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Signed 2-independence of Cartesian product of directed cycles

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Abstract

A function $f : V(D) \to \{-1, 1\}$ defined on the vertices of a digraph $D = (V(D), A(D))$ is called a signed 2-independence function if $f(N^-[v]) \leq 1$ for every v in D. The weight of a signed 2-independence function is $f(V(D)) = \sum_{v \in V(D)} f(v)$. The signed 2-independence number of a digraph D, denoted by $\alpha_s^2(D)$, equals the maximum weight of a signed 2-independence function on D. Let $C_m \times C_n$ be the Cartesian product of directed cycles C_m and C_n. In this paper, the exact values of $\alpha_s^2(C_m \times C_n)$ for $2 \leq m \leq 5$ and $n \geq 2$ are determined.

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Keywords: Signed 2-independence function, Signed 2-independence number, Cartesian product, Directed cycle.

1 Introduction

All digraphs considered in this paper are finite, without loops and multiple arcs. For notation and terminology not defined here, we generally follow [1]. Specially, let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. We say that u is an *in-neighbor* of v and v is an *out-neighbor* of u if uv is an arc of D. For a vertex $v \in V(D)$, the sets of in-neighbors and out-neighbors of v are called the *open in-neighborhood* $N_D^-(v)$ and *open out-neighborhood* $N_D^+(v)$ of v, respectively. The *closed in-neighborhood* of v is $N_D^-[v] = N_D^-(v) \cup \{v\}$. The numbers $d_D^-(v) = |N_D^-(v)|$ and $d_D^+(v) = |N_D^+(v)|$ are the *indegree* and *outdegree* of v, respectively. We use $\delta^-(D)$, $\Delta^-(D)$, $\delta^+(D)$, and $\Delta^+(D)$ to denote the *minimum indgree, maximum indgree, minimum outdegree* and *maximum outdegree* of a vertex in D, respectively. In all cases above, we omit the subscript D when no ambiguity on D is possible. For $S \subseteq V(D)$, $D[S]$ denotes the subdigraph induced by S.

Given two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, the *Cartesian product* $D_1 \times D_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in A(D_1 \times D_2)$ if and

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only if $x_1 = y_1$ and $x_2y_2 \in A_2$ or $x_2 = y_2$ and $x_1y_1 \in A_1$, where $x_i, y_i \in V_i$ for $i = 1, 2$. We use $D_1 \cong D_2$ to denote that D_1 and D_2 are isomorphic. Throughout this paper, we denote the sets of vertices of directed cycles C_m and C_n by $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively, and $A(C_m) = \{u_1u_2, u_2u_3, \ldots, u_{m-1}u_m, u_mu_1\}$ and $A(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Moreover, in Cartesian product $C_m \times C_n$ (see Figure 1), let $X_j = \bigcup_{i=1}^{n}\{(u_j, v_i)\}$ for $1 \le j \le m$ and let $Y_i = \bigcup_{j=1}^{m}\{(u_j, v_i)\}$ for $1 \le i \le n$. Throughout this paper, for Y_i, the subscript i is taken modulo n. Thus, if $i \le 0$, then $Y_i = Y_{n+i}$, and if $i > n$, then $Y_i = Y_{i-n}$. For $S \subseteq V(C_m \times C_n)$, we write $Proj_{C_m}(S)$ to indicate the natural projection of S to $V(C_m)$.

For a function $f : V(D) \to \{-1, 1\}$, the *weight* of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. For a vertex $v \in V(D)$, we denote $f(N^-[v])$ by $f[v]$ for notational convenience.

The study of signed 2-independence number of undirected graphs was studied by [2,5] and elsewhere. Recently, Volkmann [6] began to investigate this parameter in digraphs. Formally, a function $f : V(D) \to \{-1, 1\}$ is called a *signed 2-independence function* (abbreviated, S2IF) if $f[v] \le 1$ for every vertex $v \in V(D)$. The *singed 2-independence number*, denoted by $\alpha_s^2(D)$, of D is the maximum weight of a S2IF on D. We call a S2IF of weight $\alpha_s^2(D)$ a $\alpha_s^2(D)$-function on D. Volkmann [6] presented some upper bounds on $\alpha_s^2(D)$ for general digraph D, Wang and Kim [7] determined the exact values of $\alpha_s^2(P_m \times P_n)$ for Cartesian product $P_m \times P_n$, where $1 \le m \le 5$ and $n \ge 1$. Throughout this paper, if f is a S2IF of D, then we let P and M denote the sets of those vertices in D which are assigned under f the value 1 and -1, respectively. Therefore $|V(D)| = |P| + |M|$ and $\alpha_s^2(D) = |P| - |M|$.

In this paper, our aim is to determine exact values of $\alpha_s^2(C_m \times C_n)$ for $2 \le m \le 5$ and $n \ge 2$.

2 Main results

In this section exact values of $\alpha_s^2(C_m \times C_n)$ are determined for Cartesian product $C_m \times C_n$ with $2 \le m \le 5$ and $n \ge 2$. From the definition of S2IF, the following lemmas are straightforward.

Lemma 1 *Let $D = (V(D), A(D))$ be a digraph. Then $\alpha_s^2(D)$ has the same parity with $|V(D)|$.*

Lemma 2 *Let f be a S2IF of $C_m \times C_n$. For $1 \le i \le n$, if $|Y_i \cap P| = m$, then $|Y_{i-1} \cap P| = 0$.*

Volkmann [6] established the following result.

Theorem 3 (Volkmann [6]) *If D is a digraph of order n such that $d^+(v) = r$ for all $v \in V(D)$, then $\alpha_s^2(D) \le \frac{n}{r+1}$.*

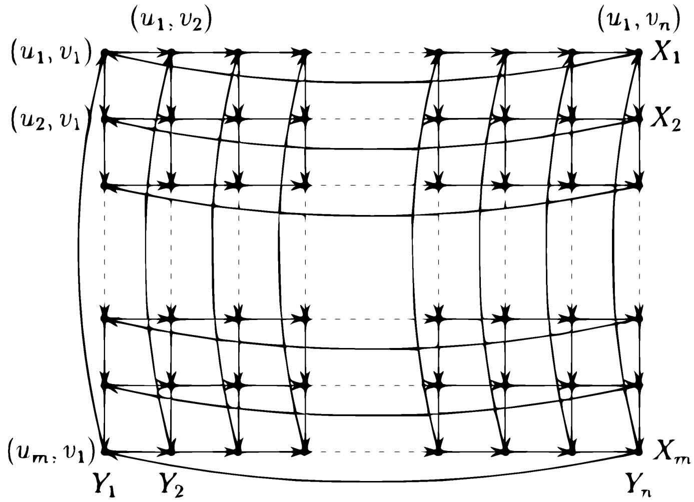

Figure 1: The Cartesian product $C_m \times C_n$

Notice that $d^+(x, y) = 2$ for every $(x, y) \in V(C_m \times C_n)$. As an immediate consequence of Theorem 3, we have the following corollary.

Corollary 4 *For integers $m \geq 2$ and $n \geq 2$, $\alpha_s^2(C_m \times C_n) \leq \frac{mn}{3}$.*

Theorem 5 *For any integer $n \geq 2$, $\alpha_s^2(C_2 \times C_n) = 0$.*

Proof. The proof is by induction on n. If $n = 2$, then $\alpha_s^2(C_2 \times C_2) \leq 0$ by Lemma 1 and Corollary 4. The function $f : V(C_2 \times C_2) \to \{-1, 1\}$ is defined as follows: $f((u_1, v_1)) = f((u_1, v_2)) = -1$ and $f((u_2, v_1)) = f((u_2, v_2)) = 1$. Obviously, f is a S2IF of $C_2 \times C_2$ with weight $w(f) = 0$, and so $\alpha_s^2(C_2 \times C_2) = 0$. Assume that $\alpha_s^2(C_2 \times C_l) = 0$ for all integers $2 \leq l < n$. Next we show that $\alpha_s^2(C_2 \times C_n) = 0$. Define $g : V(C_2 \times C_n) \to \{-1, 1\}$ by assigning to each vertex of X_1 the value -1 while to each vertex of X_2 the value 1. It is easy to check that g is a S2IF on $C_2 \times C_n$ of weight $w(g) = 0$. So $\alpha_s^2(C_2 \times C_n) \geq 0$. Suppose that $\alpha_s^2(C_2 \times C_n) = 0$ is false. Then $\alpha_s^2(C_2 \times C_n) \geq 2$ from Lemma 1. Let h be a $\alpha_s^2(C_2 \times C_n)$-function. We obtain $|P| \geq n + 1$ because $|P| - |M| \geq 2$ and $|P| + |M| = 2n$. Hence there must exist a subset Y_i in $C_2 \times C_n$ such that $|Y_i \cap P| = 2$. Without loss of generality, suppose that $|Y_n \cap P| = 2$. Then $|Y_{n-1} \cap P| = 0$ by Lemma 2. If $n = 3$, then $|P| \geq 4$. Thus $|Y_1 \cap P| = 2$, contradicting Lemma 2. So $n \geq 4$. Define $D_1 = D[V(C_2 \times C_n) \setminus \{Y_{n-1}, Y_n\}] \cup \{(u_1, v_{n-2})(u_1, v_1), (u_2, v_{n-2})(u_2, v_1)\}$. Then $D_1 \cong C_2 \times C_{n-2}$. Applying the induction hypothesis, $\alpha_s^2(D_1) = 0$. Clearly, $h_1 = h|_{D_1}$ is a S2IF of D_1 with weight $w(h_1) \geq (|P| - 2) - (|M| - 2) = |P| - |M| \geq 2$. This implies that $\alpha_s^2(D_1) \geq 2$, a contradiction. Therefore $\alpha_s^2(C_2 \times C_n) = 0$. □

Now we consider the Cartesian product $C_3 \times C_n$. To our purpose, the following result is useful.

Lemma 6 *Let f be a S2IF of $C_3 \times C_n$. For $1 \leq i \leq n$, if $|Y_i \cap P| = 3$, then $|Y_{i+1} \cap P| \leq 1$.*

Proof. The statement is trivial by the definition of S2IF. □

Theorem 7 *For any integer $n \geq 2$,*

$$\alpha_s^2(C_3 \times C_n) = \begin{cases} n & \text{if } n \equiv 0 \ (mod\ 3), \\ n-2 & \text{otherwise.} \end{cases}$$

Proof. To complete the proof, we distinguish three cases.

