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MATCHING NUMBER IN RELATION WITH MAXIMAL-MINIMAL NULLITY CONDITIONS AND CYCLOMATIC NUMBER BY COEFFICIENT RELATIONS

MERT SINAN OZ AND ISMAIL NACI CANGUL

Abstract. Let G be a simple graph. So called K_2 deletion process was recently introduced by Wang. A subgraph G' of G that is obtained as a result of some K_2 deletion process will be called as a crucial subgroup. Let $\nu(G)$ and $\nu(G')$ be the matching numbers of G and G', respectively. In this study, we study the relation between $\nu(G)$, $\nu(G')$ and the coefficients of the characteristic polynomials of G and G'. Several results are obtained on these notions. Moreover, conservation of maximal and minimal nullity conditions after applying K_2 deletion process are studied. As a result of this, when G satisfies the maximal or minimal nullity condition, we obtain the conditions for the equality $c(G) = c(G')$ where $c(G)$ and $c(G')$ denote the cyclomatic numbers of G and G', respectively. Finally, for some graphs, we state $\nu(G)$ in terms of $c(G)$, $c(G')$, $n(G)$, $n(G')$ and the coefficients of the characteristic polynomials of G and G' where $n(G)$, $n(G')$ are the numbers of vertices of G and G', respectively.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1,\ v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1,\ e_2,\ \cdots, e_m\}$, namely, G is an undirected graph without loops or multiple edges. The adjacency matrix $A(G)$ of G is defined as an $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ in the case of vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Characteristic polynomial of G is defined as $\det[xI_n - A(G)]$ and it is denoted by $P_G(x)$. The roots of $P_G(x)$ are called eigenvalues of G. The multiplicity of zero as an eigenvalue of G is defined as nullity of G and it is denoted by $\eta(G)$. The rank of G is defined as the rank of $A(G)$ and it is denoted by $r(G)$. It is a well known equality that

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Corresponding author: I. N. Cangul.

$r(G) + \eta(G) = |V(G)| = n$. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a subgraph of G. If every component of a subgraph H of G consists of a single edge or a $2-$regular subgraph of G, then H is called a Sachs subgraph of G, see e. g. [1]. To avoid ambiguity, we give another naming of Sachs subgraphs as elementary subgraphs and if the number of vertices of the subgraph H is the same with the one for G, then H is called a spanning elementary subgraph of G, see [3], [4]. If there is a path between every pair of vertices of G, then G is called connected. A maximal connected subgraph of G is called a component of G. The number of edges connected to the vertex v is called the degree of v and is denoted by $d(v)$. If $d(v) = 1$, then v is called pendant-vertex and the edge which is incident to v is called a pendant-edge. The vertex adjacent to the pendant vertex is called a quasi-pendant or a support vertex. The null graph is defined as a graph with n vertices and no edges. If G does not contain two cycles with at least a common vertex, then we call G as a cycle-vertex-disjoint graph. Similarly, if G does not contain two cycles with at least a common edge, then it is called a cycle-edge-disjoint graph. Clearly a cycle-vertex-disjoint graph must be cycle-edge-disjoint, but the converse may not be true.

A matching in a graph G is a set of edges such that none of them have a common vertex and the maximum possible number of edges in a matching is called a maximum matching. The number of edges that exist in a maximum matching of G is called the matching number of G and in this paper we denote it with $\nu(G)$, see e. g. [15]. A matching that covers all of the vertices of G is called a perfect matching and it is clear that in a graph G that contains a perfect matching, $2\nu(G)$ is equal to the number of vertices of G. The dimension of the cycle space is denoted by $c(G)$ and given by the formula $c(G) = m(G) - n(G) + \theta(G)$ where $\theta(G)$, $m(G)$ and $n(G)$ are the number of components, number of edges and number of vertices of G, respectively. A graph G with nullity $\eta(G) = n(G) - 2\nu(G) - c(G)$ is called as a graph G satisfying the minimal nullity condition in [29] and [22]. Similarly, a graph G with nullity $\eta(G) = n(G) - 2\nu(G) + 2c(G)$ is called as a graph G satisfying the maximal nullity condition, see [26]. In some papers, the dimension of the cycle space is also named as the cyclomatic number of the graph G. In [11], Delen and Cangul defined a new graph invariant which is named as the omega invariant which helps to determine numerous combinatorical and topological properties of a graph or even of the realizations of a given degree sequence. Some extremal problems related to the number of components and loops are studied in [12].

