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NEW RAMANUJAN'S REMARKABLE PRODUCT OF THETA-FUNCTION AND THEIR EXPLICIT EVALUATIONS

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ABSTRACT. In this article, we define $d_{m,n}$ for any positive real numbers m and n involving Ramanujan's product of theta-functions $\phi(q)$ and $f(-q^2)$, which is analogous to Ramanujan's remarkable product of theta-functions and establish its several properties by Ramanujan. We establish general theorems for the explicit evaluations of $d_{m,n}$ and its explicit values.

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1. INTRODUCTION

Ramanujan's general theta-function [14] $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.1}$$
$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty .$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.4}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1-aq^n), \qquad |q| < 1.$$

On page 338 in his first notebook [14],[3] Ramanujan defines

$$a_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.5}$$

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [4]. M. S. Mahadeva Naika and B. N. Dharmendra [6], also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values from it. Further results on $a_{m,n}$ can be found by Mahadeva Naika, Dharmendra and K. Shivashankar [7], and Mahadeva Naika and M. C. Mahesh Kumar [8]. Recently Nipen Saikia [12] established new properties of $a_{m,n}$.

Mahadeva Naika et al. [10], defined the product

$$b_{m,n} = \frac{ne^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}}\psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [9] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+} \, \frac{q^3+q^6}{1} \, {}_{+} \cdots, \quad |q|<1, \tag{1.7}$$

and found some particular values of $V(q)$.

In this paper, we define

$$d_{m,n} = \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{m}}}\right)\varphi\left(e^{-\pi\sqrt{mn}}\right)}{e^{-\frac{(m-1)\pi}{12}\sqrt{\frac{n}{m}}}f\left(-e^{-2\pi\sqrt{mn}}\right)\varphi\left(e^{-\pi\sqrt{\frac{n}{m}}}\right)}, \tag{1.8}$$

where m and n are positive real numbers. We establish several properties of the product $d_{m,n}$ and prove general formulas for explicit evaluations of $d_{m,n}$ and find its explicit values.

Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k, l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{1.9}$$

Then a modular equation of degree n is a relation between k and l induced by (1.5). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α.

Define

$$\chi(q) := (-q; q^2)_\infty,$$

and

$$G_n := 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(q),$$

where

$$q = e^{-\pi\sqrt{r}}.$$

Moreover, if $q = e^{-\pi\sqrt{\frac{n}{m}}}$ and β has degree n over α, then

$$G_{\frac{n}{m}} = (4\alpha(1-\alpha))^{\frac{-1}{24}} \tag{1.10}$$

and

$$G_{nm} = (4\beta(1-\beta))^{\frac{-1}{24}}. \tag{1.11}$$

The main purpose of this paper is to obtain several general theorems for the explicit evaluations of analogous of Ramanujan's product of theta-function of $d_{m,n}$ and also some new explicit evaluations from it.

2. Preliminary Results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. [1, Ch. 17, Entry 10(i) and Entry 11(iii), pp. 122 and 124] *We have*

$$\varphi(e^{-\alpha}) = \sqrt{z_1}, \tag{2.1}$$

$$\varphi(e^{-m\alpha}) = \sqrt{z_m}, \tag{2.2}$$

$$2^{1/3}\, e^{-\alpha/12}\, f(-e^{-2\alpha}) = \sqrt{z_1}\, \{\alpha(1-\alpha)\}^{1/12}, \tag{2.3}$$

$$2^{1/3}\, e^{-\alpha/12}\, f(-e^{-2m\alpha}) = \sqrt{z_m}\, \{\beta(1-\beta)\}^{1/12}. \tag{2.4}$$

Lemma 2.2. [1, Ch. 16, Entry 27 (i) and (iii), pp. 43] *We have*

$$\sqrt{\alpha}\,\varphi(e^{-\alpha^2}) = \sqrt{\beta}\,\varphi(e^{-\beta^2}), \quad if\ \alpha\beta = \pi, \tag{2.5}$$

$$e^{-\alpha/12}\sqrt[4]{\alpha}\,f(-e^{-2\alpha}) = e^{-\beta/12}\sqrt[4]{\beta}\,f(-e^{-2\beta}), \quad if\ \alpha\beta = \pi^2. \tag{2.6}$$

Lemma 2.3. [5, Theorem 2.1] *We have*

$$\frac{f^6(-q^2)}{qf^6(-q^6)} = \frac{\varphi^2(-q)}{\varphi^2(-q^3)}\left\{\frac{\varphi^4(-q) - 9\varphi^4(-q^3)}{\varphi^4(-q) - \varphi^4(-q^3)}\right\}. \tag{2.7}$$