Case 1. $n \equiv 0$ $(mod\ 3)$. Then $n = 3k$ for some integer $k \geq 1$. Let $A_i = \{(u_3, v_{3i-2}), (u_1, v_{3i-1}), (u_2, v_{3i})\}$ for $1 \leq i \leq k$, and let $A = \cup_{i=1}^{k} A_i$. Assigning to all vertices of A the value -1 and to all other vertices the value 1, we produce a S2IF f of $C_3 \times P_{3k}$ with weight $w(f) = 3k$. Hence $\alpha_s^2(C_3 \times C_{3k}) \geq 3k = n$. On the other hand, $\alpha_s^2(C_3 \times C_{3k}) \leq 3k = n$ by Corollary 4. So $\alpha_s^2(C_3 \times C_{3k}) = 3k = n$.

Case 2. $n \equiv 2$ $(mod\ 3)$. Then $n = 3k + 2$ for some integer $k \geq 0$. We proceed our proof by induction on k. If $k = 0$, then $n = 2$. Note that $C_3 \times C_2 \cong C_2 \times C_3$. By Theorem 5, $\alpha_s^2(C_3 \times C_2) = \alpha_s^2(C_2 \times C_3) = 0 = n-2$. Assume, then, that $\alpha_s^2(C_3 \times C_{3l+2}) = 3l$ for all integers $0 \leq l < k$. Now we show that $\alpha_s^2(C_3 \times C_{3k+2}) = 3k = n-2$. Let A be defined as in the proof of Case 1 of Theorem 7. Define $g : V(C_3 \times C_{3k+2}) \to \{-1, 1\}$ by assigning to each vertex of $A \cup \{(u_3, v_{3k+1}), (u_2, v_{3k+2}), (u_3, v_{3k+2})\}$ the value -1 and to each vertex of $V(C_3 \times C_{3k+2}) \setminus (A \cup \{(u_3, v_{3k+1}), (u_2, v_{3k+2}), (u_3, v_{3k+2})\})$ the value 1. It can be readily verified that g is a S2IF on $C_3 \times C_{3k+2}$ of weight $w(g) = 3k$, and so $\alpha_s^2(C_3 \times C_{3k+2}) \geq 3k = n - 2$. Suppose that $\alpha_s^2(C_3 \times C_{3k+2}) \neq 3k$. Then $\alpha_s^2(C_3 \times C_{3k+2}) = 3k + 2$ by Lemma 1 and Corollary 4. Let g_1 be a $\alpha_s^2(C_3 \times C_{3k+2})$-function.

Since $|P| - |M| = 3k + 2$ and $|P| + |M| = 9k + 6$, we have $|P| = 6k + 4$. Suppose that there is a subset Y_i for some $1 \leq i \leq 3k + 2$ such that $|Y_i \cap P| = 3$. Then $|Y_{i-1} \cap P| = 0$ by Lemma 2. Let $D_1 = D[V(C_3 \times C_{3k+2}) \setminus \{Y_{i-1}, Y_i\}] \cup (\cup_{j=1}^{3} \{(u_j, v_{i-2})(u_j, v_{i+1})\})$. Then $D_1 \cong C_3 \times C_{3k}$. According to the result of Case 1 of Theorem 7, $\alpha_s^2(D_1) = 3k$. Obviously, $g_2 = g_1|_{D_1}$ is a S2IF of D_1 with weight $w(g_2) \geq (|P|-3)-(|M|-3) = |P|-|M| = 3k+2$. This yields that $\alpha_s^2(D_1) \geq 3k+2$, a contradiction. Hence $|Y_i \cap P| = 2$ for each $1 \leq i \leq 3k+2$ as $|P| = 6k+4$. Without loss of generality, assume that $Y_{3k+2} \cap P = \{(u_1, v_{3k+2}), (u_2, v_{3k+2})\}$. Then $Y_{3k+1} \cap P = \{(u_1, v_{3k+1}), (u_3, v_{3k+1})\}$ and $Y_{3k} \cap P = \{(u_2, v_{3k}), (u_3, v_{3k})\}$. Thus $Y_{3k-1} \cap P = \{(u_1, v_{3k-1}), (u_2, v_{3k-1})\}$. Define $D_2 = D[V(C_3 \times C_{3k+2}) \setminus \{Y_{3k}, Y_{3k+1}, Y_{3k+2}\}] \cup (\cup_{j=1}^{3} \{(u_j, v_{3k-1})(u_j, v_1)\})$. Then $D_2 \cong C_3 \times C_{3(k-1)+2}$. So $\alpha_s^2(D_2) = 3k - 3$ by the induction hypothesis. It is not hard to see that $g_3 = g_1|_{D_2}$ is a S2IF on D_2 of weight $w(g_3) \geq (|P| - 6) - (|M| - 3) = |P| - |M| - 3 = 3k - 1$. This means that $\alpha_s^2(D_2) \geq 3k - 1$, a contradiction. Consequently, $\alpha_s^2(C_3 \times C_{3k+2}) = 3k = n - 2$.

Case 3. $n \equiv 1$ $(mod\ 3)$. Then $n = 3k + 1$ for some integer $k \geq 1$. We now show by induction on k that $\alpha_s^2(C_3 \times C_{3k+1}) = 3k - 1 = n - 2$. Let A_1 and A

be defined as in the proof of Case 1 of Theorem 7. If $k = 1$, then $n = 4$. Let $h : V(C_3 \times C_4) \to \{-1, 1\}$ be a function that assigns the value -1 to each vertex of $A_1 \cup \{(u_2, v_4), (u_3, v_4)\}$ and 1 to each vertex of $V(C_3 \times C_4) \setminus (A_1 \cup \{(u_2, v_4), (u_3, v_4)\})$. It is straightforward to check that h is a S2IF of $C_3 \times C_4$ with weight $w(h) = 2$. Hence $\alpha_s^2(C_3 \times C_4) \geq 2 = n - 2$. Suppose that $\alpha_s^2(C_3 \times C_4) = 2$ is not true. Then $\alpha_s^2(C_3 \times C_4) = 4$ according to Lemma 1 and Corollary 4. Let h_1 be a $\alpha_s^2(C_3 \times C_4)$-function. We obtain $|P| = 8$ because $|P| - |M| = 4$ and $|P| + |M| = 12$. If there is a subset Y_i for some $1 \leq i \leq 4$ such that $|Y_i \cap P| = 3$, then it follows from Lemma 2 that $|Y_{i-1} \cap P| = 0$. By Lemma 6, $|Y_{i+1} \cap P| \leq 1$. Thus $|P| \leq 7$, a contradiction. Hence, for each $1 \leq i \leq 4$, $|Y_i \cap P| = 2$ as $|P| = 8$. Without loss of generality, assume that $Y_4 \cap P = \{(u_1, v_4), (u_2, v_4)\}$. Thus we can get $Y_1 \cap P = \{(u_1, v_1), (u_2, v_1)\}$. Then $h_1[(u_2, v_1)] = 3$, which is a contradiction. Therefore $\alpha_s^2(C_3 \times C_4) = 2 = n - 2$.

Assume that $\alpha_s^2(C_3 \times C_{3l+1}) = 3l - 1$ for all integers $1 \leq l < k$. We will prove that $\alpha_s^2(C_3 \times C_{3k+1}) = 3k - 1 = n - 2$. Let $A' = A \cup \{(u_2, v_{3k+1}), (u_3, v_{3k+1})\})$. Clearly, the function $h_2 : V(C_3 \times C_{3k+1}) \to \{-1, 1\}$ defined by $h_2((x, y)) = -1$ for each $(x, y) \in A'$ and $h_2((x, y)) = 1$ for each $(x, y) \in V(C_3 \times C_{3k+1}) \setminus A'$ is a S2IF of $C_3 \times C_{3k+1}$ with weight $w(h_2) = 3k - 1$. Hence $\alpha_s^2(C_3 \times C_{3k+1}) \geq 3k - 1 = n - 2$. Suppose that $\alpha_s^2(C_3 \times C_{3k+1}) = 3k - 1$ is false. Applying Lemma 1 and Corollary 4, $\alpha_s^2(C_3 \times C_{3k+1}) = 3k + 1$. Let h_3 be a $\alpha_s^2(C_3 \times C_{3k+1})$-function. Since $|P| - |M| = 3k + 1$ and $|P| + |M| = 9k + 3$, we have $|P| = 6k + 2$. Suppose that there exists a subset Y_i for some $1 \leq i \leq 3k + 1$ such that $|Y_i \cap P| = 3$. Then $|Y_{i-1} \cap P| = 0$ by Lemma 2. Let $D_3 = D[V(C_3 \times C_{3k+1}) \setminus \{Y_{i-1}, Y_i\}] \cup (\cup_{j=1}^{3} \{(u_j, v_{i-2})(u_j, v_{i+1})\})$. Then $D_3 \cong C_3 \times C_{3(k-1)+2}$. By the result of Case 2 of Theorem 7, $\alpha_s^2(D_3) = 3k - 3$. It is not difficult to see that $h_4 = h_3|_{D_3}$ is a S2IF on D_3 of weight $w(h_4) \geq (|P| - 3) - (|M| - 3) = |P| - |M| = 3k + 1$. This implies that $\alpha_s^2(D_3) \geq 3k + 1$, a contradiction. Hence $|Y_i \cap P| = 2$ for each $1 \leq i \leq 3k + 1$ as $|P| = 6k + 2$. By a similar argument that used in the proof of Case 2 of Theorem 7, we can get the same contradiction. Hence $\alpha_s^2(C_3 \times C_{3k+1}) = 3k - 1 = n - 2$. This completes the proof of Theorem 7. □

We now turn our attention to the Cartesian product $C_4 \times C_n$. To determine the exact value of $\alpha_s^2(C_4 \times C_n)$, the following lemma is required.

