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ARITHMETIC IDENTITIES OF RAMANUJAN'S GENERAL PARTITION FUNCTION FOR MODULO 17

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ABSTRACT. In this paper, we prove four infinite families of congruences modulo 17 for the general partition function $p_r(n)$ for negative values of r. Our emphasis throughout this paper is to exhibit the use of q-identities to generate congruences for the general partition function.

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1. INTRODUCTION

Throughout the paper, we assume $|q| < 1$ and use the standard q-series notation

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^{k-1}).$$

In his letter written to G. H. Hardy, Ramanujan [5, pp. 192−193] introduced the general partition function $p_r(n)$ which is defined as

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{(q;q)_\infty^r}, \tag{1}$$

where r is a non-zero integer and n is a non-negative integer. For $r = 1$, $p_1(n) = p(n)$ is the usual unrestricted partition function which counts the number of unrestricted partitions of n. In [15–17], Ramanujan found nice congruence properties modulo powers of 5, 7 and 11, namely

$$p(5n+4) \equiv 0 \pmod 5, \tag{2}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}$$

for any non-negative integer n. Furthermore, K. G. Ramanathan [14] considers the generalization of these congruences modulo powers of 5 and 7 for all $p_r(n)$ but A. O. L. Atkin [1] found that Ramanathan's results are incorrect. Recently, N. D. Baruah and B. K. Sarmah [3] proved (2) and (3) by using binomial theorem. M. Newmann [11–13], studied the function $p_r(n)$ and obtained several interesting congruences and identities involving $p_r(n)$. The function $p_r(n)$ have been studied by various mathematicians. For the wonderful introduction on partitions, one can see the work of Atkin [1],

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Baruah and Ojah [2], Baruah and B. K. Sarmah [3], B. C. Berndt and R. A. Rankin [5], M. Boylan [6], H. M. Farkas and I. Kra [7], J. M. Gandhi [8], B. Gordon [9], I. Kimming and J. B. Olsson [10], M. Newmann [11–13], Ramanathan [14], Ramanujan [15–17], N. Saikia and J. Chetry [18]. In the sequel, in this paper, we demonstrate four new infinite families of congruences modulo 17 by employing q-identities, for the general partition function $p_r(n)$, r being negative. We prove the following partition identities for any non-negative integer λ and n:

Theorem 1.1. *We have for* $k = 3, 4, 8, 10, 11, 13, 14, 16$,

$$p_{-(17\lambda+1)}(17n + k) \equiv 0 \pmod{17}.$$

Theorem 1.2. *We have for* $k = 2, 5, 7, 8, 9, 12, 13, 14, 16$,

$$p_{-(17\lambda+3)}(17n + k) \equiv 0 \pmod{17}.$$

Theorem 1.3. *We have for* $k = 1, 2, \ldots, 16$,

$$p_{-(289\lambda+1)}(289n + 17k + 12) \equiv 0 \pmod{17}.$$

Theorem 1.4. *We have for* $k = 1, 2, \ldots, 16$,

$$p_{-(289\lambda+2)}(289n + 17k + 7) \equiv 0 \pmod{17}.$$

2. Preliminaries

We have, from [4, p. 397, Entry 12(i), Eqns. (12.1) and (12.3)]

(4)
$$(q; q)_\infty = (q^{289}; q^{289})_\infty (A - qB - q^2C + q^5D + q^7E - q^{12} - q^{15}F + q^{26}G - q^{40}H),$$

