Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 4. pp. 593 - 608
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.4.593

THE MINIMUM VERTEX-BLOCK DOMINATING ENERGY OF THE GRAPH

Sayinath Udupa[a] R. S. Bhat*[a] Vinay Madhusudanan[a]

[a]Department of Mathematics, Manipal Institute of Technology
A Constituent Unit of Manipal Academy of Higher Education
Manipal, Karnataka, India – 576104
sayinath.udupa@manipal.edu rs.bhat@manipal.edu vinay.m2000@gmail.com

Abstract

Let $B(G)$ denote the set of all blocks of a graph G. A vertex $v \in V$ and a block $b \in B(G)$ are said to block dominate (b-dominate) each other if v is in the block b. A set $D \subseteq V$ is said to be a vertex block dominating set (VBD-set) if every block in G is b-dominated by some vertex in D. The vertex block domination number $\gamma_{vb} = \gamma_{vb}(G)$ is the cardinality of the minimum vertex block dominating set of G. In this paper we introduce new kind of graph energy, the minimum vertex block dominating energy of the graph denoting it as $E_{vb}(G)$. It depends both on the underlying graph of G and the particular minimum vertex block dominating set (γ_{vb}-set) of G. Upper and lower bounds for $E_{vb}(G)$ are established and we also obtain energy of some family of graphs.

AMS subject classification: 05C69, 05C50

Key words: Energy of the graph, the minimum vertex block dominating energy of the graph, Corona product of the graph.

1 Introduction

The terminologies and notations used here are as in [9, 16]. By a graph $G(V,E)$ we mean a connected finite simple graph of order p and size q. A set $D \subseteq V$ is a dominating set of G if every vertex not in D is adjacent to some vertex in D. The domination number $\gamma = \gamma(G)$ is the order of a minimum dominating set of G. The domination number is a well studied parameter in literature and for a survey refer [4, 10, 15]. A vertex $v \in V$ is a cutvertex if $G - \{v\}$ is disconnected. A graph which has no cutvertex is called non separable. A maximal non-separable subgraph is a block of G. Let $B(G)$ and $C(G)$ respectively denote the set of all blocks and set of all cutvertices of G with $|B(G)| = m$ and $|C(G)| = n$. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in an non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \ldots, \lambda_p(G)$. Then the energy of the graph G is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i(G)|$. Various properties of energy of the graph may be found in [6, 7]. In connection with graph energy, energy-like quantities were considered for other matrices such as Laplacian [8], distance [11] and incidence [12]. Recently Chandrashekar Adiga et al., [1] defined a new kind of energy called minimum covering energy. In this paper we introduce a new matrix, called minimum vb-dominating matrix of a graph and study its energy.

2 Minimum vb-dominating energy of the graph

In 2013 P. G. Bhat et al. [5] initiated the study of vb-dominating sets. Two vertices are vv-adjacent if they are vertices of the same block. A vertex $v \in V$ and a block $b \in B(G)$ are said to block dominate (b-dominate) each other if v is in the block b. A set $D \subseteq V$ is said to be a vertex block dominating set (VBD-set) if every block in G is b-dominated by some vertex in D. The vertex block domination number $\gamma_{vb} = \gamma_{vb}(G)$ is the cardinality of the minimum vertex block dominating set of G. Let γ_{vb}-set be a minimum vertex block dominating set of a graph G. The minimum vertex block dominating matrix of G is the $p \times p$ matrix $A_{vb} = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are vv-adjacent} \\ 1 & \text{if } i = j \text{ and } v_i \in \gamma_{vb}\text{-set} \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{vb}(G)$ is denoted by $f_p(G, \eta) = \det(\eta I - A_{vb}(G))$. The minimum vertex block dominating eigenvalues of the graph G are the eigenvalues of $A_{vb}(G)$. Since $A_{vb}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_p$. The minimum vertex block dominating energy of G is then defined as $E_{vb}(G) = \sum_{i=1}^{p} |\eta_i(G)|$. In this paper we discuss some basic properties of minimum vertex block dominating energy of the graph $E_{vb}(G)$ and derive an upper and lower bound for $E_{vb}(G)$ and we compute minimum vb-dominating energy of some family of graphs. The minimum vb-dominating matrix is independent of the internal structure of the blocks of the graph.