Lemma 8 *Let f be a S2IF of $C_4 \times C_n$. The following statements are true:*

(1) *For $1 \leq i \leq n$, if $|Y_i \cap P| = 3$, then $|Y_{i-1} \cap P| \leq 2$ and $|Y_{i+1} \cap P| \leq 2$. Furthermore, if $|Y_{i-1} \cap P| = 2$, then $Y_{i-1} \cap P = \{(u_j, v_{i-1}), (u_{j+1}, v_{i-1})\}$ for some $1 \leq j \leq 4$, where $u_5 = u_1$.*

(2) *For $1 \leq i \leq n$, if $|Y_i \cap P| = 4$, then $|Y_{i+1} \cap P| \leq 2$. Furthermore, if $|Y_{i+1} \cap P| = 2$, then $|P \cap \{(u_j, v_{i+1}), (u_{j+1}, v_{i+1})\}| \leq 1$ for any $1 \leq j \leq 4$, where $u_5 = u_1$.*

(3) *For $1 \leq i \leq n$, $|(Y_i \cup Y_{i+1}) \cap P| \leq 6$. Furthermore, $|(Y_i \cup Y_{i+1}) \cap P| = 6$ if and only if $|Y_i \cap P| = 4$ and $|Y_{i+1} \cap P| = 2$.*

(4) *For $1 \leq i \leq n$, $|(Y_i \cup Y_{i+1} \cup Y_{i+2}) \cap P| \leq 8$.*

(5) *For $1 \leq i \leq n$, $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| \leq 11$. Furthermore, if $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| = 11$, then $|Y_i \cap P| = 4$, $|Y_{i+1} \cap P| = |Y_{i+2} \cap P| = 2$ and $|Y_{i+3} \cap P| = 3$.*

(6) *In* $C_4 \times C_n$ $(n > 9)$, *suppose that* $2 \le |Y_i \cap P| \le 3$ *and* $|Y_i \cap P| \ne |Y_{i+1} \cap P|$ *for each* $1 \le i \le n$. *If* $|Y_i \cap P| = 3$, *then* $Proj_{C_4}(Y_i \cap P) = Proj_{C_4}(Y_{i-8} \cap P)$.

Proof. (1) Suppose that $|Y_i \cap P| = 3$ for some $1 \le i \le n$. By the symmetry of $C_4 \times C_n$, we may assume, without loss of generality, that $Y_i \cap P = \{(u_1, v_i), (u_2, v_i), (u_3, v_i)\}$. Then $\{(u_2, v_{i-1}), (u_3, v_{i-1})\} \subseteq M$, which means that $|Y_{i-1} \cap P| \le 2$. Furthermore, if $|Y_{i-1} \cap P| = 2$, then $Y_{i-1} \cap P = \{(u_1, v_{i-1}), (u_4, v_{i-1})\}$. If $|Y_{i+1} \cap P| = 4$, then $|Y_i \cap P| = 0$ by Lemma 2, a contradiction. So $|Y_{i+1} \cap P| \le 3$. If $|Y_{i+1} \cap P| = 3$, then $|Y_i \cap P| \le 2$ by the first part of Lemma 8 (1), a contradiction again. Hence $|Y_{i+1} \cap P| \le 2$. The statement (1) holds.

(2) Suppose that $|Y_i \cap P| = 4$ for some $1 \le i \le n$. If $|Y_{i+1} \cap P| = 4$, then $|Y_i \cap P| = 0$ by Lemma 2, a contradiction. Hence $|Y_{i+1} \cap P| \le 3$. If $|Y_{i+1} \cap P| = 3$, then it follows from Lemma 8 (1) that $|Y_i \cap P| \le 2$, which is a contradiction. So $|Y_{i+1} \cap P| \le 2$. If $|Y_{i+1} \cap P| = 2$, then $|P \cap \{(u_j, v_{i+1}), (u_{j+1}, v_{i+1})\}| \le 1$ for any $1 \le j \le 4$ by the definition of S2IF, where $u_5 = u_1$. The statement (2) is true.

(3) If $|Y_{i+1} \cap P| = 4$ for some $1 \le i \le n$, then $|Y_i \cap P| = 0$ by Lemma 2, and so $|(Y_i \cup Y_{i+1}) \cap P| \le 4$. If $|Y_{i+1} \cap P| = 3$ for some $1 \le i \le n$, then it follows from Lemma 8 (1) that $|Y_i \cap P| \le 2$. Thus $|(Y_i \cup Y_{i+1}) \cap P| \le 5$. If $|Y_{i+1} \cap P| \le 2$ for some $1 \le i \le n$, then it is obvious that $|(Y_i \cup Y_{i+1}) \cap P| \le 6$. According to Lemma 8 (1), it is easily seen from the arguments above that $|(Y_i \cup Y_{i+1}) \cap P| = 6$ if and only if $|Y_i \cap P| = 4$ and $|Y_{i+1} \cap P| = 2$. The statement (3) follows.

(4) If $|Y_{i+2} \cap P| = 4$ for some $1 \le i \le n$, then $|Y_{i+1} \cap P| = 0$ from Lemma 2. This implies that $|(Y_i \cup Y_{i+1} \cup Y_{i+2}) \cap P| \le 8$. If $|Y_{i+2} \cap P| = 3$ for some $1 \le i \le n$, then we can claim that $|(Y_i \cup Y_{i+1}) \cap P| \le 5$. Otherwise, $|(Y_i \cup Y_{i+1}) \cap P| = 6$ by Lemma 8 (3). Then $|Y_i \cap P| = 4$ and $|Y_{i+1} \cap P| = 2$ by Lemma 8 (3) again, however, which is impossible according to Lemma 8 (1) and (2). So $|(Y_i \cup Y_{i+1} \cup Y_{i+2}) \cap P| \le 8$. If $|Y_{i+2} \cap P| \le 2$ for some $1 \le i \le n$, by Lemma 8 (3), then $|(Y_i \cup Y_{i+1} \cup Y_{i+2}) \cap P| \le 8$. The statement (4) holds.

(5) If $|Y_i \cap P| = 4$ for some $1 \le i \le n$, then $|Y_{i+1} \cap P| \le 2$ by Lemma 8 (2). Suppose $|(Y_{i+2} \cup Y_{i+3}) \cap P| \le 5$. Then the desired result follows. So $|Y_{i+2} \cup Y_{i+3}) \cap P| = 6$, and hence $|(Y_{i+2} \cap P| = 4$ from Lemma 8 (3). Thus $|Y_{i+1} \cap P| = 0$ by Lemma 2. This derives that $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| \le 10$. If $|Y_i \cap P| \le 3$ for some $1 \le i \le n$, then $|(Y_i \cup Y_{i+1}) \cap P| \le 5$ and $|(Y_{i+2} \cup Y_{i+3}) \cap P| \le 6$ by Lemma 8 (3). Therefore $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| \le 11$. Now assume that $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| = 11$ for some $1 \le i \le n$. If $|Y_i \cap P| \le 3$, then $|(Y_i \cup Y_{i+1}) \cap P| = 5$ and $|(Y_{i+2} \cup Y_{i+3}) \cap P| = 6$ from the arguments above. Thus $|(Y_{i+2} \cap P| = 4$ by Lemma 8 (3). Then $|Y_{i+1} \cap P| = 0$ from Lemma 2. We deduce that $|(Y_i \cup Y_{i+1}) \cap P| \le 4$, a contradiction. Hence $|Y_i \cap P| = 4$. If $|Y_{i+1} \cap P| \le 1$, then one can reach the same contradiction by the similar argument above. So $|Y_{i+1} \cap P| = 2$ by Lemma 8 (2). Thus $|(Y_{i+2} \cup Y_{i+3}) \cap P| = 5$. Clearly, by Lemma 2, neither $|Y_{i+2} \cap P| = 4$ nor $|Y_{i+3} \cap P| = 4$ holds. According to Lemma 8 (1) and (2), $|Y_{i+2} \cap P| = 2$ and $|Y_{i+3} \cap P| = 3$. This establishes the statement (5).

(6) Suppose that $2 \le |Y_i \cap P| \le 3$ and $|Y_i \cap P| \ne |Y_{i+1} \cap P|$ for each $1 \le i \le n$. Furthermore, assume that $|Y_i \cap P| = 3$ for some $1 \le i \le n$. Without loss of generality, suppose that $Y_i \cap P = \{(u_1, v_i), (u_2, v_i), (u_3, v_i)\}$. Then

$Y_{i-1} \cap P = \{(u_1, v_{i-1}), (u_4, v_{i-1})\}$, $Y_{i-2} \cap P = \{(u_2, v_{i-2}), (u_3, v_{i-2}), (u_4, v_{i-2})\}$ and $Y_{i-3} \cap P = \{(u_1, v_{i-3}), (u_2, v_{i-3})\}$. Thus $Y_{i-4} \cap P = \{(u_1, v_{i-4}), (u_3, v_{i-4}), (u_4, v_{i-4})\}$, $Y_{i-5} \cap P = \{(u_2, v_{i-5}), (u_3, v_{i-5})\}$, $Y_{i-6} \cap P = \{(u_1, v_{i-6}), (u_2, v_{i-6}), (u_4, v_{i-6})\}$ and $Y_{i-7} \cap P = \{(u_3, v_{i-7}), (u_4, v_{i-7})\}$. So $Y_{i-8} \cap P = \{(u_1, v_{i-8}), (u_2, v_{i-8}), (u_3, v_{i-8})\}$. Therefore the statement (6) is true. The proof of Lemma 8 is completed. □

Theorem 9 *For any integer* $n \geq 2$,

$$\alpha_s^2(C_4 \times C_n) = \begin{cases} n & \text{if } n \equiv 0 \ (mod\ 8), \\ n-1 & \text{if } n \equiv 1, 3 \ (mod\ 4), \\ n-2 & \text{otherwise.} \end{cases}$$

Proof. We consider the following cases to complete the proof of Theorem 9.

Case 1. $n \equiv 0$ $(mod\ 8)$. Then $n = 8k$ for some integer $k \geq 1$. We proceed our proof by induction on k. If $k = 1$, then $n = 8$. Let $A = \{(u_1, v_1), (u_4, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_3), (u_1, v_4), (u_2, v_5), (u_3, v_5), (u_4, v_6), (u_1, v_7), (u_2, v_7), (u_3, v_8)\}$. Assigning to each vertex of A the value -1 and to each vertex of $V(C_4 \times C_8) \setminus A$ the value 1, we produce a S2IF f of $C_4 \times C_8$ with weight $w(f) = 8$. Hence $\alpha_s^2(C_4 \times C_8) \geq 8 = n$. Suppose that $\alpha_s^2(C_4 \times C_8) = 8$ is false. Then $\alpha_s^2(C_4 \times C_8) = 10$ by Lemma 1 and Corollary 4. Thus $|P| = 21$ since $|P| - |M| = 10$ and $|P| + |M| = 32$. Suppose that there exists a subset Y_i for some $1 \leq i \leq 8$ such that $|Y_i \cap P| = 4$. Without loss of generality, assume that $|Y_8 \cap P| = 4$. Then $|Y_7 \cap P| = 0$ and $|Y_1 \cap P| \leq 2$ by Lemma 2 and Lemma 8 (2), respectively. Furthermore, $|(Y_1 \cup Y_2) \cap P| \leq 5$ from Lemma 8 (3). So $|(Y_1 \cup Y_2 \cup Y_7 \cup Y_8) \cap P| \leq 9$. On the other hand, by Lemma 8 (5), $|(Y_3 \cup Y_4 \cup Y_5 \cup Y_6) \cap P| \leq 11$. We deduce that $|P| \leq 20$, a contradiction. Hence $|Y_i \cap P| \leq 3$ for each $1 \leq i \leq 8$. Applying Lemma 8 (1), there exist at most 4 Y_i's in $C_4 \times C_8$ such that $|Y_i \cap P| = 3$. This implies that $|P| \leq 20$, which is a contradiction. Therefore $\alpha_s^2(C_4 \times C_8) = 8 = n$.

