Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 4. pp. 507 - 516

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.4.507

DELINEATION OF Ω BITOPOLOGICAL SPACES

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ABSTRACT. In the present paper, we study and analyse a function on a bitopological space which enabled us to derive a new bitopology on a given bitopological space. We explore various characterizations of the derived bitopological space and investigate the propinquity between both the bitopologies. Also we define new separation axioms on the bitopological spaces and study how a separation axiom on one bitopological space influences the other.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 54A05, 54A10, 54D10, 54D15, 54D80.

KEYWORDS AND PHRASES. R_1 closure, R_2 closure, pairwise Ω-closure, pairwise closely regular spaces.

1. INTRODUCTION

The notion of bitopological space was introduced by Kelley [5] as a consequence of study of non symmetric behaviour of quasi metric spaces which gave rise to two topologies on a set. The discovery of bitopological spaces captivated interests of many mathematicians and scientists and since then a lot of concepts have been delivered on this phenomenon. The concept of bitopological space still has a lot to explore which keeps the mathematicians engrossed in exploring new dimensions till today. Due to the presence of two topologies in a bitopological space (X, $\Im_1$, $\Im_2$), it is always possible to consider the $\Im_j$ closure of an open set with respect to the topology $\Im_i$ where i, j $\in \{1, 2\}$ i $\neq$ j. This idea of associating the closure with open sets provoked us to introduce a new type of closure operator on a bitopological space which gave rise to two new topologies and hence we derived a new bitopology for any given bitopological space. This derived bitopological space shows quite unique properties. For instance, this derived bitopology is always pairwise T_1 irrespective of the nature of the parent topology. Also its elements are completely independent to the parent topology in general. Moreover, one separation axiom in one bitopology leads to another separation axiom in the derived bitopological space. We name this derived bitopological space as Ω bitopological space and study its behaviour with respect to the existing separation axioms. Also, we introduce some new separation axioms which behave in a peculiar manner with the bitopological space. For the sake of self completeness, in section 2, we enlist the necessary definitions and concepts. In section 3, we recall the new closure property introduced earlier and show that it derives a new bitopological space. Finally in section 4 we provide some new characterizations to Ω bitopological spaces in terms of various separation axioms.

2. Preliminaries

A non empty set X equipped with two arbitrary topologies $\Im_1$ and $\Im_2$ is called a bitopological space and is written as (X, $\Im_1$, $\Im_2$). The closure and interior of a subset of X have their general sense. To make the article self contained, we recall the following well known definitions given by various topologists:

Definition 2.1 [7]: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_0 if for every pair of distinct elements of X there exists an open set in any of the topologies containing only one of the points.

Definition 2.2 [4]: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_1 if for every pair of distinct elements of X say x and y there exists an open set U in $\Im_i$ containing x but not y and an open set say V in $\Im_j$ containing y but not x, $\forall$ i, j $\in$ {1,2} i $\neq$ j.
It is obvious that if a bitopological space is T_1 then every singleton is closed both in $\Im_1$ and $\Im_2$.

Definition 2.3 [5]: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_2 if for every pair of distinct elements of X say x and y there exists an open set U in $\Im_i$ containing x and an open set say V in $\Im_i$ containing y such that U $\cap$V = $\emptyset$, $\forall$ i, j $\in$ {1,2}, i $\neq$ j.

Definition 2.4 [5]: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise regular if for every element of X say x and every open set U in $\Im_i$ containing x there exists a $\Im_i$ open neighbourhood of x say V_x whose $\Im_j$ closure is contained in U, $\forall$ i, j $\in$ {1,2} i $\neq$ j.

Definition 2.5 [8]: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_3 if it is both pairwise regular and pairwise T_2.

Definition 2.6 [10]: A cover $\eth$ ={ U_α | $\alpha \in \Lambda$} is said to be pairwise open cover of X if $U_\alpha \in \Im_1 \cup \Im_2$ and $\eth \cap \Im_i$ contains a nonempty set $\forall$ i $\in$ {1,2}.