2.1 Some basic properties of minimum vb-dominating energy of the graph

First we compute the minimum vb-dominating energy of the graph shown in Figure 1.

Example 2.1.

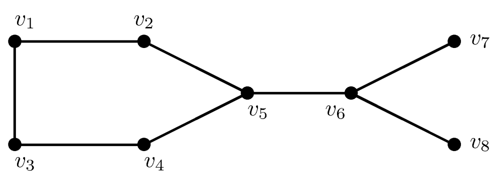

Figure 1: Graph G

Let G be a graph with 8 vertices $v_1, v_2, \ldots, v_8$ (See Figure 1) with minimum vb-dominating set $B = \{v_5, v_6\}$. Then

$$A_{vb}(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of $A_{vb}(G)$ is $\eta^8 - 2\eta^7 - 12\eta^6 - 2\eta^5 + 33\eta^4 + 42\eta^3 + 18\eta^2 + 2\eta$, the minimum vb-dominating eigenvalues are
$-1.2889, -1.0000, -1.0000, -1.0000, -0.1687, 0.0000, 2.1199, 4.3377$. Therefore the minimum vb-dominating energy of a graph G is $E_{vb}(G) = 10.9154$.
Similar to cutvertex graph, P.G. Bhat et al. [5] defined point graph as follows. The point graph $P_G(G)$ is a graph with vertex set same as that of G and any two vertices in $P_G(G)$ are adjacent if, and only if, they are vv-adjacent in G. Number of edges in the point graph is denoted by q_p.

Theorem 2.1. *If $\eta_1(G), \eta_2(G) \ldots, \eta_p(G)$ are the eigenvalues of $A_{vb}(G)$, then*

$$\sum_{i=1}^{p} \eta_i = \gamma_{vb} \tag{1}$$

$$\sum_{i=1}^{p} \eta_i^2 = 2q_p + \gamma_{vb} \tag{2}$$

where q_p is the number of edges in the point graph of G.

Proof. (1) We know that the sum of the eigenvalues of $A_{vb}(G)$ is equal to trace of $A_{vb}(G)$.
Therefore $\sum\limits_{i=1}^{p} \eta_i = \sum\limits_{i=1}^{p} a_{ii} = |\gamma_{vb}\text{-set}| = \gamma_{vb}$
(2) The sum of the squares of the eigenvalues of $A_{vb}(G)$ is just the trace of $A_{vb}(G)^2$.

Therefore

$$\begin{aligned}\sum_{i=1}^{p}\eta_i^2 &= \sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}a_{ji}\\ &= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{p}(a_{ii})^2\\ &= 2q_p + \gamma_{vb}.\end{aligned}$$

□

Theorem 2.2. *Let G be a graph with p vertices, a minimum vertex block dominating set γ_{vb}-set, and let q_p be the number of edges in the point graph of G. Let $f_p(G,\eta) = c_0\eta^p + c_1\eta^{p-1} + c_2\eta^{p-2} + \cdots + c_p$ be the characteristic polynomial of G. Then*

$$c_0 = 1 \tag{3}$$

$$c_1 = -\left|\gamma_{vb}\text{-}set\right| \tag{4}$$

$$c_2 = \binom{\gamma_{vb}}{2} - q_p \tag{5}$$

$$c_3 = \gamma_{vb}q_p - \sum_{u\in\gamma_{vb}\text{-}set} d_{vv}(u) - \binom{\gamma_{vb}}{3} - 2\triangle \tag{6}$$

where $\triangle$ and q_p respectively are the number of triangles and edges in the point graph $P_G(G)$ of G.

