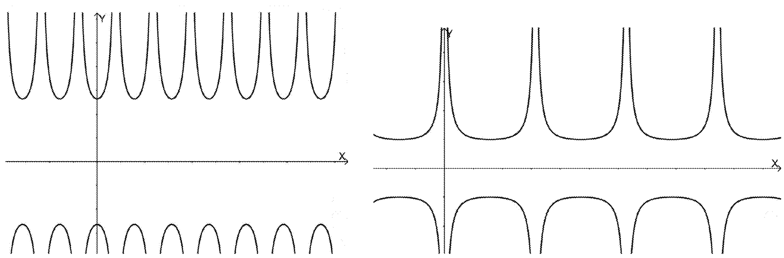

Figure 9: Graphics of $esec\alpha$ and $ecsc\alpha$

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6 Invers elliptic trigonometric functions

Now we study the inverse functions of the fundamental elliptic trigonometric functions. Note that the graphics will be changing slightly depending on a, b:

The function $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-b, b]$ given by $f(x) = esinx$ is one-to-one and onto. Its inverse is $f^{-1} : [-b, b] \longrightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is $f^{-1}(x) = arcesinx$. Similarly the function $f : [0, \pi] \longrightarrow [-a, a]$, $f(x) = ecosx$ is one-to-one and onto and its inverse is given by $f^{-1} : [-a, a] \longrightarrow [0, \pi]$, $f^{-1}(x) = arcecosx$.

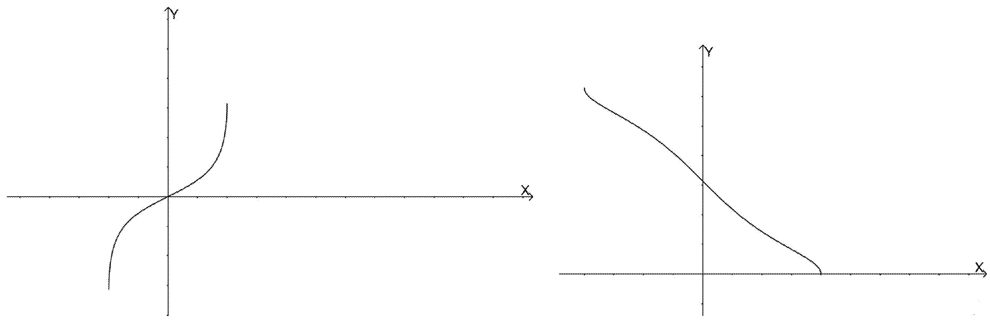

Figure 10: Graphics of $arcesinx$ and $arcecosx$

$f : \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (-\infty, \infty)$, $f(x) = etanx$ is one-to-one and onto with inverse f^{-1} : $(-\infty, \infty) \longrightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $f^{-1}(x) = arcetanx$.

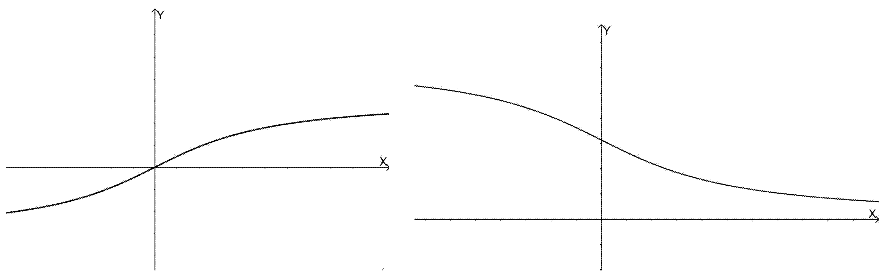

Figure 11: Graphics of $arcetanx$ and $arcecotx$

Finally the function $f : (0,\pi) \longrightarrow (-\infty,\infty)$ $f(x) = ecotx$ being one-to-one and onto has inverse $f^{-1} : (-\infty,\infty) \longrightarrow (0,\pi)$, $f^{-1}(x) = arcecotx$.

7 Derivatives and integrals of elliptic trigonometric functions

In this section, we give the derivatives and integrals of the elliptic trigonometric functions in terms of classial trigonometric functions without proof. When derivatives and integrals are considered, the situation is much more complicated than the classical trigonometric functions. It is also possible to state these derivatives and integrals in terms of elliptic trigonometric functions, but they look more complicated, so we leave them to the reader.

7.1 Derivatives of elliptic trigonometric functions

Theorem 7.1. *The derivatives of the elliptic trigonometric functions are as follows:*

i) $esin'x = \dfrac{ab^3 \cdot \cos x}{((a^2-b^2)\sin^2 x + b^2)^{\frac{3}{2}}}$,

ii) $ecos'x = \dfrac{ba^3 \cdot \cos x}{((b^2-a^2)\cos^2 x + a^2)^{\frac{3}{2}}}$,

iii) $etan'x = a \cdot \sec^2 x$,

iv) $ecot'x = -b \cdot \csc^2 x$.