Definition 2.7 [1]: In a bitopological space (X, $\Im_1$, $\Im_2$), if there is a pairwise open cover $\eth$ of X such that each element in X has a $\Im_{\mathrm{i}}$ neighbourhood with its $\Im_j$ closure intersecting only countably many elements of ð ∀ i, j ∈{1,2}, i ≠ j then ð is called pairwise locally countable open cover of X.

Definition 2.8 [6]: A topological space (X, $\Im$), is said to be antilocally countable if each non empty open set in X is uncountable.

Definition 2.9 [9]: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise antilocally countable if it is antilocally countable with respect to both the topologies.

3. Ω Bitopological Space : Definitions and Properties

Before we explore the properties of Ω Bitopological Space we recall some fundamental definitions, examples and some basic results [9] related to it in order to have better understanding of the concept.

Definition 3.1: Let (X, $\Im_1$, $\Im_2$) be a bitopological space. Define a function $\Omega_{(i,j)}$ **:** P(X) → P(X) as $\Omega_{(i,j)}$(A) =A ∪ {y ∈ X | ∀ U ∈ $\Im_j$ | y ∈ U, $\Im_i$cl-{U}∩ A ≠ Countable}. Then ∀ i, j ∈ {1,2} i ≠j, the $\Omega_{(i,\ j)}$ operator satisfies all the properties of Kuratowsi's closure operator and therefore can be applied to define a topology on a bitopological space (X, $\Im_1$, $\Im_2$) ∀ i, j ∈ {1,2} i ≠j. We denote this topology as $\Re_i$ and the topological space equipped with this topology as (X, $\Re_i$). The open and closed sets in $\Re_i$ topology are defined as:

Definition 3.2: In a bitopological space (X, $\Im_1$, $\Im_2$), A ⊂ X is called $\Re_i$ closed or $\Omega_{(i,\ j)}$ closed if $\Omega_{(i,\ j)}$ (A) = A i.e. ∀ x ∈ X-A there exists U_x ∈ $\Im_j$ containing x such that ($\Im_i$cl-{U_x})∩ A = Countable ∀ i, j∈ {1, 2} i ≠j.

Definition 3.3: If U ∈ P(X) then U ∈ $\Re_i$ or U is called $\Omega_{(i,\ j)}$ open if for each x ∈ U there exists a $\Im_j$ open neighbourhood U_x of x such that $\Im_i$cl(U_x) –U is countable.

In fact with the help of this function, we can define two topologies on a bitopological space and consequently we can derive a bitopology on a bitopological space. We denote this derived bitopological space by (X, $\Re_1$, $\Re_2$) where $\Re_1$ is the topology derived by the operator $\Omega_{(1,2)}$ and $\Re_2$ is the topology derived by the operator $\Omega_{(2,1)}$. We call this derived bitopological space as Ω bitopological space. Also we define the concept of pairwise Ω closed sets as a subset of X which is both $\Re_1$ closed and $\Re_2$ closed.

Example 3.1: Let we consider two topologies on the set of real numbers $\mathcal{R}$ as follows :

($\mathcal{R}$, $\Im_1$) = {∅, $\mathcal{R}$, {0}, (-∞, 0), (-∞, 0] }
($\mathcal{R}$, $\Im_2$) = {∅, $\mathcal{R}$, {0}, (0, ∞), [0, ∞) }

It can be checked easily that $(\mathcal{R}, \Re_1)$ consists of all the sets of the type $\emptyset$, $(-\infty, 0]$-C and $\mathcal{R}$-C where C is any countable subset of $\mathcal{R}$. Similarly $(\mathcal{R}, \Re_2)$ consists of all the sets of the type $\emptyset$, $[0, \infty)$-C and $\mathcal{R}$-C where C is any countable subset of $\mathcal{R}$.

Example 3.2: Let $\mathcal{R}$ be the set of real numbers equipped with two topologies $\Im_1$ and $\Im_2$ where $\Im_1$ is the usual topology and $\Im_2$ is the right order topology then, $\Re_1$ is the topology finer than $\Im_2$ as it consists of all the members of $\Im_2$ topology along with all the sets of the type A-C and $\mathcal{R}$-C where A is any member of $\Im_2$ and C is any countable subset of $\mathcal{R}$. Members of $(\mathcal{R}, \Re_2)$ will be of the type $\emptyset$, $(-\infty, a)$-C and $\mathcal{R}$-C where a is any element of $\mathcal{R}$ and C is any countable subset of $\mathcal{R}$.