Proof. (3) directly follows from the definition of $f_p(G,\eta)$. It follows that $c_0 = 1$. Since the sum of diagonal elements of $A_{vb}(G)$ is equal to γ_{vb}, the sum of determinants of all 1×1 principal submatrices of $A_{vb}(G)$ is the trace of $A_{vb}(G)$, which evidently is equal to γ_{vb}. Thus $(-1)^1c_1 = \gamma_{vb}$.

$(-1)^2c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of

$A_{vb}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le p} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le p} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le p} (a_{ii}a_{jj}) - \sum_{i\le i<j\le p} a_{ij}^2 \\ &= \binom{\gamma_{vb}}{2} - q_p. \end{aligned}$$

We have

$$\begin{aligned} c_3 &= (-1)^3 \sum_{1\le i<j<k\le p} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \\ &= -\sum_{1\le i<j<k\le p} [a_{ii}(a_{jj}a_{kk} - a_{jk}a_{kj}) - a_{ij}(a_{ji}a_{kk} - a_{ki}a_{jk}) + a_{ik}(a_{ji}a_{kj} - a_{ki}a_{jj})] \\ &= -\sum_{1\le i<j<k\le p} (a_{ii}a_{jj}a_{kk}) + \sum_{1\le i<j<k\le p} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}) - \\ &\quad \sum_{1\le i<j<k\le p} (a_{ij}a_{jk}a_{ki}) - \sum_{1\le i<j<k\le p} (a_{ik}a_{kj}a_{ji}) \\ &= -\binom{\gamma_{vb}}{3} + \sum_{1\le i<j<k\le p} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}) - 2\triangle \\ &= -\binom{\gamma_{vb}}{3} + \sum_{i=1}^{p} a_{ii} \sum_{1\le i<j<k\le p} (a_{jk}) - \sum_{i=1}^{p} a_{ii} \sum_{k=1,k\ne i} a_{ik} - 2\triangle \\ &= -\binom{\gamma_{vb}}{3} + \gamma_{vb}q_p - \sum_{u\in\gamma_{vb}\text{-}set} d_{vv}(u) - 2\triangle \\ c_3 &= \gamma_{vb}q_p - \sum_{u\in\gamma_{vb}\text{-}set} d_{vv}(u) - \binom{\gamma_{vb}}{3} - 2\triangle\,. \end{aligned}$$

□

Theorem 2.3. *If $\eta_1(G)$ is the largest eigenvalue of the minimum vb-dominating matrix $A_{vb}(G)$, then*

$$\eta_1(G) \ge \frac{2q_p + \gamma_{vb}}{p}. \tag{7}$$

Proof. Let X be any non zero vector, then we have

$$\eta_1\left(A_{vb}(G)\right) = \begin{matrix} \max \\ X \neq 0 \end{matrix} \left[\frac{X' A_{vb} X}{X' X}\right] \text{ (see [2])}$$

$$\eta_1(A_{vb}(G)) \geq \left[\frac{J' A_{vb} J}{J' J}\right] = \frac{2q_p + \gamma_{vb}}{p}, \text{ where } J \text{ is the all one's vector.}$$ □

Bapat and S. Pati [3] proved that if energy of the graph is a rational number, then it is an even integer. Similar result for minimum vb-dominating energy is established in the following theorem.

Theorem 2.4. *Let G be a graph with vb-domination number γ_{vb} and if the minimum vb-dominating energy $E_{vb}(G)$ is a rational number, then*

$$E_{vb}(G) \equiv \gamma_{vb} \pmod 2. \tag{8}$$

Proof. Let $\eta_1, \eta_2 \ldots, \eta_p$ be the eigenvalues of the minimum vb-dominating matrix $A_{vb}(G)$ of a graph G of which $\eta_1, \eta_2 \ldots, \eta_r$ are positive and the rest of the eigenvalues are non-positive, then

$$\sum_{i=1}^{p} |\eta_i| = (\eta_1 + \eta_2 + \cdots + \eta_r) - (\eta_{r+1} + \eta_{r+2} \cdots + \eta_p)$$