7.2 Integrals of elliptic trigonometric functions

Theorem 7.2. *Let us assume that $b > a$. Then the integrals of the elliptic trigonometric functions are as follows:*

i) $\displaystyle\int ecosx\, dx = \frac{ab \cdot \arcsin\left(\frac{\sqrt{b^2-a^2}\sin x}{|b|}\right)}{\sqrt{b^2-a^2}} + C$,

ii) $\displaystyle\int esinx\, dx = -\frac{ab \cdot arcsinh\left(\frac{\sqrt{b^2-a^2}\cos x}{|a|}\right)}{\sqrt{b^2-a^2}} + C$

iii) $\displaystyle\int etanx\, dx = a \cdot \ln|\sec x| + C$, *iv)* $\int ecotx\, dx = b \cdot \ln|\sin x| + C.$

8 Sum-difference and double angle formulae for elliptic trigonometric functions

Theorem 8.1. *The sum and difference formulae for the elliptic trigonometric functions are as follows:*

i) $esin(x \pm y) = \dfrac{abK}{\sqrt{b^2 + (a^2 - b^2)K^2}}$, *where* $K = \dfrac{esinxecosy \pm esinyecosx}{\sqrt{(esin^2x + ecos^2x)(esin^2y + ecos^2y)}}$,

ii) $ecos(x \pm y) = \dfrac{abT}{\sqrt{a^2 - (a^2 - b^2)T^2}}$, *where* $T = \dfrac{ecosxecosy \mp esinxesiny}{\sqrt{(esin^2x + ecos^2x)(esin^2y + ecos^2y)}}$,

iii) $etan(x \pm y) = \dfrac{a^2(etanx \pm etany)}{a^2 \mp etanxetany}$,

iv) $ecot(x \pm y) = \dfrac{ecotxecoty \mp b^2}{ecoty \pm ecotx}$.

Proof. The results can be obtained by the help of the above equalities and identities. □

If we substitute $y = x$ at the sum formulae for the above functions, we get

Theorem 8.2. *The double angle formulae for the elliptic trigonometric functions are as follows:*

i) $esin2x = \dfrac{abM}{\sqrt{b^2 + (a^2 - b^2)M^2}}$, *where* $M = \dfrac{2ecosxesinx}{esin^2x + ecos^2x}$,

ii) $ecos2x = \dfrac{abN}{\sqrt{a^2 - (a^2 - b^2)N^2}}$, *where* $N = \dfrac{ecos^2x - esin^2x}{esin^2x + ecos^2x}$,

iii) $etan2x = \dfrac{2a^2etanx}{a^2 - etan^2x}$,

iv) $ecot2x = \dfrac{ecot^2x - b^2}{2ecotx}$.

9 Applications of elliptic trigonometric functions

We know from classical trigonometry that in a triangle as in Fig. 12, we have the relation

$$z^2 = x^2 + y^2 - 2xy\cos\alpha$$

which is known as the cosine theorem.

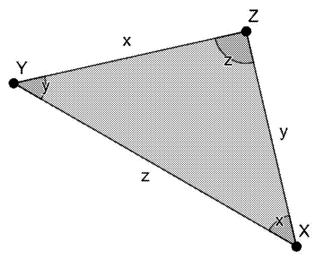

Figure 12

Recalling Eqn. 8, we obtain

Theorem 9.1. *(ecosine theorem)*

$$z^2 = x^2 + y^2 - 2xy\frac{a \cdot ecos\alpha}{\sqrt{a^2b^2 + (a^2 - b^2)ecos^2\alpha}} = x^2 + y^2 - 2xy\frac{ecos\alpha}{\sqrt{esin^2\alpha + ecos^2\alpha}}. \quad (9)$$

We know that in the triangle in Fig. 12, we have a relation called as sinus theorem:

$$\frac{z}{\sin Z} = \frac{x}{\sin X} = \frac{y}{\sin Y}$$

and also using Eqn. 7, we obtain

Theorem 9.2. *(esine theorem)*

$$z\frac{\sqrt{esin^2Z + ecos^2Z}}{esinZ} = x\frac{\sqrt{esin^2X + ecos^2X}}{esinX} = y\frac{\sqrt{esin^2Y + ecos^2Y}}{esinY}. \quad (10)$$

Finally we know that $A(XYZ) = \frac{1}{2}xy\sin Z$. By Eqn. 7, we obtain

Theorem 9.3. *(Triangle area)*

$$A(XYZ) = \frac{1}{2}xy\frac{esin\alpha}{\sqrt{esin^2\alpha + ecos^2\alpha}}. \quad (11)$$

10 Relation between elliptic trigonometric functions and complex numbers

As $e^{i\alpha} = \cos\alpha + i\sin\alpha$, using Eqns. 7 and 8, we can obtain $e^{i\alpha}$ in terms of elliptic trigonometric functions:

$$e^{i\alpha} = \frac{ecos\alpha + i\,esin\alpha}{\sqrt{esin^2\alpha + ecos^2\alpha}}. \quad (12)$$

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References

[1] Bayeh, C., Bernard, M., Moubayed, N., Introduction to the Elliptical Trigonometry, *WSEAS Transactions on Mathematics*, 8(9): 551-560 (2009)

[2] Christopher, M., From Eudoxus to Einstein: A History of Mathematical Astronomy, Cambridge University Press, 2004

[3] Dutta, M., Debnath, L., Elements of the Theory of Elliptic and Associated Functions with Applications, The World Press, 1965

[4] Du Val, P., Elliptic Functions and Elliptic Curves, Cambridge University Press, 1973

[5] Hancock, H., Lectures on the Theory of Elliptic Functions, Dover Publications, 1958

[6] Mathieu, E., Memoire sur le mouvement vibratoire d'une membrane de forme elliptique. *J. Math. Pure Appl.*, 13: 137-203 (1868)

[7] Thielman, H.P., *A Generalization of Trigonometry*, *National Mathematics Magazine*, 11(8): 349-351 (1937)