Theorem 3.1: If a bitopological space $(X, \Im_1, \Im_2)$ is pairwise antilocally countable then the bitopological space $(X, \Re_1, \Re_2)$ is also pairwise antilocally countable.

Theorem 3.2: Every countable set in a bitopological space $(X, \Im_1, \Im_2)$ is pairwise Ω closed.

Theorem 3.3: If a bitopological space $(X, \Im_1, \Im_2)$ is pairwise antilocally countable and A is $\Im_i$ open then $A \subset \Im_i cl(A) \subset \Re_j cl(A)$ $\forall$ i, j $\in \{1, 2\}$, $i \neq j$.

4. Separation Axioms in Pairwise Ω Bitopological Spaces

Theorem 4.1: If a bitopological space $(X, \Im_1, \Im_2)$ is pairwise regular then $A \subset \Re_i cl\ (A) \subset \Im_j cl(A)$ $\forall$ i, j$\in\{1,2\}$, $i \neq j$.
Proof : Let $x \in \Re_i cl\ (A)$. Let $U \in \Im_j \mid x \in U$. Since $(X, \Im_1, \Im_2)$ is pairwise regular there exists a $\Im_j$ neighbourhood V of x such that $\Im_i cl(V) \subset U$. Also $\Im_i cl(V) \cap A$ is uncountable $\Rightarrow U \cap A$ is uncountable $\Rightarrow U \cap A$ is non empty. Hence $x \in \Im_j cl(A)$.

Corollary 4.1: In a pairwise regular bitopological space $(X, \Im_1, \Im_2)$, every $\Im_j$ closed set is $\Re_i$ closed $\forall$ i, j$\in\{1,2\}$, $i \neq j$.
Proof : By the above theorem, $A \subset \Re_i cl\ (A) \subset \Im_j cl(A)$ $\forall$ i, j$\in\{1,2\}$, $i \neq j$. Therefore A is $\Im_j$ closed $\Rightarrow A = \Im_j cl(A) \Rightarrow A = \Re_i cl\ (A)$. Hence, A is $\Re_i$ closed.

Corollary 4.2: In a pairwise regular bitopological space $(X, \Im_1, \Im_2)$ every $\Im_i$ open set is $\Re_j$ open $\forall$ i, j $\in$ { 1,2 }, $i \neq j$.

Corollary 4.3: In a pairwise regular bitopological space (X, $\Im_1$, $\Im_2$) each set open in both the topologies is pairwise Ω open.

Theorem 4.2: In a bitopological space (X, $\Im_1$, $\Im_2$) the bitopological space (X, $\Re_1$, $\Re_2$) is always pairwise T_1.
Proof : Since a countable set is closed with respect to both $\Re_1$ and $\Re_2$, in particular for every pair of elements say x, y $\in$ X such that x $\neq$ y, each of {x} and {y} is closed both in $\Re_1$ and $\Re_2$ and therefore X-{x} is the open set in both $\Re_1$ and $\Re_2$ containing y but not x and X-{y} is the open set in both $\Re_1$ and $\Re_2$ containing x but not y.

The result is of great importance in the sense that it shows that any bitopological space can be embedded into another bitopological space that is pairwise T_1 as every singleton is closed in it. It also implies that if U is $\Re_i$ open then, U-{x} is also $\Re_i$ open. The result can be generalized to the fact that if U is $\Re_i$ open then, U- C is also $\Re_i$ open for any countable subset C of X. Also if bitopological space (X, $\Im_1$, $\Im_2$) is pairwise regular then the derived bitopological space (X, $\Re_1$, $\Re_2$) contains all the open sets of the parent space. Whereas in general there is no relation between the open sets of both the bitopological spaces.

Definition 4.1: A bitopological space (X,$\Im_1$, $\Im_2$) is pairwise closely regular if for every element of X say x and every subset U of X containing x such that U is either $\Im_i$ open or $\Im_j$ closed there exists a $\Im_i$ open neighborhood of x say V_x whose $\Im_j$ closure is contained in U, $\forall$ i, j $\in$ {1,2} i $\neq$ j.