Lemma 2.4. [15, p.56][13] *We have*

$$\frac{f^3(-q^2)}{qf^3(-q^{10})} = \frac{\varphi(-q)}{\varphi(-q^5)}\left\{\frac{5\varphi^2(-q^5) - \varphi^2(-q)}{\varphi^2(-q) - \varphi^2(-q^5)}\right\}. \tag{2.8}$$

Lemma 2.5. [5, Theorem 2.2] *We have*

$$\frac{f^3(-q^2)}{q^2f^3(-q^{18})} = \frac{\varphi(-q)}{\varphi(-q^9)}\left\{\frac{\varphi(-q) - 3\varphi(-q^9)}{\varphi(-q) - \varphi(-q^9)}\right\}^2. \tag{2.9}$$

Lemma 2.6. [1, Ch. 19, Entry 5(xii), pp. 231]
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$*, then*

$$Q + \frac{1}{Q} = 2\sqrt{2}\left(\frac{1}{P} - P\right). \tag{2.10}$$

Lemma 2.7. [1, Ch. 19, Entry 13(xiv), pp. 282]
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$ *and* $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/8}$*, then*

$$Q + \frac{1}{Q} = 2\left(\frac{1}{P} - P\right). \tag{2.11}$$

Lemma 2.8. [1, Ch. 19, Entry 19(ix), pp. 315]
If $P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/6}$*, then*

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right). \tag{2.12}$$

3. Some Properties of $d_{m,n}$

In this section, we derive some properties of $d_{m,n}$.

Theorem 3.1. *We have*

$$d_{m,n} = d_{n,m}. \tag{3.1}$$

Proof. Substituting the equations (2.5) and (2.6) in (1.8), we obtain (3.1). □

Theorem 3.2. *We have*

$$d_{m,n} d_{m,1/n} = 1. \tag{3.2}$$

Proof. Using the equations (2.5) and (2.6) in (1.8), we obtain (3.2). □

Corollary 3.3. *We have*

$$d_{m,1} = 1. \tag{3.3}$$

Proof. Putting $n = 1$ in the equation (3.2), we get (3.3). □

Remark 1. By using the definition of $\varphi(q)$, $f(-q)$ and $d_{m,n}$, it can be seen that $d_{m,n}$ has positive real value and that the values of $d_{m,n}$ increases as n increase when $m > 1$. Thus by the above corollary, $d_{m,n} > 1$ for all $n > 1$ if $m > 1$.

Theorem 3.4. *We have*

$$\frac{d_{km,n}}{d_{nm,k}} = d_{m,\frac{n}{k}}. \tag{3.4}$$

Proof. Employing the definition of $d_{m,n}$, we obtain

$$\frac{d_{km,n}}{d_{nm,k}} = e^{\frac{\pi\left(\sqrt{\frac{k}{mn}}-\sqrt{\frac{n}{mk}}\right)}{12}} \frac{f\left(-e^{-2\pi\sqrt{\frac{n}{mk}}}\right)\varphi\left(e^{-\pi\sqrt{\frac{k}{mn}}}\right)}{f\left(-e^{-2\pi\sqrt{\frac{k}{mn}}}\right)\varphi\left(e^{-\pi\sqrt{\frac{n}{mk}}}\right)}. \tag{3.5}$$

Using the Lemma **2.2** in the above equation (3.5) and simplifying using the Theorems **3.1** and **3.2**, we obtain (3.4). □

Corollary 3.5. *We have*

$$d_{m^2,n} = d_{nm,n} d_{m,\frac{n}{m}}. \tag{3.6}$$

Proof. Putting $m = n$ in the above Theorem **3.3** and simplifying using the equation (3.2), we get

$$d_{m^2,k} = d_{mk,m} d_{m,\frac{k}{m}}. \tag{3.7}$$

Replace k by n, we obtain (3.6). □

Theorem 3.6. *If* $mn = rs$, *then*

$$d_{m,n}d_{kr,ks} = d_{r,s}d_{km,kn}. \tag{3.8}$$

Proof. Using the definition of $d_{m,n}$ and letting $mn = rs$ for positive real numbers m, n, r, s and k, we find that

$$\frac{d_{km,kn}}{d_{m,n}} = \frac{d_{kr,ks}}{d_{r,s}}. \tag{3.9}$$

On rearranging the above equation (3.9) we obtain the required result.