We now assume that $\alpha_s^2(C_4 \times C_{8l}) = 8l$ for all integers $1 \leq l < k$. We need to prove $\alpha_s^2(C_4 \times C_{8k}) = 8k = n$. Let $V_i = \cup_{r=0}^{7} Y_{8i-r}$ for $1 \leq i \leq k$. Let $f_1 : V(C_4 \times C_{8k}) \to \{-1, 1\}$ be a function on $C_4 \times C_{8k}$ such that V_i has the same assignment of function values under f_1 as that of $V(C_4 \times C_8)$ under f, for $1 \leq i \leq k$. It is easy to check that f_1 is a S2IF of $C_4 \times C_{8k}$ with weight $w(f_1) = 8k$. Hence $\alpha_s^2(C_4 \times C_{8k}) \geq 8k = n$. Suppose that $\alpha_s^2(C_4 \times C_{8k}) = 8k$ is not true. Then $\alpha_s^2(C_4 \times C_{8k}) \geq 8k + 2$ by Lemma 1. Let f_2 be a $\alpha_s^2(C_4 \times C_{8k})$-function. Since $|P| - |M| \geq 8k + 2$ and $|P| + |M| = 32k$, we obtain $|P| \geq 20k + 1$. Thus there must exist a subset V_i for some $1 \leq i \leq k$ such that $|V_i \cap P| \geq 21$. Set $V_i^1 = \cup_{j=0}^{3} Y_{8i-j}$ and $V_i^2 = \cup_{j=4}^{7} Y_{8i-j}$ for $1 \leq i \leq k$. According to Lemma 8 (5), there is a V_i^r for some $1 \leq r \leq 2$ such that $|V_i^r \cap P| = 11$. We distinguish two subcases depending on $|V_i^1 \cap P| = 11$ or $|V_i^2 \cap P| = 11$.

Case 1.1. $|V_i^1 \cap P| = 11$. Then it follows from Lemma 8 (5) that $|Y_{8i-3} \cap P| = 4$, and so $|Y_{8i-4} \cap P| = 0$ by Lemma 2. Since $|V_i \cap P| \geq 21$, $|(Y_{8i-7} \cup Y_{8i-6} \cup Y_{8i-5}) \cap P| \geq 10$, which contradicts to Lemma 8 (4).

Case 1.2. $|V_i^2 \cap P| = 11$. Then it follows from Lemma 8 (5) that $|Y_{8i-7} \cap P| = 4$ and $|Y_{8i-4} \cap P| = 3$. Thus $|Y_{8i-8} \cap P| = 0$ and $|Y_{8i-3} \cap P| \leq 2$ by Lemma 2 and Lemma 8 (1), respectively. Further, by Lemma 8 (4), $|V_{(i-1)}^1 \cap P| \leq 8$. Let $D_1 = D[V(C_4 \times C_{8k}) \setminus \{V_{(i-1)}^1, V_i^2\}] \cup (\cup_{j=1}^4 \{(u_j, v_{8i-12})(u_j, v_{8i-3})\})$. Then $D_1 \cong C_4 \times C_{8(k-1)}$, and hence $\alpha_s^2(D_1) = 8k - 8$ by the induction hypothesis. Suppose $|Y_{8i-3} \cap P| \leq 1$. Clearly, $f_3 = f_2|_{D_1}$ is a S2IF on D_1. So $\alpha_s^2(D_1) \geq w(f_3) \geq (|P|-19)-(|M|-13) = |P|-|M|-6 \geq 8k-4$, a contradiction. Hence $|Y_{8i-3} \cap P| = 2$. Let $Y_{8i-3} \cap P = \{(u_{j_1}, v_{8i-3}), (u_{j_2}, v_{8i-3})\}$, where $1 \leq j_1, j_2 \leq 4$. The function $f_4 : V(D_1) \to \{-1, 1\}$ is defined as follows: $f_4((u_{j_1}, v_{8i-3})) = -1$ and $f_4((x, y)) = f_2((x, y))$ for each $(x, y) \in V(D_1) \setminus \{(u_{j_1}, v_{8i-3})\}$. It is not hard to see that f_4 is a S2IF of D_1 with weight $w(f_4) \geq (|P|-19)-1-(|M|-13+1) = |P|-|M|-8 \geq 8k-6$. Thus $\alpha_s^2(D_1) \geq 8k - 6$, a contradiction. Consequently, $\alpha_s^2(C_4 \times C_{8k}) = 8k = n$.

Case 2. $n \equiv 0$ $(mod\ 4)$ and $n \not\equiv 0$ $(mod\ 8)$. Then $n = 8k+4$ for some integer $k \geq 0$. We now show by induction on k that $\alpha_s^2(C_4 \times C_{8k+4}) = 8k+2 = n-2$. If $k = 0$, then $n = 4$. Let $A' = \{(u_1, v_1), (u_4, v_1), (u_2, v_2), (u_4, v_2), (u_1, v_3), (u_2, v_3), (u_3, v_4)\}$. Define $g : V(C_4 \times C_4) \to \{-1, 1\}$ by assigning to all vertices of A' the value -1 and to all other vertices the value 1. It is easy to verify that g is a S2IF on $C_4 \times C_4$ of weight $w(g) = 2$, which means that $\alpha_s^2(C_4 \times C_4) \geq 2 = n-2$. Assume that $\alpha_s^2(C_4 \times C_4) = 2$ is false. Then $\alpha_s^2(C_4 \times C_4) = 4$ by Lemma 1 and Corollary 4. Let g_1 be a $\alpha_s^2(C_4 \times C_4)$-function. Thus $|P| = 10$ as $|P| - |M| = 4$ and $|P| + |M| = 16$. Suppose that there exists a subset Y_i for some $1 \leq i \leq 4$ such that $|Y_i \cap P| = 4$. Without loss of generality, assume that $|Y_4 \cap P| = 4$. Then $|Y_3 \cap P| = 0$ and $|Y_1 \cap P| \leq 2$ by Lemma 2 and Lemma 8 (2), respectively. Since $|P| = 10$, it follows that $|Y_2 \cap P| = 4$ and $|Y_1 \cap P| = 2$, which contradicts to Lemma 2. So $|Y_i \cap P| \leq 3$ for each $1 \leq i \leq 4$. According to Lemma 8 (1), $2 \leq |Y_i \cap P| \leq 3$ and $|Y_i \cap P| \neq |Y_{i+1} \cap P|$ for each $1 \leq i \leq 4$ as $|P| = 10$. Without loss of generality, assume that $Y_4 \cap P = \{(u_1, v_4), (u_2, v_4), (u_3, v_4)\}$. Then $Y_3 \cap P = \{(u_1, v_3), (u_4, v_3)\}$ and $Y_2 \cap P = \{(u_2, v_2), (u_3, v_2), (u_4, v_2)\}$. Thus $Y_1 \cap P = \{(u_1, v_1), (u_2, v_1)\}$, which implies that $g_1[(u_2, v_1)] = 3$, contradicting the definition of S2IF. Therefore $\alpha_s^2(C_4 \times C_4) = 2 = n - 2$.

Assume that $\alpha_s^2(C_4 \times C_{8l+4}) = 8l + 2$ for all integers $0 \leq l < k$. We will show that $\alpha_s^2(C_4 \times C_{8k+4}) = 8k + 2 = n - 2$. Let V_i be defined as in the proof of Case 1 of Theorem 9, where $1 \leq i \leq k$. Define $g_2 : V(C_4 \times C_{8k+4}) \to \{-1, 1\}$ as follows: V_i $(1 \leq i \leq k)$ and $V(C_4 \times C_{8k+4}) \setminus (\cup_{i=1}^k V_i)$ have the same assignments of function values under g_2 as those of $V(C_4 \times C_8)$ under f and $V(C_4 \times C_4)$ under g, respectively. It is not hard to check that g_2 is a S2IF of $C_4 \times C_{8k+4}$ with weight $w(g_2) = 8k + 2$. Thus $\alpha_s^2(C_4 \times C_{8k+4}) \geq 8k + 2 = n - 2$. Suppose that $\alpha_s^2(C_4 \times C_{8k+4}) \neq 8k + 2$. Then $\alpha_s^2(C_4 \times C_{8k+4}) \geq 8k + 4$ by Lemma 1. Let g_3 be a $\alpha_s^2(C_4 \times C_{8k+4})$-function. Since $|P| - |M| \geq 8k + 4$ and $|P| + |M| = 32k + 16$, we obtain $|P| \geq 20k + 10$. We have the following claim.

Claim 1. For each $1 \leq i \leq 8k + 4$, $|Y_i \cap P| \leq 3$.

Otherwise, there is a subset Y_i for some $1 \leq i \leq 8k + 4$ such that $|Y_i \cap P| = 4$. Then $|Y_{i-1} \cap P| = 0$ by Lemma 2. Note that $|(Y_{i-4} \cup Y_{i-3} \cup Y_{i-2}) \cap P| \leq 8$ and $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| \leq 11$ from Lemma 8 (4) and (5), respectively. Let

$D_2 = D[V(C_4 \times C_{8k+4}) \setminus (\cup_{r=-4}^{3} Y_{i+r})] \cup (\cup_{j=1}^{4}\{(u_j, v_{i-5})(u_j, v_{i+4})\})$. Then $D_2 \cong C_4 \times C_{8(k-1)+4}$. Applying the induction hypothesis, $\alpha_s^2(D_2) = 8k-6$. We proceed our proof by considering the value of $|Y_{i+4} \cap P|$.

Suppose $|Y_{i+4} \cap P| \leq 2$. With a proof similar to that of Case 1.2 of Theorem 9, we can obtain the same contradictions.