$$= 2(\eta_1 + \eta_2 + \cdots + \eta_r) - (\eta_1 + \eta_2 + \cdots + \eta_p)$$

$$E_{vb}(G) = 2(\eta_1 + \eta_2 + \cdots + \eta_r) - \sum_{i=1}^{p} \eta_i$$

$$E_{vb}(G) = 2(\eta_1 + \eta_2 + \cdots + \eta_r) - \gamma_{vb}.$$

Therefore $E_{vb}(G) \equiv \gamma_{vb} \pmod 2$. □

2.2 Some bounds for minimum vb-dominating energy of the graph

Bounds for $E_{vb}(G)$ similar to McClelland's inequalities [14] for graph energy are given in the following theorems.

Theorem 2.5. *Let G be a graph with p vertices, minimum vertex block dominating set γ_{vb}-set, and let q_p be the number of edges in the point graph of G. Then*

$$E_{vb}(G) \leq \sqrt{p\left(2q_p + \gamma_{vb}\right)}. \tag{9}$$

Proof. Let $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_p$ be the eigenvalues of $A_{vb}(G)$.
Using Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right)$$

choose $a_i = 1$ and $b_i = |\eta_i|$,

$$E_{vb}(G)^2 = \left(\sum_{i=1}^{p} |\eta_i|\right)^2 \leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} |\eta_i|^2\right) = p\left(2q_p + \gamma_{vb}\right),$$

by using Theorem 2.1.
Therefore $E_{vb}(G) \leq \sqrt{p(2q_p + \gamma_{vb})}$. □

Theorem 2.6. *Let G be a graph with p vertices, minimum vb-dominating set γ_{vb}-set, and let q_p be the number of edges in the point graph of G. If K is the determinant of $A_{vb}(G)$, then*

$$E_{vb}(G) \geq \sqrt{2q_p + \gamma_{vb} + p(p-1)K^{\frac{2}{p}}}. \tag{10}$$

Proof.

$$\begin{aligned} E_{vb}(G)^2 &= \left(\sum_{i=1}^{p} |\eta_i|\right)^2 = \left(\sum_{i=1}^{p} |\eta_i|\right)\left(\sum_{j=1}^{p} |\eta_j|\right) \\ &= \sum_{i=1}^{p} |\eta_i|^2 + \sum_{i \neq j} |\eta_i||\eta_j| \end{aligned}$$

Now employing the inequality between the arithmetic and geometric means, we obtain

$$\frac{1}{p(p-1)}\sum_{i\neq j}|\eta_i||\eta_j| \geq \left(\prod_{i\neq j}|\eta_i||\eta_j|\right)^{\frac{1}{p(p-1)}}$$

$$\sum_{i\neq j}|\eta_i||\eta_j| \geq p(p-1)\left(\prod_{i\neq j}|\eta_i||\eta_j|\right)^{\frac{1}{p(p-1)}}$$

$$\text{Thus } E_{vb}(G)^2 \geq \sum_{i=1}^{p}|\eta_i|^2 + p(p-1)\left(\prod_{i\neq j}|\eta_i||\eta_j|\right)^{\frac{1}{p(p-1)}}$$

$$\geq \sum_{i=1}^{p}|\eta_i|^2 + p(p-1)\left(\prod_{i=1}^{p}|\eta_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}}$$

$$= \sum_{i=1}^{p}|\eta_i|^2 + p(p-1)\left(\prod_{i=1}^{p}|\eta_i|\right)^{\frac{2}{p}}$$

$$E_{vb}(G)^2 \geq 2q_p + \gamma_{vb} + p(p-1)K^{\frac{2}{p}}$$

Therefore $E_{vb}(G) \geq \sqrt{2q_p + \gamma_{vb} + p(p-1)K^{\frac{2}{p}}}$. □

Similar to Koolen and Moultons's [13] upper bound for energy of the graph, upper bound for $E_{vb}(G)$ is given in the following theorem.