□

Corollary 3.7. *If* $mn = rs$, *then*

$$d_{np,np} = d_{np^2,n}d_{p,p}. \tag{3.10}$$

Proof. Letting $m = p^2$, $n = 1$, $r = s = p$ and $k = n$ in above Theorem **3.4**, we deduced the equation (3.10). □

Theorem 3.8. *For all positive real numbers* m, n, r *and* s, *we have*

$$d_{m/n,r/s} = \frac{d_{ms,nr}}{d_{mr,ns}}. \tag{3.11}$$

Proof. Employing the equation (3.2) in equation (3.4), we find that, for all positive real numbers m, n and k

$$d_{m/n,k} = d_{m,nk}d^{-1}_{n,mk}. \tag{3.12}$$

Letting $k = r/s$ and again using the equation (3.4) and (3.1) in (3.12), we get (3.11). □

Theorem 3.9. *We have*

$$d_{m/n,m/n} = d_{n,n}d_{m,m/n^2}. \tag{3.13}$$

Proof. Using the Theorems **3.2** and **3.5**, we get (3.13). □

Theorem 3.10. *We have*

$$d_{m,m}d_{m,n^2/m} = d_{n,n}d_{n,m^2/n}. \tag{3.14}$$

Proof. Putting $k = m/n$ in the equation (3.12) and employing Theorems **3.2** and **3.6**, we obtain (3.14). □

Theorem 3.11. *We have*

$$d_{m,m}d_{n,m^2n} = d_{n,n}d_{m,mn^2}. \tag{3.15}$$

Proof. Employing the Theorems **3.1, 3.2, 3.6** and **3.7**, we obtain (3.15). □

4. Some General Theorems on $d_{m,n}$ and their explicit evaluations

In this section, we establish some general theorems on $d_{m,n}$ and their explicit evaluations.

Theorem 4.1. *If* $P := \{G_{n/3}G_{3n}\}^{-3}$ *and* $Q := d_{3,n}^{-3}$, *then*

$$Q + \frac{1}{Q} = 2\sqrt{2}\left\{\frac{1}{P} - P\right\}. \tag{4.1}$$

Proof. Using the Lemma **2.1** with the definition of $d_{m,n}$, we obtain

$$d_{m,n} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/12}. \tag{4.2}$$

Using the above equation (4.2) and definition of class invariant (1.10), (1.11) in the Lemma **2.6** with $m = 3$, we obtain (4.1). □

Corollary 4.2. *We have*

$$d_{3,9} = \left\{1 + 2^{2/3} - 2^{4/3}\right\}^{-1/3}. \tag{4.3}$$

Proof. Putting $n = 9$ in the above Theorem **4.1**, we obtain

$$d_{3,9}^3 + d_{3,9}^{-3} = 2\sqrt{2}\left\{G_3^3G_{27}^3 - G_3^{-3}G_{27}^{-3}\right\}. \tag{4.4}$$

Solving the above equation (4.4) with from the table of Chapter 34 of Ramanujan notebooks [3, p.189,190] $G_3 = 2^{1/12}$ and $G_{27} = 2^{1/12}\left(\sqrt[3]{2} - 1\right)^{-1/3}$, we obtain (4.3). □

Theorem 4.3. *If* $P := \{G_{n/5}G_{5n}\}^{-2}$ *and* $Q := d_{5,n}^{-3/2}$, *then*

$$Q + \frac{1}{Q} = 2\left\{\frac{1}{P} - P\right\}. \tag{4.5}$$

Proof. Using the equation (4.2) and definition of class invariant (1.10), (1.11) in the Lemma **2.7** with $m = 5$, we obtain (4.5). □

Theorem 4.4. *If* $P := \{G_{n/7}G_{7n}\}^{-3}$ *and* $Q := d_{7,n}^{-2}$, *then*

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left\{P + \frac{1}{P}\right\}. \tag{4.6}$$

Proof. Using the equation (4.2) and definition of class invariant (1.10), (1.11) in the Lemma **2.8** with $m = 7$, we obtain (4.6). □

Theorem 4.5. *Let*

$$d_{3,n} = \frac{f(-q^2)\varphi(q^3)}{q^{1/6}f(-q^6)\varphi(q)}; \quad q := e^{-\pi\sqrt{\frac{n}{3}}}. \tag{4.7}$$

If

$$P := \frac{\varphi(q)}{\varphi(q^3)} \quad \textit{and} \quad Q := \frac{f(-q^2)}{q^{1/6}f(-q^6)}, \quad \textit{then} \tag{4.8}$$

$$d_{3,n}^6 = \frac{P^4 - 9}{P^4(1 - P^4)}, \quad \textit{if} \quad P^4 \neq 1. \tag{4.9}$$

