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DISCONTINUITY SET OF A LOCALLY RELATIVELY COMPACT PURE PSEUDOREPRESENTATION OF A TOPOLOGICAL GROUP

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Abstract. We introduce the notion of discontinuity set for a locally relatively compact pure pseudorepresentation of a topological group and establish its simplest properties.

§ 1. Introduction

The notion of discontinuity group of a (not necessarily continuous) locally relatively compact representation of a topological group in a Banach space was introduced in [1] and studied in detail in [2]. For the notions of quasirepresentations, pseudorepresentations, and pure pseudorepresentations, see [3–5]. In the present paper, using results of [3–5], we introduce the notion of discontinuity set of a locally relatively compact pure pseudorepresentation of a topological group and study its simplest general properties.

§ 2. Preliminaries

Let G be a topological group and let π be a (not necessarily continuous) pure pseudorepresentation of G in a Banach space E (i.e.,

$$\|\pi(g_1g_2) - \pi(g_1)\pi(g_2)\|_{\mathcal{L}(E)} \leq q$$

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for all $g_1, g_2 \in G$ and some $q \geq 0$; q is called the *defect* of π). We say that π is *locally relatively compact* if there is a neighborhood V of the identity element e in G such that the closure of the set $\{\pi(g) \mid g \in V\}$ in the strong operator topology is a compact subset of the space $\mathcal{L}(E)$ of bounded linear operators on E equipped with the strong operator topology.

Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of the identity element in a topological group G. For any (not necessarily continuous) locally relatively compact pure pseudorepresentation π of the group G on a Banach space E, introduce the notation

$$\mathrm{DS}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}.$$

Here and henceforth, the bar stands for the closure in the strong operator topology. The set $\mathrm{DS}(\pi)$ is called the *discontinuity set* of π.

As is known [2], if π is an ordinary representation of G on E, then $\mathrm{DS}(\pi)$ is denoted by $\mathrm{DG}(\pi)$ and is a compact (in the strong operator topology) subgroup of the group $\mathrm{GL}(E)$ of invertible bounded linear operators on E and a normal subgroup of the closure (in the strong operator topology) of the image of G in $\mathrm{GL}(E)$; moreover, $\mathrm{DG}(\pi) = 1_E$ if and only if the representation π is (strongly) continuous. See [6, 7] for applications of this notion.

In the present paper, we modify the above results for the more general case of pure pseudorepresentations.

§ 3. Main results

Theorem 1. *Let G be a topological group and let π be a locally relatively compact pure pseudorepresentation of G on a Banach space E. The set $\mathrm{DS}(\pi)$ is a compact subset of $\mathcal{L}(E)$ which contains the identity operator on E, is symmetric ($A \in \mathrm{DS}(\pi)$ if and only if $A^{-1} \in \mathrm{DS}(\pi)$), and is invariant with respect to conjugations generated by the elements of $\pi(G)$. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DS}(\pi)$, and the homomorphism π is continuous if and only if $\mathrm{DS}(\pi) = \{1_E\}$.*

Proof. The filter $\mathfrak{U}$ admits a basis of symmetric neighborhoods V of the identity element ($V = V^{-1}$), and every symmetric neighborhood V contains a symmetric neighborhood W such that $WW \subset V$. By assumption, there is a neighborhood $U \in \mathfrak{U}$ such that the closure $\overline{\pi(U)}$ is compact. This implies that the above set $\mathrm{DS}(\pi)$ is compact and contains the inverses (since the restriction of π to every commutative subgroup is, by the definition of pure pseudorepresentation, an ordinary representation of the subgroup; in particular, $\pi(e) = 1_E$ for the identity element e of G). Thus, the set $\mathrm{DS}(\pi)$ is a compact symmetric subset of H containing the identity operator. Since the set gVg^{-1}, $g \in G$, is a symmetric neighborhood of the identity element in G together with V, it follows that the intersection $\mathrm{DS}(\pi)$ of all sets of the form $\overline{\pi(U)}$, $U \in \mathfrak{U}$, is invariant with respect to the conjugation with any element of the form $\pi(g)$, $g \in G$, as was claimed.

Since the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ contains a compact element and the intersection of all elements of the filter is $\mathrm{DS}(\pi)$, it follows that the filter basis converges to $\mathrm{DS}(\pi)$.

By the definition of continuity we have $\mathrm{DS}(\pi) = \{e_H\}$ if π is continuous. Conversely, if $\mathrm{DS}(\pi) = \{e_H\}$, then the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\{1_E\}$, and therefore the homomorphism π is continuous.

This completes the proof of the theorem.

It cannot be stated that $\mathrm{DS}(\pi)$ is a subgroup. Instead, we have the following assertion.

Theorem 2. *Let G be a topological group and let π be a locally relatively compact pure pseudorepresentation of G on a Banach space E with defect $q < 1/12$. The symmetric invariant* (*with respect to the conjugations generated by the operators in the image of π*) *compact set $K = \mathrm{DS}(\pi)$ of $\mathcal{L}(E)$ satisfies the relation*

$$K^2 \subset K + B(1_E, q), \tag{1}$$

where $B(1_E, q)$ is the ball in $\mathcal{L}(E)$ with respect to the operator norm with the center at 1_E and of radius q; this ball consists of invertible elements.

Proof. Every symmetric neighborhood V contains a symmetric neighborhood W such that $WW \subset V$. By assumption, there is a neighborhood $U \in \mathfrak{U}$ such that the closure $\overline{\pi(U)}$ is compact. This implies that the set $\mathrm{DS}(\pi)$ is compact; if $w \in W$, then $ww \in V$, and therefore it follows from the inequality

$$\|\pi(ww) - \pi(w)^2\| \le q$$

that the difference between every element of K^2 and some element of K has the norm not exceeding q, which proves (1) and completes the proof of the theorem.

§ 4. Discussion

Relation (1), which replaces for pure pseudorepresentations the subgroup condition for ordinary representations, is a kind of "small doubling;" however, for example, the conditions of small doubling studied in [8, 9] have another nature.

The discontinuity set of a pure pseudorepresentation is an ordinary subgroup for many finite-dimensional pure pseudorepresentations of connected locally compact groups.

Theorem 3. *The discontinuity set of a locally bounded finally precontinuous finite-dimensional pure pseudorepresentation of a connected locally compact group is an ordinary subgroup.*

Proof. The proof immediately follows from the explicit description of canonical forms of these pseudorepresentations in [4].

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