Theorem 2.7. *If G is a graph with p vertices with point graph $P_G(G)$ having q_p edges, and $2q_p + \gamma_{vb} \geq p$, then*

$$E_{vb}(G) \leq \frac{2q_p + \gamma_{vb}}{p} + \sqrt{(p-1)\left((2q_p + \gamma_{vb}) - \left(\frac{2q_p + \gamma_{vb}}{p}\right)^2\right)}. \tag{11}$$

Proof. Using Cauchy-schwarz inequality, $\left(\sum\limits_{i=2}^{p} a_i b_i\right)^2 \leq \left(\sum\limits_{i=2}^{p} a_i^2\right)\left(\sum\limits_{i=2}^{p} b_i^2\right)$

choose $a_i = 1$ and $b_i = |\eta_i|$, $i = 2, 3, \ldots, p$,

$$\left(\sum_{i=2}^{p} |\eta_i|\right)^2 \leq \left(\sum_{i=2}^{p} 1\right)\left(\sum_{i=2}^{p} |\eta_i|^2\right)$$

$$\left(\sum_{i=1}^{p} |\eta_i| - \eta_1\right)^2 \leq (p-1)\left(\sum_{i=1}^{p} \eta_i^2 - \eta_1^2\right)$$

$$(E_{vb}(G) - \eta_1)^2 \leq (p-1)\left(2q_p + \gamma_{vb} - \eta_1^2\right)$$

$$(E_{vb}(G) - \eta_1) \leq \sqrt{(p-1)\left(2q_p + \gamma_{vb} - \eta_1^2\right)}$$

$$E_{vb}(G) \leq \eta_1 + \sqrt{(p-1)(2q_p + \gamma_{vb} - \eta_1^2)}$$

Let $f(t) = t + \sqrt{(p-1)(2q_p + \gamma_{vb} - t^2)}$

for decreasing function $f'(t) \leq 0$

$$f'(t) = 1 - \frac{t(p-1)}{\sqrt{(p-1)(2q_p + \gamma_{vb} - t^2)}} \leq 0$$

Therefore $t \geq \sqrt{\dfrac{2q_p + \gamma_{vb}}{p}}$

Since $2q_p + \gamma_{vb} \geq p$, we have

$$\sqrt{\frac{2q_p + \gamma_{vb}}{p}} \leq \frac{2q_p + \gamma_{vb}}{p} \leq \eta_1$$

$$f(\eta_1) \leq f\left(\frac{2q_p + \gamma_{vb}}{p}\right)$$

Therefore $E_{vb}(G) \leq f(\eta_1) \leq f\left(\dfrac{2q_p + \gamma_{vb}}{p}\right)$

$$E_{vb}(G) \leq f\left(\frac{2q_p + \gamma_{vb}}{p}\right)$$

$$E_{vb}(G) \leq \frac{2q_p + \gamma_{vb}}{p} + \sqrt{(p-1)\left((2q_p + \gamma_{vb}) - \left(\frac{2q_p + \gamma_{vb}}{p}\right)^2\right)}.$$

□

3 Minimum vb-dominating energies of some families of graphs

Definition 3.1. *The corona product of G and H is the graph $G \cdot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, called the outer* *graph, and making the i^{th} vertex of G adjacent to every vertex of the i^{th} copy of H, where $1 \leq i \leq |V(G)|$.*

Definition 3.2. *If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then their Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Consider the corona product $G = K_m \cdot rK_n$ of the complete graph K_m and the disjoint union of r copies of the complete graph K_n, $r > 0$. We label the vertices in the following manner, as shown in Figure 2. The vertices of K_m are labelled