Proof. Employing the definition of $d_{m,n}$ with $m = 3$, we get

$$d_{3,n} = \frac{f(-q^2)\varphi(q^3)}{q^{1/6}f(-q^6)\varphi(q)}. \tag{4.10}$$

Raising the power 6 on the (4.10) with the Lemma **2.3**, we deduced that

$$d_{3,n}^6 = \frac{f^6(-q^2)\varphi^6(q^3)}{qf^6(-q^6)\varphi^6(q)}, \tag{4.11}$$

$$d_{3,n}^6 = \frac{P^2\left\{\frac{P^4 - 9}{1 - P^4}\right\}}{P^6}. \tag{4.12}$$

On simplifying the above equation (4.12), we obtain (4.9). □

Corollary 4.6. *We have*

$$d_{3,3} = \left\{2 + \sqrt{3}\right\}^{1/3}. \tag{4.13}$$

Proof. Putting $n = 3$ in the equation (4.8) and from Ramanujan's Notebooks [3, p. 327], we have

$$P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3} - 9}. \tag{4.14}$$

Substituting (4.14) in (4.9), we obtain the required result. □

Theorem 4.7. *Let*

$$d_{5,n} = \frac{f(-q^2)\varphi(q^5)}{q^{1/3}f(-q^{10})\varphi(q)}; \quad q := e^{-\pi\sqrt{\frac{n}{5}}}. \tag{4.15}$$

If

$$P := \frac{\varphi(q)}{\varphi(q^5)} \quad and \quad Q := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}, \quad then \tag{4.16}$$

$$d_{5,n}^3 = \frac{5-P^2}{P^2(P^2-1)}, \quad if \quad P^2 \neq 1. \tag{4.17}$$

Proof. Employing the definition of $d_{m,n}$ with $m = 5$, we get

$$d_{5,n} = \frac{f(-q^2)\varphi(q^5)}{q^{1/3}f(-q^{10})\varphi(q)}. \tag{4.18}$$

Raising the power 3 on the (4.18) with the Lemma **2.4**, we deduced that

$$d_{5,n}^3 = \frac{f^3(-q^2)\varphi^3(q^5)}{qf^3(-q^{10})\varphi^3(q)}, \tag{4.19}$$

$$d_{5,n}^3 = \frac{P\left\{\dfrac{5-P^2}{P^2-1}\right\}}{P^3}. \tag{4.20}$$

On simplifying the above equation (4.20), we obtain (4.15). □

Corollary 4.8. *We have*

$$d_{5,5} = \left\{2+\sqrt{5}\right\}^{2/3}. \tag{4.21}$$

Proof. Putting $n = 5$ in the equation (4.16) and from Ramanujan's Notebooks [3, p. 327], we have

$$P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-5\pi})} = \sqrt{5\sqrt{5}-10}. \tag{4.22}$$

Substituting (4.22) in (4.17), we obtain the required result. □

Theorem 4.9. *Let*

$$d_{9,n} = \frac{f(-q^2)\varphi(q^9)}{q^{2/3}f(-q^{18})\varphi(q)}; \quad q = e^{-\pi\sqrt{\frac{n}{9}}}. \tag{4.23}$$

If

$$P := \frac{\varphi(q)}{\varphi(q^9)} \quad and \quad Q := \frac{f(-q^2)}{q^{2/3}f(-q^{18})}, \quad then \tag{4.24}$$

$$d_{9,n}^3 = \left\{\frac{P-3}{P(P-1)}\right\}^2, \quad if \quad P \neq 1. \tag{4.25}$$

Proof. Employing the definition of $d_{m,n}$ with $m = 9$, we get

$$d_{9,n} = \frac{f(-q^2)\varphi(q^3)}{q^{1/6} f(-q^6)\varphi(q)}. \tag{4.26}$$

Raising the power 3 on the (4.26) with the Lemma **2.5**, we deduced that

$$d_{9,n}^3 = \frac{f^3(-q^2)\varphi^3(q^9)}{q^2 f^3(-q^{10})\varphi^3(q)}, \tag{4.27}$$

$$d_{9,n}^3 = \frac{P\left\{\dfrac{P-3}{P-1}\right\}^2}{P^3}. \tag{4.28}$$

On simplifying the above equation (4.28), we obtain (4.23). □

Corollary 4.10. *We have*

$$d_{9,9} = \left\{\frac{\left(13\sqrt{3}+22\right)\left[23a^2 + \left(39+\sqrt{3}\right)a + \left(42+17\sqrt{3}\right)\right]}{23}\right\}^{1/3}, \tag{4.29}$$

where $a = (2\sqrt{3}+2)^{1/3}$.