Suppose $|Y_{i+4} \cap P| = 3$. Then $|Y_{i+3} \cap P| \leq 2$ by Lemma 8 (1). Thus it follows from Lemma 8 (5) that $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| \leq 10$. Let $Y_{i+4} \cap P = \{(u_{j_1}, v_{i+4}), (u_{j_2}, v_{i+4}), (u_{j_3}, v_{i+4})\}$, where $1 \leq j_1, j_2, j_3 \leq 4$. Define $g_4 : V(D_2) \to \{-1, 1\}$ by $g_4((u_{j_1}, v_{i+4})) = g_4((u_{j_2}, v_{i+4})) = -1$ and $g_4((x, y)) = g_3((x, y))$ for each $(x, y) \in V(D_2) \setminus \{(u_{j_1}, v_{i+4}), (u_{j_2}, v_{i+4})\}$. It can be readily verified that g_4 is a S2IF of D_2 with weight $w(g_4) \geq (|P|-18)-2-(|M|-14+2) = |P|-|M|-8 \geq 8k-4$. This yields that $\alpha_s^2(D_2) \geq 8k-4$, which is a contradiction.

Suppose $|Y_{i+4} \cap P| = 4$. Then $|Y_{i+3} \cap P| = 0$ by Lemma 2. Hence $|(Y_i \cup Y_{i+1} \cup Y_{i+2} \cup Y_{i+3}) \cap P| \leq 8$ from Lemma 8 (4). The function $g_5 : V(D_2) \to \{-1, 1\}$ is defined as follows: $g_5((u_1, v_{i+4})) = g_5((u_2, v_{i+4})) = g_5((u_3, v_{i+4})) = -1$ and $g_5((x, y)) = g_3((x, y))$ for each $(x, y) \in V(D_2) \setminus \{(u_1, v_{i+4}), (u_2, v_{i+4}), (u_3, v_{i+4})\}$. It is easily seen that g_5 is a S2IF on D_2 of weight $w(g_5) \geq (|P|-16)-3-(|M|-16+3) = |P|-|M|-6 \geq 8k-2$. This implies that $\alpha_s^2(D_2) \geq 8k-2$, a contradiction. In either case, we always arrive at a contradiction. Therefore Claim 1 holds.

By Claim 1 and $|P| \geq 20k+10$, there exist at least $4k+2$ Y_i's in $C_4 \times C_{8k+4}$ such that $|Y_i \cap P| = 3$. According to Lemma 8 (1), there exist exactly $4k+2$ Y_i's in $C_4 \times C_{8k+4}$ such that $|Y_i \cap P| = 3$. Thus, by Claim 1, $2 \leq |Y_i \cap P| \leq 3$ for each $1 \leq i \leq 8k+4$ as $|P| \geq 20k+10$. Moreover, $|Y_i \cap P| \neq |Y_{i+1} \cap P|$ for each $1 \leq i \leq 8k+4$ by Lemma 8 (1) again. Note that $n = 8k+4 \geq 12$. Without loss of generality, we may assume that $|Y_{8k+4} \cap P| = 3$. Applying Lemma 8 (6), $Proj_{C_4}(Y_{8k+4} \cap P) = Proj_{C_4}(Y_{8k-4} \cap P)$. Let $D_3 = D[V(C_4 \times C_{8k+4}) \setminus (\cup_{r=0}^{7} Y_{8k+4-r})] \cup (\cup_{j=1}^{4}\{(u_j, v_{8k-4})(u_j, v_1)\})$. Then $D_3 \cong C_4 \times C_{8(k-1)+4}$. So $\alpha_s^2(D_3) = 8k-6$ by the induction hypothesis. It is easy to see that $g_6 = g_3|_{D_3}$ is a S2IF of D_3 with weight $w(g_6) \geq (|P|-20)-(|M|-12) = |P|-|M|-8 \geq 8k-4$. This derives that $\alpha_s^2(D_3) \geq 8k-4$, a contradiction. Hence $\alpha_s^2(C_4 \times C_{8k+4}) = 8k+2 = n-2$.

Case 3. $n \equiv 1$ $(mod\ 4)$. Then $n = 4k+1$ for some integer $k \geq 1$. To prove $\alpha_s^2(C_4 \times C_{4k+1}) = 4k = n-1$, we employ induction on k. If $k = 1$, then $n = 5$. Let $B = \{(u_1, v_1), (u_3, v_1), (u_2, v_2), (u_3, v_2), (u_4, v_3), (u_1, v_4), (u_2, v_4), (u_3, v_5)\}$. Define $h : V(C_4 \times C_5) \to \{-1, 1\}$ by assigning to each vertex of B the value -1 and to each vertex of $V(C_4 \times C_5) \setminus B$ the value 1. It is not difficult to check that h is a S2IF of $C_4 \times C_5$ with weight $w(h) = 4$. So $\alpha_s^2(C_4 \times C_5) \geq 4 = n-1$. Suppose that $\alpha_s^2(C_4 \times C_5) = 4$ is not true. Then $\alpha_s^2(C_4 \times C_5) = 6$ by Lemma 1 and Corollary 4. Thus $|P| = 13$ because $|P|-|M| = 6$ and $|P|+|M| = 20$. If there is a subset Y_i for some $1 \leq i \leq 5$ such that $|Y_i \cap P| = 4$, then $|Y_{i-1} \cap P| = 0$ and $|Y_{i+1} \cap P| \leq 2$ by Lemma 2 and Lemma 8 (2), respectively. Furthermore, by Lemma 8 (3), $|(Y_{i+2} \cup Y_{i+3}) \cap P| \leq 6$. This leads to $|P| \leq 12$, a contradiction. Hence $|Y_i \cap P| \leq 3$ for each $1 \leq i \leq 5$. According to Lemma 8 (1), there are at most 2 Y_i's in $C_4 \times C_5$ such that $|Y_i \cap P| = 3$. This implies that $|P| \leq 12$, which is a contradiction. Therefore $\alpha_s^2(C_4 \times C_5) = 4 = n - 1$.

Assume that $\alpha_s^2(C_4 \times C_{4l+1}) = 4l$ for all integers $1 \le l < k$. We shall show that $\alpha_s^2(C_4 \times C_{4k+1}) = 4k = n - 1$.

If k is odd, then $k = 2l + 1$ for some integer $l \ge 1$, and $n = 8l + 5$. We write $V_i^{'} = \cup_{r=0}^{7} Y_{8i-r}$ for $1 \le i \le l$. Let $h_1 : V(C_4 \times C_{4k+1}) \to \{-1, 1\}$ be a function on $C_4 \times C_{4k+1}$ such that $V_i^{'}$ $(1 \le i \le l)$ and $V(C_4 \times C_{4k+1}) \setminus (\cup_{i=1}^{l} V_i^{'})$ have the same assignments of function values under h_1 as those of $V(C_4 \times C_8)$ under f and $V(C_4 \times C_5)$ under h, respectively. It is easy to verify that h_1 is a S2IF of $C_4 \times C_{4k+1}$ with weight $w(h_1) = 4k$. Thus $\alpha_s^2(C_4 \times C_{4k+1}) \ge 4k = n - 1$.

If k is even, then $k = 2l$ for some integer $l \ge 1$, and $n = 8l + 1$. The function $h_2 : V(C_4 \times C_{4k+1}) \to \{-1, 1\}$ is defined as follow: $V_i^{'}$ $(1 \le i \le l)$ has the same assignment of function values under h_2 as that of $V(C_4 \times C_8)$ under f, $h_2((u_1, v_{8l+1})) = h_2((u_3, v_{8l+1})) = -1$ and $h_2((u_2, v_{8l+1})) = h_2((u_4, v_{8l+1})) = 1$. It can be readily checked that h_2 is a S2IF of $C_4 \times C_{4k+1}$ with weight $w(h_2) = 4k$. So $\alpha_s^2(C_4 \times C_{4k+1}) \ge 4k = n - 1$.

Suppose that $\alpha_s^2(C_4 \times C_{4k+1}) \ne 4k$. Then $\alpha_s^2(C_4 \times C_{4k+1}) \ge 4k + 2$ by Lemma 1. Let h_3 be a $C_4 \times C_{4k+1}$-function. Since $|P| - |M| \ge 4k + 2$ and $|P| + |M| = 16k + 4$, we obtain $|P| \ge 10k + 3$. We have the following claim.

Claim 2. For each $1 \le i \le 4k + 1$, $|Y_i \cap P| \le 3$.

Suppose on the contrary that there exists a subset Y_i for some $1 \le i \le 4k + 1$ such that $|Y_i \cap P| = 4$. Then $|Y_{i-1} \cap P| = 0$ by Lemma 2. Recall that $|(Y_{i-3} \cup Y_{i-2}) \cap P| \le 6$ from Lemma 8 (3). Thus $|(Y_{i-3} \cup Y_{i-2} \cup Y_{i-1} \cup Y_i) \cap P| \le 10$. Let $D_4 = D[V(C_4 \times C_{4k+1}) \setminus \{Y_{i-3}, Y_{i-2}, Y_{i-1}, Y_i\}] \cup (\cup_{j=1}^{4} \{(u_j, v_{i-4})(u_j, v_{i+1})\})$. Then $D_4 \cong C_4 \times C_{4(k-1)+1}$. Applying the induction hypothesis, $\alpha_s^2(D_4) = 4k - 4$. Obviously, $h_4 = h_3|_{D_4}$ is a S2IF on D_4 of weight $w(h_4) \ge (|P| - 10) - (|M| - 6) = |P| - |M| - 4 \ge 4k - 2$, implying that $\alpha_s^2(D_4) \ge 4k - 2$, a contradiction. So Claim 2 is true.

By Claim 2, we deduce that there exist as least $2k + 1$ Y_i's in $C_4 \times C_{4k+1}$ such that $|Y_i \cap P| = 3$ as $n = 4k + 1$ and $|P| \ge 10k + 3$. Thus there must exist a subset Y_i for some $1 \le i \le 4k + 1$ such that $|Y_i \cap P| = |Y_{i+1} \cap P| = 3$ since $n = 4k + 1$, which contradicts to Lemma 8 (1). Consequently, $\alpha_s^2(C_4 \times C_{4k+1}) = 4k = n - 1$.