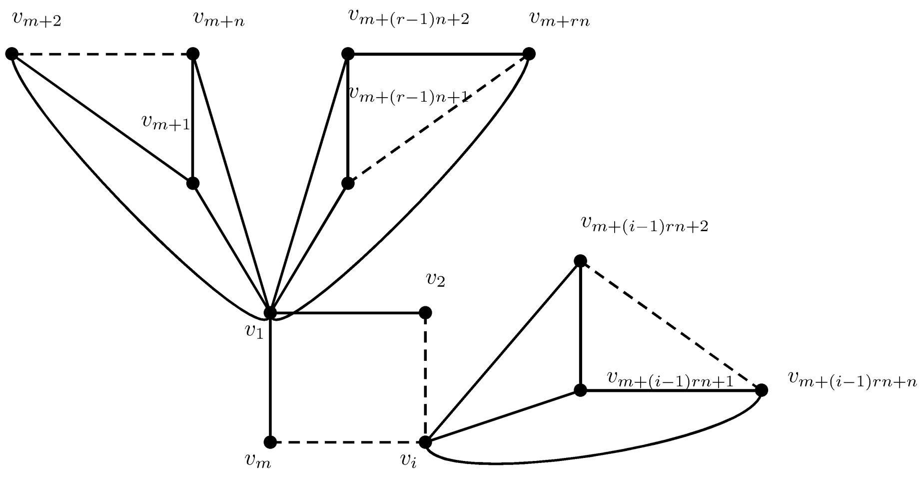

Figure 2: $K_m \cdot rK_n$

$v_1, v_2, \dots, v_m$. Each such vertex v_i has r pendant blocks incident with it, each a copy of K_n. Let the vertices of the j^{th} pendant block incident with v_i be labelled $v_{m+(i-1)nr+(j-1)n+1}, \dots, v_{m+(i-1)nr+(j-1)n+n}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then the vb-dominating matrix of G with minimum vb-dominating set $\{v_1, \dots, v_m\}$ is

$$A = \begin{bmatrix} J_m & I_m \otimes \mathbf{1}_{nr}^T \\ I_m \otimes \mathbf{1}_{nr} & I_{mr} \otimes (J_n - I_n) \end{bmatrix}. \tag{12}$$

Theorem 3.1. *Let $G = K_m \cdot rK_n$ with vertices labelled as in Figure 2. Then the characteristic polynomial of the $A_{vb}(G)$ with minimum vb-dominating set $\{v_1, \ldots, v_m\}$ is*

$$(\eta+1)^{mr(n-1)}(\eta-n+1)^{m(r-1)}(\eta^2-(n-1)\eta-nr)^{m-1}(\eta^2-(m+n-1)\eta+m(n-1)-nr). \tag{13}$$

Proof. Let A be the vb-dominating matrix of G given in (12). We prove the result by showing that $AX = \eta X$ for certain vectors X, and by making use of the fact that for each such eigenvalue η, its geometric multiplicity is equal to its algebraic multiplicity, since A is real and symmetric. In what follows, let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be an $m(nr+1) \times 1$ vector partitioned conformally with A.

First, let $X_1 = 0$ and $X_2 = Y \otimes Z \otimes W$, where Y, Z, and W are non-zero column vectors of lengths m, r, and n respectively. Then

$$\begin{aligned} AX &= \begin{bmatrix} (I_m \otimes \mathbf{1}_{nr}^T)X_2 \\ (I_{mr} \otimes (J_n - I_n))X_2 \end{bmatrix} \\ &= \begin{bmatrix} (I_m \otimes \mathbf{1}_r^T \otimes \mathbf{1}_n^T)(Y \otimes Z \otimes W) \\ (I_{mr} \otimes (J_n - I_n))(Y \otimes Z \otimes W) \end{bmatrix} \\ &= \begin{bmatrix} Y \otimes \mathbf{1}_r^T Z \otimes \mathbf{1}_n^T W \\ Y \otimes Z \otimes (J_n - I_n)W \end{bmatrix} \end{aligned}$$