Theorem 4.3: In a pairwise closely regular space (X, $\Im_1$, $\Im_2$), A is pairwise Ω closed iff $\forall$ x $\in$ X-A there exists a closed set $P_i \in \Im_i$ such that x $\in P_i$ and $P_i \cap$ A is countable $\forall$ i $\in$ {1,2}.
Proof: If A is pairwise Ω closed then $\forall$ x$\in$ X-A there exists an open set $U_j \in \Im_j$ such that x $\in U_j$ and $\Im_i$cl(U_j) $\cap$ A is countable. Take $\Im_i$cl(U_j)=P_i and we are done. Conversely, $\forall$ x $\in$ X-A let there exist a closed set $P_i \in \Im_i$ such that x$\in P_i$ and $P_i \cap$ A is countable $\forall$ i $\in${1,2}. Since X is pairwise closely regular, each P_i contains a U_j nbd of x in $\Im_j$ whose $\Im_i$ closure is contained in P_i $\forall$ i, j$\in${1,2}, i $\neq$ j. $\Rightarrow$ $\Im_i$cl(U_j) $\cap$ A is countable $\Rightarrow$ A is pairwise Ω closed.

Theorem 4.4: In a pairwise regular space (X, $\Im_1$, $\Im_2$), A is pairwise Ω closed iff $\forall$ x$\in$ X-A there exists an open set $V_i \in \Im_i$ such that x $\in V_i$ and $V_i \cap$ A is countable $\forall$ i $\in${1,2} .
Proof: If A is pairwise Ω closed then $\forall$ x$\in$ X-A there exists an open set U_i $\in\Im_i$ such that x$\in U_i$ and $\Im_j$ cl(U_i) $\cap$ A is countable $\Rightarrow$ $U_i \cap$ A is countable. Conversely, let $\forall$ x$\in$ X-A there exist an open set $V_i \in \Im_i$ such that x$\in V_i$ and $V_i \cap$ A is countable $\forall$ i $\in${1,2}. Now x $\in V_i$ and X is pairwise regular

therefore there exists a $\Im_i$ neighbourhood U_i of x such that $\Im_j cl(U_i) \subset V_i$. Then $\Im_j cl(U_i) \cap A$ is countable.

Theorem 4.5: In a pairwise closely regular space (X, $\Im_1$, $\Im_2$), countable union of $\Re_i$ closed sets in (X, $\Re_1$, $\Re_2$) is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.
Proof: Let $\{A_{in} \mid n \in N\}$ be a countable collection of $\Re_i$ closed sets. Obviously, $\bigcup_{n\in N} A_{in} \subset \Re_i cl \bigcup_{n\in N} A_{in}$. If $x \notin \bigcup_{n\in N} A_{in} \Rightarrow x \notin A_{in}$ $\forall$ i $\in\{1,2\}$ and $\forall$ n $\in$ N $\Rightarrow$ There exists $U_{in} \in \Im_i$ such that $x \in U_{in}$ and $\Im_j$ cl(U_{in}) $\cap A_{in}$ is countable j $\neq$ i. Let $P_j = \cap_{n\in N} \Im_j$ cl(U_{in}) $\Rightarrow P_j$ is $\Im_j$ closed and $P_j \cap \bigcup_{n\in N} A_{in}$ is countable. Now X is pairwise closely regular therefore P_j contains $\Im_j$ closure of some $\Im_i$ neighbourhood U_i of x$\Rightarrow \Im_j$ cl(U_i) $\cap \bigcup_{n\in N} A_{in}$ is countable$\Rightarrow$ countable union of $\Re_i$ closed sets in (X, $\Re_1$, $\Re_2$) is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.

Corollary 4.4: In a pairwise closely regular space (X, $\Im_1$, $\Im_2$), union of $\Re_i$ closure of countably many subsets of X in (X, $\Re_1$, $\Re_2$) is equal to $\Re_i$ closure of their union $\forall$ i $\in\{1,2\}$.

Corollary 4.5: In a pairwise closely regular space (X, $\Im_1$, $\Im_2$), countable intersection of $\Re_i$ open sets of X in (X, $\Re_1$, $\Re_2$) is $\Re_i$ open $\forall$ i $\in\{1,2\}$.