Case 4. $n \equiv 2$ $(mod\ 4)$. Then $n = 4k + 2$ for some integer $k \ge 0$. The proof is by induction on k. If $k = 0$, then $n = 2$. Note that $C_4 \times C_2 \cong C_2 \times C_4$. By Theorem 5, $\alpha_s^2(C_4 \times C_2) = \alpha_s^2(C_2 \times C_4) = 0 = n - 2$. If $k = 1$, then $n = 6$. Let $B^{'} = \{(u_1, v_1), (u_4, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_3), (u_1, v_4), (u_2, v_5), (u_3, v_5), (u_1, v_6), (u_3, v_6)\}$. Assigning to each vertex of $B^{'}$ the value -1 and to each vertex of $V(C_4 \times C_6) \setminus B^{'}$ the value 1, we produce a S2IF p of $C_4 \times C_6$ with weight $w(p) = 4$. Thus $\alpha_s^2(C_4 \times C_6) \ge 4 = n - 2$. Suppose that $\alpha_s^2(C_4 \times C_6) = 4$ is false. Then $\alpha_s^2(C_4 \times C_6) \ge 6$ by Lemma 1. Let p_1 be a $\alpha_s^2(C_4 \times C_6)$-function. It follows that $|P| \ge 15$ as $|P| - |M| \ge 6$ and $|P| + |M| = 24$. If there exists a subset Y_i for some $1 \le i \le 6$ such that $|Y_i \cap P| = 4$, then $|Y_{i-1} \cap P| = 0$ and $|Y_{i+1} \cap P| \le 2$ by Lemma 2 and Lemma

8 (2), respectively. Without loss of generality, assume that $|Y_6 \cap P| = 4$. Then $|(Y_5 \cup Y_6 \cup Y_1) \cap P| \leq 6$. By Lemma 8 (4), $|(Y_2 \cup Y_3 \cup Y_4) \cap P| \leq 8$. Hence $|P| \leq 14$, a contradiction. So $|Y_i \cap P| \leq 3$ for each $1 \leq i \leq 6$. Recall that $|P| \geq 15$ and $n = 6$. Applying Lemma 8 (1), we can claim that $2 \leq |Y_i \cap P| \leq 3$ and $|Y_i \cap P| \neq |Y_{i+1} \cap P|$ for each $1 \leq i \leq 6$. Without loss of generality, suppose that $Y_6 \cap P = \{(u_1, v_6), (u_2, v_6), (u_3, v_6)\}$. Then $Y_5 \cap P = \{(u_1, v_5), (u_4, v_5)\}$, $Y_4 \cap P = \{(u_2, v_4), (u_3, v_4), (u_4, v_4)\}$ and $Y_3 \cap P = \{(u_1, v_3), (u_2, v_3)\}$. Thus $Y_2 \cap P = \{(u_1, v_2), (u_3, v_2), (u_4, v_2)\}$ and $Y_1 \cap P = \{(u_2, v_1), (u_3, v_1)\}$. This would imply that $p_1[(u_3, v_1)] = 3$, contradicting the definition of S2IF. Therefore $\alpha_s^2(C_4 \times C_6) = 4 = n - 2$.

Now we assume that $\alpha_s^2(C_4 \times C_{4l+2}) = 4l$ for all integers $1 \leq l < k$. Next we need to prove $\alpha_s^2(C_4 \times C_{4k+2}) = 4k = n - 2$.

If k is odd, then $k = 2l + 1$ for some integer $l \geq 1$, and $n = 8l + 6$. Let $V_i^{'}$ be defined as in the proof of Case 3 of Theorem 9, where $1 \leq i \leq l$. Define $p_2 : V(C_4 \times C_{4k+2}) \to \{-1, 1\}$ as follows: $V_i^{'}$ $(1 \leq i \leq l)$ and $V(C_4 \times C_{4k+2}) \setminus (\cup_{i=1}^{l} V_i^{'})$ have the same assignments of function values under p_2 as those of $V(C_4 \times C_8)$ under f and $V(C_4 \times C_6)$ under p, respectively. It is not hard to verify that p_2 is a S2IF of $C_4 \times C_{4k+2}$ with weight $w(p_2) = 4k$, and hence $\alpha_s^2(C_4 \times C_{4k+2}) \geq 4k = n - 2$.

If k is even, then $k = 2l$ for some integer $l \geq 1$, and $n = 8l + 2$. Let $C = \{(u_1, v_{8l+1}), (u_4, v_{8l+1}), (u_2, v_{8l+2}), (u_3, v_{8l+2})\}$. Let $p_3 : V(C_4 \times C_{4k+2}) \to \{-1, 1\}$ be a function on $C_4 \times C_{4k+2}$ such that $V_i^{'}$ $(1 \leq i \leq l)$ has the same assignment of function values under p_3 as that of $V(C_4 \times C_8)$ under f, each vertex of C is assigned to the value -1 under p_3 and each vertex of $\{Y_{8l+1}, Y_{8l+2}\} \setminus C$ is assigned to the value 1 under p_3. It is easy to check that p_3 is a S2IF on $C_4 \times C_{4k+2}$ of weight $w(p_3) = 4k$. So $\alpha_s^2(C_4 \times C_{4k+2}) \geq 4k = n - 2$.

Suppose that $\alpha_s^2(C_4 \times C_{4k+2}) = 4k$ is false. Then $\alpha_s^2(C_4 \times C_{4k+2}) \geq 4k + 2$ by Lemma 1. Let p_4 be a $\alpha_s^2(C_4 \times C_{4k+2})$-function. Thus $|P| \geq 10k + 5$ since $|P| - |M| \geq 4k + 2$ and $|P| + |M| = 16k + 8$. Like the argument of Claim 2, we have that $|Y_i \cap P| \leq 3$ for each $1 \leq i \leq 4k + 2$. Then one can reach the same contradiction by a similar argument that used in the proof of Case 2 of Theorem 9. Therefore $\alpha_s^2(C_4 \times C_{4k+2}) = 4k = n - 2$.

Case 5. $n \equiv 3 \ (mod\ 4)$. Then $n = 4k + 3$ for some integer $k \geq 0$. We show by induction on k that $\alpha_s^2(C_4 \times C_{4k+3}) = 4k + 2 = n - 1$. If $k = 0$, then $n = 3$. Notice that $C_4 \times C_3 \cong C_3 \times C_4$. By Theorem 7, $\alpha_s^2(C_4 \times C_3) = \alpha_s^2(C_3 \times C_4) = 2 = n - 1$. Assume that $\alpha_s^2(C_4 \times C_{4l+3}) = 4l + 2$ for all integers $0 \leq l < k$. Now we show that $\alpha_s^2(C_4 \times C_{4k+3}) = 4k + 2 = n - 1$.

If k is odd, then $k = 2l+1$ for some integer $l \geq 0$, and $n = 8l+7$. If $l = 0$, then $n = 7$. Let $C^{'} = \{(u_1, v_1), (u_4, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_3), (u_1, v_4), (u_2, v_5), (u_3, v_5), (u_4, v_6), (u_1, v_7), (u_3, v_7)\}$. Assigning to all vertices of $C^{'}$ the value -1 and to all vertices of $V(C_4 \times C_7) \setminus C^{'}$ the value 1, a S2IF q of $C_4 \times C_7$ is produced. So $\alpha_s^2(C_4 \times C_7) \geq w(q) = 6 = n - 1$. Next we may assume that $l \geq 1$. Let $V_i^{'}$ be defined as in the proof of Case 3 of Theorem 9, where $1 \leq i \leq l$. The function $q_1 : V(C_4 \times C_{4k+3}) \to \{-1, 1\}$

is defined as follows: $V_i^{'}$ $(1 \leq i \leq l)$ and $V(C_4 \times C_{4k+3}) \setminus (\cup_{i=1}^{l} V_i^{'})$ have the same assignments of function values under q_1 as those of $V(C_4 \times C_8)$ under f and $V(C_4 \times C_7)$ under q, respectively. It can be readily verified that q_1 is a S2IF of $C_4 \times C_{4k+3}$ with weight $w(q_1) = 4k+2$, and so $\alpha_s^2(C_4 \times C_{4k+3}) \geq 4k+2 = n-1$.

If k is even, then $k = 2l$ for some integer $l \geq 1$, and $n = 8l+3$. Let $C^{''} = \{(u_1, v_{8l+1}), (u_4, v_{8l+1}), (u_2, v_{8l+2}), (u_3, v_{8l+3}), (u_4, v_{8l+3})\}$. Let $q_2 : V(C_4 \times C_{4k+3}) \to \{-1, 1\}$ be a function on $C_4 \times C_{4k+3}$ such that $V_i^{'}$ $(1 \leq i \leq l)$ has the same assignment of function value under q_2 as that of $V(C_4 \times C_8)$ under f, each vertex of $C^{''}$ is assigned to the value -1 under q_2 and each vertex of $\{Y_{8l+1}, Y_{8l+2}, Y_{8l+3}\} \setminus C^{''}$ is assigned to the value 1 under q_2. It is not difficult to check that q_2 is a S2IF on $C_4 \times C_{4k+3}$ of weight $w(q_2) = 4k+2$. Hence $\alpha_s^2(C_4 \times C_{4k+3}) \geq 4k+2 = n-1$.

Suppose that $\alpha_s^2(C_4 \times C_{4k+3}) \neq 4k+2$. Then $\alpha_s^2(C_4 \times C_{4k+3}) \geq 4k+4$ by Lemma 1. Let q_3 be a $\alpha_s^2(C_4 \times C_{4k+3})$-function. It follows that $|P| \geq 10k+8$ because $|P| - |M| \geq 4k+4$ and $|P| + |M| = 16k+12$. With a proof similar to that of Claim 2, we obtain that $|Y_i \cap P| \leq 3$ for each $1 \leq i \leq 4k+3$. Using a similar method as that in the proof of Case 3 of Theorem 9, we can get the same contradiction. So $\alpha_s^2(C_4 \times C_{4k+3}) = 4k+2 = n-1$. This complete the proof of Theorem 9. □

We are now ready to establish our last result. First we present a lemma that will prove to be useful in our proof.