where

$$A := A(q^{17}) = \frac{f(-q^{102}, -q^{187})}{f(-q^{51}, -q^{238})}, \qquad B := B(q^{17}) = \frac{f(-q^{68}, -q^{221})}{f(-q^{34}, -q^{255})},$$
$$C := C(q^{17}) = \frac{f(-q^{136}, -q^{153})}{f(-q^{68}, -q^{221})}, \qquad D := D(q^{17}) = \frac{f(-q^{34}, -q^{255})}{f(-q^{17}, -q^{272})},$$
$$E := E(q^{17}) = \frac{f(-q^{119}, -q^{170})}{f(-q^{85}, -q^{204})}, \qquad F := F(q^{17}) = \frac{f(-q^{85}, -q^{204})}{f(-q^{102}, -q^{187})},$$
$$G := G(q^{17}) = \frac{f(-q^{51}, -q^{238})}{f(-q^{98}, -q^{170})}, \qquad H := H(q^{17}) = \frac{f(-q^{17}, -q^{272})}{f(-q^{136}, -q^{153})}$$

and

$$CD - AE - BH + FG = 1. \tag{5}$$

Further, on squaring (4), we obtain

$$
\begin{aligned}
(q;q)_\infty^2 \equiv (q^{289};q^{289})_\infty^2 &(A^2-2D+2CF+q(G^2-2AB+2H)\\
&+q^2(B^2-2AC-2E)+2q^3(BC-DF)+q^4(C^2-2G+2FH)\\
&+2q^5(AD-EF)-2q^6(D+AH)+q^7(1+2AE-2CD+2BH\\
&-2FG)+2q^8(CH-BE)+2q^9(AG-CE)+q^{10}(D^2-2BG\\
&+2F)-2q^{11}(DH+CG)+q^{12}(H^2-2A+2DE)\\
&+q^{13}(F^2+2B-2EH)+q^{14}(E^2+2C+2DG)\\
&-2q^{15}(AF+GH)+2q^{16}(BF+EG)) \pmod{17}. \qquad (6)
\end{aligned}
$$

Again from [4, p. 39, Entry 24(ii)], we have

$$
(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \qquad (7)
$$

From (7), we deduce that

$$
\begin{aligned}
(q;q)_\infty^3 =&J_0(q^{17})-3qJ_1(q^{17})+5q^3J_2(q^{17})-7q^6J_3(q^{17})+9q^{10}J_4(q^{17})\\
&-11q^{15}J_5(q^{17})+13q^{21}J_6(q^{17})-15q^{28}J_7(q^{17})+17J_8(q^{17}),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
(q;q)_\infty^3 \equiv&J_0(q^{17})-3qJ_1(q^{17})+5q^3J_2(q^{17})-7q^6J_3(q^{17})+9q^{10}J_4(q^{17})\\
&-11q^{15}J_5(q^{17})+13q^{21}J_6(q^{17})-15q^{28}J_7(q^{17})) \pmod{17}, \qquad (8)
\end{aligned}
$$

where $J_0, J_1, \ldots, J_8$ are the series with integral powers of q^{17}. Also from the binomial theorem, it follows that

$$
(q;q)_\infty^{17} \equiv (q^{17};q^{17})_\infty \pmod{17}. \qquad (9)
$$

3. Proof of Theorem 1.1-1.4

All the congruences in this section are to the modulus 17.

Proof of Theorem 1.1. In (1), put $r=-(17\lambda+1)$, we have

$$
\sum_{n=0}^{\infty}p_{-(17\lambda+1)}(n)q^n=(q;q)_\infty^{17\lambda+1}=(q;q)_\infty^{17\lambda}(q;q)_\infty. \qquad (10)
$$

Using (9) in (10), we have

$$
\sum_{n=0}^{\infty}p_{-(17\lambda+1)}(n)q^n\equiv(q^{17};q^{17})_\infty^{\lambda}(q;q)_\infty. \qquad (11)
$$

Employing (4) in (11), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}p_{-(17\lambda+1)}(n)q^n &\equiv (q^{17};q^{17})_\infty^{\lambda}(q^{289};q^{289})_\infty\\
&\times(A-qB-q^2C+q^5D+q^7E-q^{12}-q^{15}F+q^{26}G-q^{40}H) \qquad (12)
\end{aligned}
$$

Extracting the terms containing q^{17n+k} on both sides of (12) for $k=3,4,8,10,11,13,14,16$, we arrive at Theorem 1.1. □