Theorem 4.6: In a pairwise closely regular space (X, $\Im_1$, $\Im_2$), if every element in X has a $\Im_i$ open neighbourhood intersecting only countably many$\Re_i$ closed sets then arbitrary union of $\Re_i$ closed sets in (X, $\Re_1$, $\Re_2$) is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.
Proof: Let $\{A_{i\alpha} \mid \alpha \in \Delta\}$ be a collection of $\Re_i$ closed sets. Obviously, $\bigcup_{\alpha\in\Delta} A_{i\alpha} \subset \Re_i cl \bigcup_{\alpha\in\Delta} A_{i\alpha}$. Let $x \notin \bigcup_{\alpha\in\Delta} A_{i\alpha} \Rightarrow x \notin A_{i\alpha}$ $\forall$ i $\in\{1,2\}$ and $\forall \alpha \mid \alpha \in \Delta\} \Rightarrow$ There exists $U_{i\alpha} \in \Im_i$ such that $x \in U_{i\alpha}$ and $\Im_j$ cl($U_{i\alpha}$) $\cap A_{i\alpha}$ is countable $\forall$ i $\in\{1,2\}$, j $\neq$ i. Let U_i be a $\Im_i$ neighbourhood of x intersecting with $\{A_{in} \mid n \in N\}$ where N is countable then due to pairwise regularity of X, without loss of generality we may consider that $\Im_j cl(U_i)$ intersects with $\{A_{in} \mid n \in N\}$. Let $P_j = \cap_{\alpha\in N} \Im_j$ cl($U_{i\alpha}$) $\cap \Im_j$ cl(U_i) $\Rightarrow P_j$ is $\Im_j$ closed and $P_j \cap \bigcup_{\alpha\in\Delta} A_{i\alpha}$ is countable. Now X is pairwise closely regular therefore P_j contains $\Im_j$ closure of $\Im_i$ neighbourhood U_i of x $\Rightarrow \Im_j$ cl(U_i) $\cap \bigcup_{\alpha\in\Delta} A_{i\alpha}$ is countable $\Rightarrow$ countable union of $\Re_i$closed sets in (X, $\Re_1, \Re_2$) is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.

Corollary 4.6: In a pairwise closely regular space (X, $\Im_1$, $\Im_2$), if every element in x has a $\Im_i$ neighbourhood intersecting only finitely many $\Re_i$ closed sets then arbitrary union of $\Re_i$closed sets in (X,$\Re_1, \Re_2$) is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.

Theorem 4.7: In a bitopological space (X, $\Im_1$, $\Im_2$), if every element in x has a $\Im_i$ neighbourhood with its $\Im_j$ closure intersecting only finitely many $\Re_i$ closed sets then arbitrary union of $\Re_i$closed sets in (X, $\Re_1, \Re_2$) is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.

Proof: Let $\{A_{i\alpha} \mid \alpha \in \Delta\}$ be a collection of $\Re_i$ closed sets. Obviously, $\bigcup_{\alpha\in\Delta} A_{i\alpha} \subset \Re_i cl \bigcup_{\alpha\in\Delta} A_{i\alpha}$. If $x \notin \cup \{\Re_i cl(A_{i\alpha}) \mid \alpha \in \Delta\}$ then $x \notin \bigcup_{\alpha\in\Delta} A_{i\alpha} \Rightarrow x \notin A_{i\alpha}$ $\forall$ i $\in\{1,2\}\Rightarrow$ There exists $U_{i\alpha} \in \Im_i$ such that $x\in U_{i\alpha}$ and $\Im_j\ cl(U_{i\alpha}) \cap A_{i\alpha}$ is countable $j \neq i$. Let U_i be a $\Im_i$ neighbourhood of x such that $\Im_j\ cl(U_i)$ intersects with $\{A_{in} \mid n \in N\}$ where N is finite. Let $V_i = \cap_{\alpha\in N}(U_{i\alpha}) \cap (U_i)$ Then, $\Im_j cl(V_i) = \Im_j cl(\cap_{\alpha\in N}(U_{i\alpha}) \cap (U_i)) \subset \Im_j cl(\cap_{\alpha\in N}(U_{i\alpha})) \cap \Im_j cl(U_i)$ $\Rightarrow \Im_j cl(V_i) \cap \bigcup_{\alpha\in\Delta} A_{i\alpha}$ is countable $\Rightarrow$ arbitrary union of $\Re_i$closed sets in $(X, \Re_1, \Re_2)$ is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.