Proof. Putting $n = 9$ in the equation (4.24) and from Ramanujan's Notebooks [3, p. 327], we have

$$P := \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} = \frac{3}{1+\sqrt[3]{2(\sqrt{3}+1)}}. \tag{4.30}$$

Substituting (4.30) in (4.25), we obtain the required result. □

Theorem 4.11. *We have*

$$d_{m,n} = \left\{\frac{G_{mn}}{G_{n/m}}\right\}^2. \tag{4.31}$$

Proof. Employing the Lemma **2.1** in the definition of $d_{m,n}$, we obtain

$$d_{m,n} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/12}. \tag{4.32}$$

Using the equation (1.10) and (1.11), we get

$$\frac{G_{nm}}{G_{n/m}} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \tag{4.33}$$

By observing the equations (4.32) and (4.33), we obtain (4.31). □

Corollary 4.12. *We have*

$$d_{n,n} = G_{n^2}^2. \tag{4.34}$$

Proof. Setting $m = n$ in the above Theorem **4.7** with the value $G_1 = 1$, we obtain required result. □

Corollary 4.13. *We have*

$$(i)\ d_{2,2} = \frac{\sqrt{1+\sqrt{2}}}{2^{3/8}}, \tag{4.35}$$

$$(ii)\ d_{3,3} = \left\{2+\sqrt{3}\right\}^{1/3}, \tag{4.36}$$

$$(iii)\ d_{5,5} = \left\{2+\sqrt{5}\right\}^{2/3}, \tag{4.37}$$

$$(iv)\ d_{9,9} = \sqrt[3]{\frac{\left(13\sqrt{3}+22\right)\left[23a^2+\left(39+\sqrt{3}\right)a+\left(42+17\sqrt{3}\right)\right]}{23}} \tag{4.38}$$

(4.39) *where* $a := 2\sqrt{3}+2.$

Proof. For (i), we use the values of G_4 from [11, p.107, Theorem 3.5]. For $(ii)-(iv)$, we use corresponding values of G_n from [3, p.189-193]. □

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References

[1] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[2] B. C. Berndt, Ramanujan's Notebooks, Part IV, Springer-Verlag, New York, 1994.

[3] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York, 1997.

[4] B. C. Berndt, H. H. Chan and L.-C. Zhang, Ramanujan's remarkable product of the theta-function, Proc. Edinburgh Math. Soc., 40 (1997), 583–612.

[5] M. S. Mahadeva Naika, Some theorems on Ramanujan's cubic continued fraction and related identities. Tamsui Oxf. J. Math. Sci. 24(3) (2008), 243–256.

[6] M. S. Mahadeva Naika and B. N. Dharmendra, On some new general theorems for the explicit evaluations of Ramanujan's remarkable product of theta-function Ramanujan J. 15(3) (2008), 349–366.

[7] M. S. Mahadeva Naika, B. N. Dharmendra and K. Shivashankara, On some new explicit evaluations of Ramanujan's remarkable product of theta-function, South East Asian J. Math. Math. Sci. 5(1) (2006), 107–119.

[8] M. S. Mahadeva Naika and M. C. Maheshkumar, Explicit evaluations of Ramanujan's remarkable product of theta-function, Adv. Stud. Contemp. Math., 13(2) (2006), 235–254.

[9] M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, General formulas for explicit evaluations of Ramanujan's cubic continued fraction, Kyungpook Math. J., 49(3) (2009), 435–450.

[10] M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, On some remarkable product of theta-function, Aust. J. Math. Anal. Appl., 5(1) (2008), 1–15.

[11] M. S. Mahadeva Naika and K. Sushan Bairy, Some Modular Equations in The Form of Schläfli, Italian Journal of Pure and Applied Mathematics, 30 (2013). 233–252.

[12] Nipen Saikia, Some Properites, Explicit Evaluation, and Applications of Ramanujan's Remarkable Product of Theta-Functions, Acta Math Vietnam, Journal of Mathematics, DOI 10.1007/s40306-014-0106-8, 2015.

[13] S. -Y. Kang, Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook. Ramanujan J., 3(1) (1999), 91–11.

[14] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[15] S. Ramanujan, The lost notebook and other unpublished papers, Narosa, New Delhi, 1988.

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