Lemma 10 *Let f be a S2IF of $C_5 \times C_n$. The following statements are true:*

(1) *For $1 \leq i \leq n$, if $|Y_i \cap P| = 4$, then $|Y_{i-1} \cap P| \leq 2$ and $|Y_{i+1} \cap P| \leq 3$. Furthermore, if $|Y_{i-1} \cap P| = 2$, then $Y_{i-1} \cap P = \{(u_j, v_{i-1}), (u_{j+1}, v_{i-1})\}$ for some $1 \leq j \leq 5$, where $u_6 = u_1$.*

(2) *For $1 \leq i \leq n$, if $|Y_i \cap P| = 5$, then $|Y_{i+1} \cap P| \leq 2$. Furthermore, if $|Y_{i+1} \cap P| = 2$, then $|P \cap \{(u_j, v_{i+1}), (u_{j+1}, v_{i+1})\}| \leq 1$ for any $1 \leq j \leq 5$, where $u_6 = u_1$.*

(3) *For $1 \leq i \leq n$, $|(Y_i \cup Y_{i+1}) \cap P| \leq 7$.*

(4) *For $1 \leq i \leq n$, $|(Y_i \cup Y_{i+1} \cup Y_{i+2}) \cap P| \leq 10$.*

Proof. By a similar argument that used in the proof of Lemma 8, we can show that Lemma 10. □

Theorem 11 *For any integer $n \geq 2$,*

$$\alpha_s^2(C_5 \times C_n) = \begin{cases} 0 & \text{if } n = 2, \\ n & \text{if } n \geq 3. \end{cases}$$

Proof. Note that $C_m \times C_n \cong C_n \times C_m$. If $n = 2$, the assertion is true by Theorem 5. Next we proceed our proof by induction on n $(n \geq 3)$. Applying Theorem 7 and 9, the assertions are trivial for $3 \leq n \leq 4$. Suppose $n = 5$. Let $F =$

$\{(u_2,v_1),(u_5,v_1),(u_1,v_2),(u_3,v_2),(u_2,v_3),(u_4,v_3),(u_3,v_4),(u_5,v_4),(u_1,v_5),(u_4,v_5)\}$. Clearly, the function $f:V(C_5\times C_5)\to\{-1,1\}$ defined by $f((x,y))=-1$ for every $(x,y)\in F$ and $f((x,y))=1$ for every $(x,y)\in V(C_5\times C_5)\setminus F$ is a S2IF on $C_5\times C_5$ of weight $w(f)=5$. This yields that $\alpha_s^2(C_5\times C_5)\geq 5=n$. Assume that $\alpha_s^2(C_5\times C_5)=5$ is false. By Lemma 1 and Corollary 4, we have $\alpha_s^2(C_5\times C_5)=7$. Thus $|P|=16$ as $|P|-|M|=7$ and $|P|+|M|=25$. If there exists a subset Y_i for some $1\leq i\leq 5$ such that $|Y_i\cap P|=5$, then $|Y_{i-1}\cap P|=0$ and $|Y_{i+1}\cap P|\leq 2$ according to Lemma 2 and Lemma 10 (2). Furthermore, by Lemma 10 (3), $|(Y_{i+2}\cup Y_{i+3})\cap P|\leq 7$. This implies that $|P|\leq 14$, a contradiction. So $|Y_i\cap P|\leq 4$ for each $1\leq i\leq 5$. Since $|P|=16$, there exists at least a subset Y_i for some $1\leq i\leq 5$ such that $|Y_i\cap P|=4$. Without loss of generality, let $|Y_5\cap P|=4$. Then it follows from Lemma 10 (1) that $|Y_4\cap P|\leq 2$ and $|Y_1\cap P|\leq 3$. Thus $|(Y_2\cup Y_3)\cap P|=7$ as $|P|=16$. By Lemma 10 (1), $|Y_2\cap P|=4$, and so $|Y_1\cap P|\leq 2$. We deduce that $|P|\leq 15$, a contradiction. Hence $\alpha_s^2(C_5\times C_5)=5=n$. Assume that $\alpha_s^2(C_5\times C_l)=l$ for all integers $5\leq l<n$. We next show that $\alpha_s^2(C_5\times C_n)=n$.

If $n\equiv 0\ (mod\ 3)$, then $n=3k$ for some integer $k\geq 2$. Let $W_i^1=\{(u_2,v_{3i-2}),(u_5,v_{3i-2}),(u_1,v_{3i-1}),(u_3,v_{3i-1}),(u_2,v_{3i}),(u_4,v_{3i})\}$ for $1\leq i\leq k$. Further, we write $W^1=\cup_{i=1}^k W_i^1$. Define $f_1:V(C_5\times C_{3k})\to\{-1,1\}$ by assigning to all vertices of W^1 the value -1 and to all other vertices the value 1. It can be readily verified that f_1 is a S2IF of $C_5\times C_{3k}$ with weight $w(f_1)=3k$. Thus $\alpha_s^2(C_5\times C_{3k})\geq 3k=n$.

If $n\equiv 1\ (mod\ 3)$, then $n=3k+1$ for some integer $k\geq 2$. Let $W_i^2=\{(u_1,v_{3i-2}),(u_3,v_{3i-2}),(u_2,v_{3i-1}),(u_4,v_{3i-1}),(u_2,v_{3i}),(u_5,v_{3i})\}$ for $1\leq i\leq k$, and let $W^2=\cup_{i=1}^k W_i^2$. Assigning to every vertex of $W^2\cup\{(u_3,v_{3k+1}),(u_5,v_{3k+1})\}$ the value -1 and to every vertex of $V(C_5\times C_{3k+1})\setminus(W^2\cup\{(u_3,v_{3k+1}),(u_5,v_{3k+1})\})$ the value 1, we produce a S2IF f_2 on $C_5\times C_{3k+1}$ of weight $w(f_2)=3k+1$. Hence $\alpha_s^2(C_5\times C_{3k+1})\geq 3k+1=n$.

If $n\equiv 2\ (mod\ 3)$, then $n=3k+2$ for some integer $k\geq 2$. Let $F'=\{(u_3,v_{3k+1}),(u_5,v_{3k+1}),(u_1,v_{3k+2}),(u_4,v_{3k+2})\}$. The function $f_3:V(C_5\times C_{3k+2})\to\{-1,1\}$ is defined as follows: the value -1 is assigned to each vertex of $W^1\cup F'$ and the value 1 is assigned to each vertex of $V(C_5\times C_{3k+2})\setminus(W^1\cup F')$. It is easy to check that f_3 is a S2IF of $C_5\times C_{3k+2}$ with weight $w(f_3)=3k+2$. This means that $\alpha_s^2(C_5\times C_{3k+2})\geq 3k+2=n$.

Suppose that $\alpha_s^2(C_5\times C_n)=n$ is not true. Then $\alpha_s^2(C_5\times C_n)\geq n+2$ by Lemma 1. Let g be a $\alpha_s^2(C_5\times C_n)$-function. Thus $|P|\geq 3n+1$ because $|P|-|M|\geq n+2$ and $|P|+|M|=5n$. We have the following claims.

Claim 3. For each $1\leq i\leq n$, $|Y_i\cap P|\leq 4$.

Otherwise, there exists a subset Y_i for some $1\leq i\leq n$ such that $|Y_i\cap P|=5$. Then $|Y_{i-1}\cap P|=0$ from Lemma 2. Let $D_1=D[V(C_5\times C_n)\setminus\{Y_{i-1},Y_i\}]\cup(\cup_{j=1}^5\{(u_j,v_{i-2})(u_j,v_{i+1})\})$. Then $D_1\cong C_5\times C_{n-2}$. According to the induction hypothesis, $\alpha_s^2(D_1)=n-2$. Obviously, $g_1=g|_{D_1}$ is a S2IF of D_1 with weight $w(g_1)\geq(|P|-5)-(|M|-5)=|P|-|M|\geq n+2$. This implies that $\alpha_s^2(D_1)\geq n+2$, a contradiction. Therefore Claim 3 holds.

Claim 4. There exists a subset Y_i $(1\leq i\leq n)$ such that $|(Y_i\cup Y_{i+1}\cup Y_{i+2})\cap P|=10$.

Suppose $n \equiv 0$ (mod 3). Then $n = 3k$ for some integer $k \geq 2$, and $|P| \geq 9k+1$. Let $W_i = \cup_{r=0}^{2} Y_{3i-r}$ for $1 \leq i \leq k$. Thus, by Lemma 10 (4), there must be a W_i for some $1 \leq i \leq n$ such that $|W_i \cap P| = 10$ as $|P| \geq 9k+1$. Hence Claim 4 is true in this case.

Suppose $n \equiv 1$ (mod 3). Then $n = 3k+1$ for some integer $k \geq 2$, and $|P| \geq 9k+4$. According to Claim 3 and Lemma 10 (1), there exists a subset Y_i for some $1 \leq i \leq n$ such that $|Y_i \cap P| \leq 3$. By the structure of $C_5 \times C_n$, without loss of generality, assume that $|Y_n \cap P| \leq 3$. Thus, by Lemma 10 (4), there must exist a W_i for some $1 \leq i \leq k$ such that $|W_i \cap P| = 10$ since $|P| \geq 9k+4$. So Claim 4 holds in this case.

Suppose $n \equiv 2$ (mod 3). Then $n = 3k+2$ for some integer $k \geq 2$, and $|P| \geq 9k+7$. By Claim 3, there exists a subset Y_i for some $1 \leq i \leq n$ such that $|Y_i \cap P| = 4$ as $|P| \geq 9k+7$. By the structure of $C_5 \times C_n$, without loss of generality, suppose that $|Y_n \cap P| = 4$. Then $|Y_{n-1} \cap P| \leq 2$ from Lemma 10 (1). Thus, by Lemma 10 (4), there must be a W_i for some $1 \leq i \leq k$ such that $|W_i \cap P| = 10$ as $|P| \geq 9k+7$. Therefore Claim 4 holds.

By Claim 4, there exists a subset Y_i for some $1 \leq i \leq n$ such that $|(Y_i \cup Y_{i+1} \cup Y_{i+2}) \cap P| = 10$. Applying Claim 3 and Lemma 10 (1), it follows that $|Y_i \cap P| = |Y_{i+2} \cap P| = 4$ and $|Y_{i+1} \cap P| = 2$ or $|Y_i \cap P| = 4$ and $|Y_{i+1} \cap P| = |Y_{i+2} \cap P| = 3$. We consider two cases.

Case 1. $|Y_i \cap P| = |Y_{i+2} \cap P| = 4$ and $|Y_{i+1} \cap P| = 2$. By the structure of $C_5 \times C_n$, without loss of generality, we may assume that $Y_{i+2} \cap P = \{(u_1, v_{i+2}), (u_2, v_{i+2}), (u_3, v_{i+2}), (u_4, v_{i+2})\}$. Then $Y_{i+1} \cap P = \{(u_1, v_{i+1}), (u_5, v_{i+1})\}$ and $Y_i \cap P = \{(u_2, v_i), (u_3, v_i), (u_4, v_i), (u_5, v_i)\}$. Thus $\{(u_3, v_{i-1}), (u_4, v_{i-1}), (u_5, v_{i-1})\} \subseteq M$. Let $D_2 = D[V(C_5 \times C_n) \setminus \{Y_i, Y_{i+1}, Y_{i+2}\}]$
$\cup (\cup_{j=1}^{5} \{(u_j, v_{i-1})(u_j, v_{i+3})\})$. Then $D_2 \cong C_5 \times C_{n-3}$. By the induction hypothesis, $\alpha_s^2(D_2) = n-3$. Define $g_2 : V(D_2) \rightarrow \{-1, 1\}$ by $g_2((u_4, v_{i-1})) = 1$ and $g_2((x, y)) = g((x, y))$ for each $(x, y) \in V(D_2) \setminus \{(u_4, v_{i-1})\}$. It is easy to see that g_2 is a S2IF of D_2 with weight $w(g_2) \geq (|P|-10)+1-(|M|-5-1) = |P|-|M|-3 \geq n-1$. Hence $\alpha_s^2(D_2) \geq n-1$, which is a contradiction.