Theorem 4.8: In a bitopological space $(X, \Im_1, \Im_2)$, arbitrary intersection of $\Re_i$ closed sets in $(X, \Re_1, \Re_2)$ is $\Re_i$ closed $\forall$ i $\in\{1, 2\}$.
Proof : Let $\{A_{i\alpha} \mid \alpha \in \Delta\}$ be a collection of $\Re_i$ closed sets. Obviously, $\bigcap_{\alpha\in\Delta} A_{i\alpha} \subset \Re_i cl \bigcap_{\alpha\in\Delta} A_{i\alpha}$. If $x\notin \{\bigcap_{\alpha\in\Delta} A_{i\alpha} \mid \alpha \in \Delta\}$ then $x\notin A_{i\alpha}$ for some $\alpha \in \Delta \Rightarrow$There exists $U_{i\alpha} \in \Im_i$ such that $x\in U_{i\alpha}$ and $\Im_j\ cl(U_{i\alpha}) \cap A_{i\alpha}$ is countable for $j \neq i \Rightarrow cl(U_{i\alpha}) \cap \bigcap_{\alpha\in\Delta} A_{i\alpha}$ is countable for $j \neq i$ $\Rightarrow x\notin\Re_i cl \bigcap_{\alpha\in\Delta} A_{i\alpha} \Rightarrow \bigcap_{\alpha\in\Delta} A_{i\alpha} \supset \Re_i cl \bigcap_{\alpha\in\Delta} A_{i\alpha}$.

Definition 4.2: A bitopological space $(X, \Im_1, \Im_2)$ is pairwise Ω regular if $\forall$ $x \in U_i \in \Re_i$ there exists an open set in $\Re_i$ containing x whose $\Re_j$ closure is contained in U_i $\forall$ i $\in\{1,2\}$.

Definition 4.3: A bitopological space $(X, \Im_1, \Im_2)$ is pairwise Ω^* regular if $\forall$ $x\in U_i \in\Im_i$ there exists an open set in $\Re_j$ containing x whose $\Re_i$ closure is contained in U_i $\forall$ i $\in\{1,2\}$.

Theorem 4.9: A pairwise regular bitopological space $(X, \Im_1, \Im_2)$ is pairwise Ω^* regular.
Proof: Let $x \in U\in \Im_i$ Since X is pairwise regular there exists a $\Im_i$ neighbourhood V of x such that $\Im_j\ cl(V) \subset U$. But in a pairwise regular space a $\Im_i$ open set is $\Re_j$ open and $\Re_i cl(V)\subset \Im_j\ cl(V)$ $\forall$ i, j$\in\{1,2\}$, $i \neq j \Rightarrow (X, \Im_1, \Im_2)$ is pairwise Ω^* regular.

Theorem 4.10: If $\eth$ is pairwise Ω locally countable open cover of X in a pairwise closely regular space $(X, \Im_1, \Im_2)$, then arbitrary union of elements of $\eth$ is closure preserving in the sense that arbitrary union of $\Re_i$ closure of $\Re_j$ open sets in $\eth$ is $\Re_i$ closed $\forall$ i $\in\{1,2\}$.
Proof: The theorem can be easily proved with the help of theorem 4.6.

Theorem 4.11: In a pairwise closely regular space $(X, \Im_1, \Im_2)$, every locally countable pairwise open cover of X has a pairwise Ω open refinement of locally finite disjoint equivalence classes.
Proof : Let $\eth$ be a pairwise locally countable open cover of X. Let U_{xj} be a $\Im_j$ neighbourhood of $x \in X$ intersecting only countably many elements of $\eth$. Then $\eth$ is a cover of X such that x is contained in countable many elements of $\mathcal{U} \cap \Im_j$. Let we denote this collection by $\eth_{xj} = \{U_{jn} \mid n\in N\}$.