Now, if W is any eigenvector of J_n with corresponding eigenvalue μ, then

$$AX = \begin{bmatrix} Y \otimes \mathbf{1}_r^T Z \otimes \mathbf{1}_n^T W \\ Y \otimes Z \otimes (\mu - 1)W \end{bmatrix} = \begin{bmatrix} Y \otimes \mathbf{1}_r^T Z \otimes \mathbf{1}_n^T W \\ (\mu - 1)X_2 \end{bmatrix}. \tag{14}$$

If $\mu = 0$, then W is a vector in the null space of J_n, and therefore, $\mathbf{1}_n^T W = 0$. Thus, from (14), $AX = -X$. Here Y and Z are arbitrary vectors of $\mathbb{R}^m$ and $\mathbb{R}^r$ respectively, and W is in the null space of J_n, which is of dimension $n-1$. Thus, -1 is an eigenvalue of A with multiplicity at least $mr(n-1)$. Now, let $\mu = n$, and let Z be a vector in the null space of J_r. Then $\mathbf{1}_r^T Z = 0$, and from (14), we have

$AX = (n-1)X$. Reasoning as in the previous case, we see that $n-1$ is an eigenvalue of A with multiplicity at least $m(r-1)$.

Next, let $X_1 = (\eta - n + 1)U$ and $X_2 = U \otimes \mathbf{1}_{nr}$, where U is an eigenvector of J_m with corresponding eigenvalue μ, and η is a solution of the quadratic equation $\eta^2 - (\mu + n - 1)\eta + \mu(n-1) - nr = 0$. Observe that U can also be written as $U \otimes 1$, considering the scalar 1 as a 1×1 matrix. Then

$$\begin{aligned} AX &= \begin{bmatrix} J_m X_1 + (I_m \otimes \mathbf{1}_{nr}^T)X_2 \\ (I_m \otimes \mathbf{1}_{nr})X_1 + (I_{mr} \otimes (J_n - I_n))X_2 \end{bmatrix} \\ &= \begin{bmatrix} J_m((\eta - n + 1)U) + U \otimes \mathbf{1}_{nr}^T \mathbf{1}_{nr} \\ (\eta - n + 1)U \otimes \mathbf{1}_{nr} + U \otimes \mathbf{1}_r \otimes (J_n - I_n)\mathbf{1}_n \end{bmatrix} \\ &= \begin{bmatrix} \mu(\eta - n + 1)U + nrU \\ (\eta - n + 1)U \otimes \mathbf{1}_r \otimes \mathbf{1}_n + U \otimes \mathbf{1}_r \otimes ((n-1)\mathbf{1}_n) \end{bmatrix} \\ &= \begin{bmatrix} (\mu(\eta - n + 1) + nr)U \\ \eta U \otimes \mathbf{1}_{nr} \end{bmatrix}. \end{aligned}$$

But $\eta^2 - (\mu + n - 1)\eta + \mu(n-1) - nr = 0 \implies \eta(\eta - n + 1) = \mu(\eta - n + 1) + nr$. Therefore, $AX = \eta X$. Clearly, the number of such linearly independent vectors X is equal to the number of linearly independent vectors U. If $\mu = 0$, then we have $\eta^2 - (n-1)\eta - nr = 0$, and each of the two solutions is an eigenvalue of A with multiplicity at least $m - 1$ (the multiplicity of $\mu = 0$ as an eigenvalue of J_m). If $\mu = m$, then $\eta^2 - (m + n - 1)\eta + m(n-1) - nr = 0$, and each of the two solutions has multiplicity at least 1.

Thus, -1, $n-1$, the two solutions of $\eta^2 - (n-1)\eta - nr = 0$, and the two solutions of $\eta^2 - (m+n-1)\eta + m(n-1) - nr = 0$ are eigenvalues of A with respective multiplicities at least $mr(n-1)$, $m(r-1)$, $m-1$, and 1. But $mr(n-1) + m(r-1) + 2(m-1) + 2 = mnr + m$, which is the order of A. Therefore these constitute all eigenvalues of A with multiplicities exactly equal to the given respective lower bounds.