Proof of Theorem 1.2. In (1), put $r = -(17\lambda + 3)$ and then using (9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(17\lambda+3)}(n)q^n &= (q;q)_\infty^{17\lambda+3} = (q;q)_\infty^{17\lambda}(q;q)_\infty^3 \\ &\equiv (q^{17};q^{17})_\infty^{\lambda}(q;q)_\infty^3. \end{aligned} \tag{13}$$

Employing (8) in (13), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(17\lambda+3)}(n)q^n \equiv (q^{17};q^{17})_\infty^{\lambda}(&J_0 - 3qJ_1 + 5q^3J_2 - 7q^6J_3 + 9q^{10}J_4 \\ &- 11q^{15}J_5 + 13q^{21}J_6 - 15q^{28}J_7) \end{aligned} \tag{14}$$

Extracting the terms containing q^{17n+k} on both sides of (14) for $k = 2, 5, 7, 8, 9, 12, 13, 14, 16$, we complete the proof. □

Proof of Theorem 1.3. In (1), put $r = -(289\lambda + 1)$ and then using (9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(289\lambda+1)}(n)q^n &= (q;q)_\infty^{289\lambda+1} = (q;q)_\infty^{289\lambda}(q;q)_\infty \\ &\equiv (q^{289};q^{289})_\infty^{\lambda}(q;q)_\infty. \end{aligned} \tag{15}$$

Employing (4) in (15), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(289\lambda+1)}(n)q^n &\equiv (q^{289};q^{289})_\infty^{\lambda+1} \\ &\times (A - qB - q^2C + q^5D + q^7E - q^{12} - q^{15}F + q^{26}G - q^{40}H) \end{aligned} \tag{16}$$

Extracting the terms containing q^{17n+12} on both sides of (16), dividing throughout by q^{12} and then letting q by $q^{1/17}$, we have

$$\sum_{n=0}^{\infty} p_{-(289\lambda+1)}(17n+12)q^n \equiv (-1)(q^{17};q^{17})_\infty^{\lambda+1}. \tag{17}$$

Extracting the terms containing q^{17n+k} on both sides of (17) for $k = 1, 2, \ldots, 16$, we obtain Theorem 1.3. □

Proof of Theorem 1.4. In (1), put $r = -(289\lambda + 2)$ and then using (9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(289\lambda+2)}(n)q^n &= (q;q)_\infty^{289\lambda+2} = (q;q)_\infty^{289\lambda}(q;q)_\infty^2 \\ &\equiv (q^{289};q^{289})_\infty^{\lambda}(q;q)_\infty^2. \end{aligned} \tag{18}$$

Employing (6) in (18), we obtain

$$\sum_{n=0}^{\infty} p_{-(289\lambda+2)}(n)q^n \equiv (q^{289};q^{289})_\infty^{\lambda+2}(A^2-2D+2CF+q(G^2-2AB+2H) + q^2(B^2-2AC-2E)+2q^3(BC-DF)+q^4(C^2-2G + 2FH)+2q^5(AD-EF)-2q^6(D+AH)+q^7(1+2AE - 2CD+2BH-2FG)+2q^8(CH-BE)+2q^9(AG-CE) + q^{10}(D^2-2BG+2F)-2q^{11}(DH+CG)+q^{12}(H^2-2A + 2DE)+q^{13}(F^2+2B-2EH)+q^{14}(E^2+2C+2DG) - 2q^{15}(AF+GH)+2q^{16}(BF+EG)) \tag{19}$$

Extracting the terms containing q^{17n+7} on both sides of (19), dividing by q^7, employing (5) and then letting q to $q^{1/17}$, we arrive at

$$\sum_{n=0}^{\infty} p_{-(289\lambda+2)}(17n+7)q^n \equiv (-1)(q^{17};q^{17})_\infty^{\lambda+2}.$$

On extracting the terms containing q^{17n+k} on both sides of the above for $k=1,2,\dots,16$, we obtain Theorem 1.4. □

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