Next, define $V_{xj} = U_{xj} \cap \{\cap U_{ji} \mid U_{ji} \in \eth_{xj}\}$ -$\{\bigcup_{(U_{j\alpha} \cap U_{xj}=\emptyset)} \Re_i cl(U_{j\alpha}) \mid U_{j\alpha} \in \eth \cap \Im_j\}$. We observe that,

(1) $x \in V_{xj}$ as $x \in U_{xj} \cap \{\cap U_{ji} \mid U_{ji} \in \eth_{xj}\}$ and $x \notin U_{j\alpha}$ for $U_{j\alpha} \cap U_{xj} = \emptyset$. Further, $x \notin \Re_i cl(U_{j\alpha})$ as U_{xj} is R_i neighbourhood of x not intersecting with $U_{j\alpha}$.
(2) V_{xj} is $\Re_i$ open since each $\Im_j$ neighbourhood is R_i open. Also $\eth$ being locally countable, is pairwise Ω locally countable in which arbitrary union of $\Re_i$closure of sets $\Re_j$ open sets in $\eth$ is $\Re_i$ closed.
(3) $V_{xj} \subset U_{xj}$
(4) V_{xj} forms partition of X : If $y \in V_{xj}$ then $y \in U_{xj} \cap \{\cap U_{ji} \mid U_{ji} \in \eth_{xj}\}$ and $y \notin \Re_i cl(U_{j\alpha})$ for $U_{j\alpha} \cap U_{xj} = \emptyset$ as U_{xj} is $\Re_i$ neighbourhood of y not intersecting with $U_{j\alpha}$. Hence y belongs to exactly the same $\Im_j$ open elements of $\eth$ as $x \Rightarrow V_{xj} = V_{yj}$.
(5) V_{xj} is $\Re_i$ open neighbourhood of x intersecting V_{xj} only and therefore the collection $\{ V_{xj} \mid x \in X$ and $\{i,j\} \in \{1, 2\}, i \neq j \}$ is pairwise Ω open in $(X, \Re_1, \Re_2)$.

Theorem 4.12: Let **B** be a collection of subsets of a pairwise closely regular space $(X, \Im_1, \Im_2)$. If there exists a countable cover of $\Re_i$ closed sets with each of its elements intersecting with countably many elements of **B** $\forall$ i, j$\in\{1, 2\}$, i $\neq$ j then, there exists a locally countable collection of $\Re_i$ open sets say **E** such that each element of **B** is contained in some element of **E**.
Proof: Let$(X, \Im_1, \Im_2)$ be a bitopological space. Let **B** be a collection of subsets of X. Let **C** be a countable cover of $\Re_i$ closed sets with each of its elements intersecting countably many elements of **B.** Let B be an element of **B**. Define $\mathbf{C}_B = \{C \in \mathbf{C} \mid C \cap B = \emptyset\}$. Define $E_B = X - \cup\{C \mid C \in \mathbf{C}_B\}$. Let $\mathbf{E} = \{ E_B \mid B \in \mathbf{B}\}$, then,

(1) X being pairwise closely regular space countable union of $\Re_i$ closed sets is closed. E_B is $\Re_i$ open being complement of $\Re_i$ closed sets.
(2) Since no element of B is contained in $C \in \mathbf{C}$, $B \subset E_B$.
(3) $C \cap B = \emptyset$ if and only if $C \cap E_B = \emptyset$ as $C \cap B = \emptyset \Rightarrow C \in \mathbf{C}_B \Rightarrow C \cap E_B = \emptyset$ by definition of E_B.
(4) Let U be a $\Im_i$ open set in X. Then U intersects countably many elements of **C**. Also by (iii) each C intersects countably many E_B. Thus each U intersects countably many elements of **E** . Hence **E** is pairwise locally countable.

Corollary 4.7: Let **B** be a collection of subsets covering a pairwise closely regular space $(X, \Im_1, \Im_2)$. If there exists a countable cover of $\Re_i$ closed sets with each of its elements intersecting countably many elements of **B** $\forall$ i, j$\in\{1,2\}$, i $\neq$ j then, there exists a countable cover of $\Re_i$ open sets say $\mathbf{E}_C$ such that **B** refines $\mathbf{E}_C$.