Case 2. $|Y_i \cap P| = 4$ and $|Y_{i+1} \cap P| = |Y_{i+2} \cap P| = 3$. By the symmetrical structure of $C_5 \times C_n$, without loss of generality, we may assume that $Y_i \cap P = \{(u_1, v_i), (u_2, v_i), (u_3, v_i), (u_4, v_i)\}$. Then $\{(u_2, v_{i-1}), (u_3, v_{i-1}), (u_4, v_{i-1})\} \subseteq M$. If $(u_1, v_{i+1}) \in P$, then $\{(u_2, v_{i+1}), (u_5, v_{i+1})\} \subseteq M$ by the definition of S2IF. Thus $\{(u_3, v_{i+1}), (u_4, v_{i+1})\} \subseteq P$ as $|Y_{i+1} \cap P| = 3$. This leads to $g[(u_4, v_{i+1})] = 3$, a contradiction. So $(u_1, v_{i+1}) \in M$. Since $g[(u_3, v_{i+1})] \leq 1$ and $g[(u_4, v_{i+1})] \leq 1$, $(u_3, v_{i+1}) \in M$. So $Y_{i+1} \cap P = \{(u_2, v_{i+1}), (u_4, v_{i+1}), (u_5, v_{i+1})\}$. If $(u_4, v_{i+2}) \in P$, then $\{(u_3, v_{i+2}), (u_5, v_{i+2})\} \subseteq M$ by the definition of S2IF. As $|Y_{i+2} \cap P| = 3$, it follows that $\{(u_1, v_{i+2}), (u_2, v_{i+2})\} \subseteq P$, which means that $g[(u_2, v_{i+2})] = 3$, a contradiction. Hence $(u_4, v_{i+2}) \in M$. Further, we have that $|\{(u_1, v_{i+2}), (u_2, v_{i+2})\} \cap P| \leq 1$ because $g[(u_2, v_{i+2})] \leq 1$. Then $\{(u_3, v_{i+2}), (u_5, v_{i+2})\} \subseteq P$ as $|Y_{i+2} \cap P| = 3$ again. Suppose $|Y_{i-1} \cap P| \leq 1$. The function $g_3 : V(D_1) \rightarrow \{-1, 1\}$ is defined as follows: $g_3((u_5, v_{i+1})) = -1$ and $g_3((x, y)) = g((x, y))$ for each $(x, y) \in V(D_1) \setminus \{(u_5, v_{i+1})\}$. It is not hard to check that g_3 is a S2IF of D_1 with weight $w(g_3) \geq$

$(|P|-5)-1-(|M|-5+1) = |P|-|M|-2 \geq n$, and hence $\alpha_s^2(D_1) \geq n$, a contradiction. So $|Y_{i-1} \cap P| \geq 2$. Recall that $\{(u_2, v_{i-1}), (u_3, v_{i-1}), (u_4, v_{i-1})\} \subseteq M$. Therefore $Y_{i-1} \cap P = \{(u_1, v_{i-1}), (u_5, v_{i-1})\}$. Then $(u_1, v_{i-2}) \in M$. We proceed the proof by distinguishing the following subcases.

Case 2.1. $(u_1, v_{i+2}) \in P$. Then $Y_{i+2} \cap P = \{(u_1, v_{i+2}), (u_3, v_{i+2}), (u_5, v_{i+2})\}$. Let $g_4((u_3, v_{i-1})) = 1$ and $g_4((x,y)) = g((x,y))$ for each $(x,y) \in V(D_2) \setminus \{(u_3, v_{i-1})\}$. Clearly, g_4 is a S2IF of D_2 with weight $w(g_4) \geq (|P|-10)+1-(|M|-5-1) = |P|-|M|-3 \geq n-1$, which implies that $\alpha_s^2(D_2) \geq n-1$, a contradiction.

Case 2.2. $(u_2, v_{i+2}) \in P$. Then $Y_{i+2} \cap P = \{(u_2, v_{i+2}), (u_3, v_{i+2}), (u_5, v_{i+2})\}$. Let $D_3 = D[V(C_5 \times C_n) \setminus \{Y_{i-1}, Y_i, Y_{i+1}\}] \cup (\cup_{j=1}^5 \{(u_j, v_{i-2})(u_j, v_{i+2})\})$. Then $D_3 \cong C_5 \times C_{n-3}$, and $\alpha_s^2(D_3) = n-3$ by the induction hypothesis. Suppose $(u_3, v_{i-2}) \in M$. Obviously, $g_5 = g|_{D_3}$ is a S2IF on D_3 of weight $w(g_5) \geq (|P|-9)-(|M|-6) = |P|-|M|-3 \geq n-1$. We deduce that $\alpha_s^2(D_3) \geq n-1$, which is a contradiction. Hence $(u_3, v_{i-2}) \in P$. Suppose $|Y_{i-2} \cap P| = 4$. Then $Y_{i-2} \cap P = \{(u_2, v_{i-2}), (u_3, v_{i-2}), (u_4, v_{i-2}), (u_5, v_{i-2})\}$. By a similar argument that used in the proof of Case 1 of Theorem 11, one reaches the same contradiction. So $|Y_{i-2} \cap P| \leq 3$. Suppose $|Y_{i-2} \cap P| \leq 2$. Let $D_4 = D[V(C_5 \times C_n) \setminus \{Y_{i-2}, Y_{i-1}, Y_i\}] \cup (\cup_{j=1}^5 \{(u_j, v_{i-3})(u_j, v_{i+1})\})$. Then $D_4 \cong C_5 \times C_{n-3}$. Applying the induction hypothesis, $\alpha_s^2(D_4) = n-3$. Define $g_6 : V(D_4) \to \{-1, 1\}$ by $g_6((u_5, v_{i+1})) = -1$ and $g_6((x,y)) = g((x,y))$ for each $(x,y) \in V(D_4) \setminus \{(u_5, v_{i+1})\}$. It can be readily verified that g_6 is a S2IF of D_4 with weight $w(g_6) \geq (|P|-8)-1-(|M|-7+1) = |P|-|M|-3 \geq n-1$. This means that $\alpha_s^2(D_4) \geq n-1$, a contradiction. Therefore $|Y_{i-2} \cap P| = 3$.

If $Y_{i-2} \cap P = \{(u_2, v_{i-2}), (u_3, v_{i-2}), (u_4, v_{i-2})\}$, then $\{(u_1, v_{i-2}), (u_5, v_{i-2})\} \subseteq M$. Clearly, $g_7 = g|_{D_1}$ is a S2IF on D_1 of weight $w(g_7) \geq (|P|-6)-(|M|-4) = |P|-|M|-2 \geq n$. So $\alpha_s^2(D_1) \geq n$, a contradiction.

If $Y_{i-2} \cap P = \{(u_2, v_{i-2}), (u_3, v_{i-2}), (u_5, v_{i-2})\}$, then $\{(u_1, v_{i-2}), (u_4, v_{i-2})\} \subseteq M$. Let $D_5 = D[V(C_5 \times C_n) \setminus \{Y_{i-1}, Y_i, Y_{i+1}, Y_{i+2}\}] \cup (\cup_{j=1}^5 \{(u_j, v_{i-2})(u_j, v_{i+3})\})$. Then $D_5 \cong C_5 \times C_{n-4}$. Applying the induction hypothesis, $\alpha_s^2(D_5) = n-4$ for $n \geq 7$ ($\alpha_s^2(D_5) = n-6$ when $n = 6$). It is easy to see that $g_8 = g|_{D_5}$ is a S2IF of D_5 with weight $w(g_8) \geq (|P|-12)-(|M|-8) = |P|-|M|-4 \geq n-2$. Thus $\alpha_s^2(D_5) \geq n-2$, which is a contradiction.

If $Y_{i-2} \cap P = \{(u_3, v_{i-2}), (u_4, v_{i-2}), (u_5, v_{i-2})\}$, then $\{(u_1, v_{i-2}), (u_2, v_{i-2})\} \subseteq M$. The function $g_9 : V(D_3) \to \{-1, 1\}$ is defined as follows: $g_9((u_2, v_{i-2})) = 1$, $g_9((u_3, v_{i-2})) = -1$ and $g_9((x,y)) = g((x,y))$ for each $(x,y) \in V(D_3) \setminus \{(u_2, v_{i-2}), (u_3, v_{i-2})\}$. It is not difficult to check that g_9 is a S2IF of D_3 with weight $w(g_9) \geq (|P|-9)-(|M|-6) = |P|-|M|-3 \geq n-1$. This derives that $\alpha_s^2(D_3) \geq n-1$, a contradiction.

In either case, we always arrive at a contradiction. Consequently, $\alpha_s^2(C_5 \times C_n) = n$ for $n \geq 3$. This completes the proof of Theorem 11. □

3 Further Remark

A set of vertices $B \subseteq V$ is called k-limited packing in G if $|N[v] \bigcap B| \leq k$, for all $v \in V$. The k-limited packing number, denoted $L_k(G)$, is the largest number of vertices in a k-limited packing set. Gallant et al. exhibited some real-world applications of the k-limited packing number to network security, market saturation and codes [3]. In 2017, Moghaddam et al. proved the signed 2-independence number of a regular graph of order n is two times the limited packing number and subtracts n and exhibited real-world applications of it to signed 2-independence number in graphs [4]. In this paper we determine the exact values of $\alpha_s^2(C_m \times C_n)$ for $m \leq 5, n \geq 3$. However, the proofs of the results obtained in this paper depend heavily on the fact that m is small. Thus it seems to be more difficult to determine the exact values of $\alpha_s^2(C_m \times C_n)$ for $m \geq 6$. We will study to establish some algorithm using our proofs in this paper and limited packing numbers for $m \geq 6$.

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