Note that if $n = r = 1$, then the equation $\eta^2 - (n-1)\eta - nr = 0$ reduces to $\eta^2 - 1 = 0$, and the corresponding eigenvalues are 1 and -1 again. However, the above analysis holds in this case as well, since the eigenvectors we have found in the two different cases of $\eta = 1$ are linearly independent, and likewise for $\eta = -1$. □

Corollary 3.1. *The minimum vb-dominating energy of the corona product $K_m \cdot rK_n$ is equal to*

$$(n-1)(2mr-m) + \left(\sqrt{(n-1)^2+4nr}\right)(m-1) + m + n - 1, \quad \text{if } m(n-1) > nr$$
$$(n-1)(2mr-m) + \left(\sqrt{(n-1)^2+4nr}\right)(m-1) + \sqrt{(m+n-1)^2 - 4(m(n-1)-nr)},$$

if $m(n-1) < nr$.

Definition 3.3. *A graph G with m blocks is said to be B-complete if any two blocks in G are adjacent, and is denoted by B_m.*

Corollary 3.2. *For any B-complete graph B_m with each block having r vertices, $E_{vb}(B_m) = (2m-1)(r-2) + \sqrt{(r-1)^2 + 4[(m-1)(r-1)+1]}$.*

Proof. It follows from the Theorem 3.1 when $m = 1, r = m$ and $n = r - 1$ □

Corollary 3.3. *If G is the $K_m \cdot \overline{K_r}$ graph, then $E_{vb}(G) = 2(m-1)\sqrt{r} + \sqrt{m^2+4r}$.*

Proof. It follows from the Theorem 3.1 when $n = 1$. □

Acknowledgment

We are indebted to the referee for his/her invaluable suggestions which has improved the overall presentation of the paper.

References

[1] C. Adiga, A. Bayad, I. Gutman, S.A. Srinivas, *The minimum covering energy of a graph*, Kragujevac J. Sci, **34** (2012), 39–56.

[2] R.B. Bapat, *Graphs and Matrices*, Hindustan Book Agency, New Delhi (2011).

[3] R.B. Bapat, S. Pati, *Energy of a graph is never be an odd integer*, Bull. Kerala Math. Assoc., **1** (2011), 129–132.

[4] C. Berge, *Theory of Graphs and its Applications*, Methuen, London (1962).

[5] P.G. Bhat, R.S. Bhat, S.R. Bhat, *Relationship between Block Domination parameters of a graph*, Discrete Mathematics, Algorithms and Applications, **5(3)** (2013) 13500181–10.

[6] I. Gutman, *The energy of a graph: Old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer-Verlag, Berlin (2001), 196–211.

[7] I. Gutman, X. Li, J. Zhang, *Graph energy*, in: M. Dehmer, F. Emmert-Streib (Eds.), Analysis of Complex Networks. From Biology to Linguistics, Wiley-VCH, Weinheim (2009), 145–174.

[8] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Lin. Algebra Appli. **414** (2006), 29–37.

[9] F. Harary, *Graph Theory*, Addison Wesley, Reading, Massachusetts (1969).

[10] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[11] G. Indulal, I. Gutman, A. Vijaykumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60** (2008), 461–472.

[12] M.R. Jooyandeh, D. Kiani, M. Mirzakhah, *Incidence energy of a graph*, MATCH Commun. Math. Comput. Chem. **62** (2009), 561–572.

[13] J.H. Koolen, V. Moulton, *Maximal energy graphs*, Adv. Appl. Math., **26** (2001), 47–52.

[14] B.J. McClelland, Properties of the latent roots of a matrix : The estimation of π-electron energies, J. Chem. Phys **54** (1971), 640–643.

[15] H.B. Walikar, B.D. Acharya, E. Sampathkumar, *Recent Developments in the Theory of Domination in graphs*, M.R.I. Lecture Notes No. 1, The Mehta Research Institute of Mathematics and Mathematical Physics Allahabad (1979).

[16] D.B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, New Jersey (1996).