Proof : **C** is a countable cover of $\Re_i$ each C intersects countably many E_B. The collection $\mathbf{E}_C = \{ E_B \mid E_B \cap C \neq \emptyset, C \in \mathbf{C}_B \}$ is countable and covers X.

Corollary 4.8: Let **B** be a collection of subsets covering a pairwise closely regular space (X, $\Im_1$, $\Im_2$). If there exists a pairwise locally countable cover of $\Re_i$ closed sets with each of its elements intersecting countably many elements of **B** $\forall$ i, j$\in\{1,2\}$, i $\neq$ j then, there exists a pairwise Ω locally countable collection of $\Re_i$ open sets say **E** such that **B** refines a countable subcollection of **E** covering X.
Proof: Let x $\in$ X. Then x has a $\Im_j$ neighbourhood U_x intersecting countably many elements of **C**. Since X is pairwise closely regular,U_x is $\Re_i$ open also. Also each C intersects countably many E_B. Thus each U_x intersects countably many elements of **E** . Hence **E** is pairwise Ω locally countable.

Theorem 4.13: If a bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_3 , then the bitopological space (X, $\Re_1, \Re_2$) is pairwise T_2.
Proof: If X is pairwise regular then every $\Im_i$ open set is $\Re_j$ open $\forall$ i, j$\in\{1,2\}$, i $\neq$ j. Also pairwise T_3 is pairwise T_2. Now let x $\neq$ y then there exist U_x and U_y in $\Im_i$ and $\Im_j$ respectively, such that x $\in U_x$, y $\in U_y$ and $U_x \cap U_y = \emptyset \Rightarrow$ there exist U_x and U_y in $\Re_j$ and $\Re_i$ respectively, such that x $\in U_x$, y $\in U_y$ and $U_x \cap U_y = \emptyset$.

Definition 4.4: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise first countable if both the topologies (X, $\Im_1$) and (X, $\Im_2$) are first countable i.e. if there exists a countable base at each point in X in both the topologies.

Theorem 4.14: If a pairwise hausdorff bitopological space (X, $\Im_1$, $\Im_2$) is pairwise first countable and pairwise closely regular, then the bitopological space (X, $\Re_1, \Re_2$) is pairwise regular.
Proof: Let x $\in$ X and let F be $\Re_1$ closed set disjoint from x. Then, $\forall$ y $\in$ F there exists ${U_y}^x \in \Im_1$ and $U_y \in \Im_2$ such that x $\in {U_y}^x$, y $\in U_y$ and ${U_y}^x \cap U_y = \emptyset$. Since (X, $\Im_1$, $\Im_2$) is pairwise first countable, $\forall$ ${U_y}^x$ there exists a $\Im_1$ open base element B_{xi} containing x and contained in ${U_y}^x$. Since (X, $\Im_1$, $\Im_2$) is pairwise regular, ${U_y}^x \in \Re_2$ and $U_y \in \Re_1$. Now, let $W_k = \cup\{ U_y \mid Bi_k \subset {U_y}^x \}$ Then, W_k is $\Re_1$ open as it is union of $\Re_1$ open sets. Let W= $\cup W_k \Rightarrow$ W is $\Re_1$ open. F$\subset$W. Let U= X-$\Re_2$-cl(W). U is $\Re_2$ open. x$\in$U such that $B_{ix} \cap$ Uy = $\emptyset \Rightarrow B_{ix} \cap R_2$ - cl(Uy) = $\emptyset \Rightarrow$ x $\notin \Re_2$- cl(Uy). Finally, $\Re_2$-cl(W)= $\Re_2$- cl(U W_k)=$\cup$ $\Re_2$- cl(W_k)(X pairwise closely regular and k countable) $\Re_2$- cl(W_k) = $\Re_2$ – cl$\cup$(Uy) = $\cup\Re_2$ - cl(Uy). Hence, x $\in$ U $\in \Re_2$, F $\subset$ W $\in \Re_1$ and is such that U$\cap$W = $\emptyset$.

Acknowledgement: The authors are thankful to the referee for his useful comments and valuable suggestions for the improvement of the paper.

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