Using the results in those papers, many relations on nullity, dimension of cycle space, the number of vertices and edges of the graph, and the matching number can be restated in terms of omega invariant which is very easy to calculate from the vertex degrees of the graph. In [11], the following relation between the cyclomatic number $c(G)$, the number of components $\theta(G)$ and the invariant omega is given by the following relation:

Lemma 1.1. [11] *Let G be any graph with $\theta(G)$ components. Then the cyclomatic number $c(G)$ of G satisfies the following relation:*

$$c(G) = \frac{\Omega(G)}{2} + \theta(G).$$

We can imply $P_G(x)$ by emphasizing coefficients as the following: $P_G(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}x + c_n$. Let us denote by $c_-(H)$ and $c_\circ(H)$ the number of components in a subgraph H which are single edges and $2-$regular subgraphs, respectively. The relations

$$|A(G)| = \sum (-1)^{n-c_-(H)-c_\circ(H)} 2^{c_\circ(H)} \quad \text{and} \quad c_k = \sum (-1)^{c_-(H)+c_\circ(H)} 2^{c_\circ(H)}$$

where the first summation is taken over all spanning elementary subgraphs H of G and the second summation is taken over all elementary subgraphs H with k vertices for $1 \leq k \leq n$, are given by Harary in 1962, see e.g. [3], [4], [9] and [10].

Nullity and rank of a molecular graph which are closely related to each other are related to the nonzero coefficients of $P_G(x)$ and are prominent notions in chemistry. The chemical importance of nullity originate from Hückel molecular orbital theory. In 1931, a procedure for approaching molecular orbitals for conjugated molecules is publicised by E. Hückel [19]. In chemistry, a conjugated hydrocarbon can be considered as a molecular graph, where the carbon atoms and bonds that exist between carbon atoms are represented by vertices and edges of the corresponding molecular graph, respectively. Hückel theory needs the detection of eigenvalues and eigenvectors of the corresponding molecular graph. In Hückel theory, the eigenvalues and eigenvectors of $A(G)$ correspond to the energy of the corresponding molecular orbital and the Hückel molecular orbitals, respectively, see e.g. [29], [30]. The number of nonbonding molecular orbitals is the nullity of $A(G)$. Hence, actually Hückel theory and spectral graph theory are "isomorphic", see [6] and [28]. Hence, if $\eta(G) > 0$ for the molecular graph, then there is at least one nonbonding molecular orbital in G and the corresponding chemical molecule is reactive and unstable or nonexistent, see [6, 20, 27, 29, 30].

If $\eta(G) > 0$, then G is called singular and if $\eta(G) = 0$, then G is called nonsingular. Collatz and Sinogowitz came up with characterizing all singular graphs in 1957, see [8]. This idea is really important in chemistry as explained above and difficult to solve as well. However, some results for specific graph types and cases are mentioned in [2, 5, 6, 7, 13, 14, 16, 17, 18, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32]. In this study, our purpose is to take attention to the connection between $\nu(G)$ and nonzero coefficients of $P_G(x)$, relatively $\eta(G)$, for any graph G and for particular graph types by using processes defined in section 2.

2. Matching Number, Nullity and Coefficients

In this section, we start with the definitions of K_2 deletion process and shrinking process. After that we mention a lemma given by Tam and Huang in 2017, see [27]. We study the relevance of $\nu(G)$ and the coefficients of $P_G(x)$ by using this lemma and K_2 deletion process. Secondly, we recall two prominent theorems that give the conditions for satisfying the maximal and minimal nullity conditions for any graph G, see e.g. [22], [26]. We make use of these theorems and shrinking process to show the conservation of maximal and minimal nullity conditions after applying K_2 deletion process. Finally, we obtain some results for some graphs that satisfy the maximal and minimal nullity conditions. As of now, let us denote by k_G^{end} and k_G^{even} the maximum index of all nonzero coefficients and the maximum index of all even nonzero coefficients in $P_G(x)$, respectively. Analogously, we let $k_{G'}^{end}$ and $k_{G'}^{even}$ denote these two numbers for $P_{G'}(x)$, respectively. Observe that $n(G) - k_G^{end} = \eta(G)$ and $\eta(G')$ can be stated similarly. Now, we are ready to give the following definitions:

Let G be a graph having at least one pendant vertex and thereby one pendant edge. The process of taking out a pendant vertex and its adjacent vertex from G is called K_2 deletion, see [27] and [29]. This process is called "Pendant Edge Deletion" in [29] but we prefer "K_2 deletion" in our paper. Moreover, we need the notion of a crucial subgraph of G. A crucial subgraph is defined as follows: If there is no pendant vertex in G, then the crucial subgraph of G is itself. If there are pendant vertices in G, then K_2 deletion process is applied consecutively until reaching up to a subgraph of G that has no pendant vertices. When the desired subgraph is obtained, K_2 deletion process will end and the subgraph will be called a crucial subgraph of G. A crucial subgraph of G is denoted by G' and we denote the number of K_2 deletions that we need to get G' from G by $\gamma(G)$ in this paper. All crucial subgraphs of G are shown to be isomorphic, see for details [27, 29]. In Figure 1, we have a graph G in the first step and we apply K_2 deletion process to G until the fifth step where we get G'.

Let G be a cycle-vertex-disjoint graph. Let $\mathbf{C(G)}$ be the set of all cycles in G. We shall call the process of shrinking each cycle in G to a single vertex to get an acyclic graph T_G as shrinking process. The vertex set of T_G is taken to be $U_G \cup W_G$ where U_G consists of all vertices of G which does not belong to any cycle of G and W_G consists of all the vertices v_{C_i} that are formed by shrinking a cycle in G. There are three types of adjacency in T_G. Any two vertices in U_G are adjacent in T_G if and only if they are adjacent in G. A vertex u in U_G is adjacent to $v_{C_i} \in W_G$ if and only if u is adjacent to a vertex in some cycle C_i. The vertices v_{C_i} and v_{C_j} are adjacent in T_G if and only if there is an edge between a vertex of $C_i \in \mathbf{C(G)}$ and a vertex of $C_j \in \mathbf{C(G)}$, see [29], [26]. In Figure 2, we have a cycle-vertex-disjoint graph G in the first step and we get T_G in the second step by using the shrinking process.

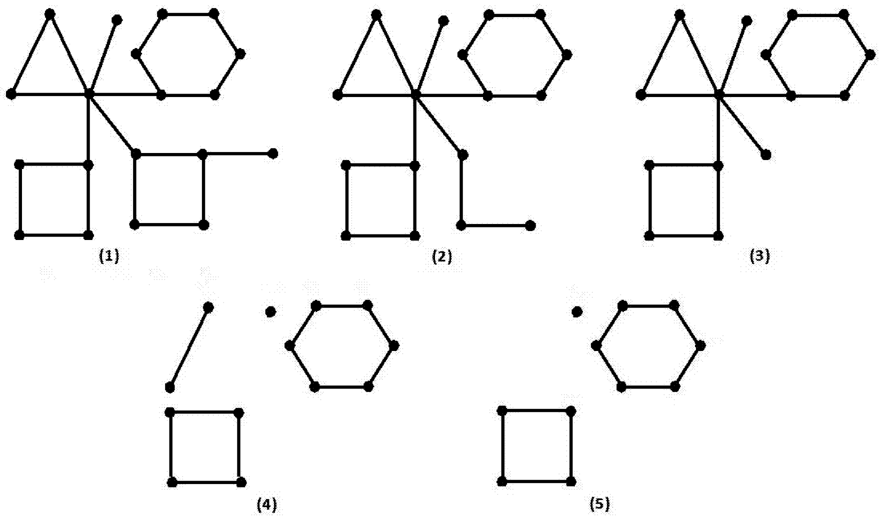

FIGURE 1. K_2 Deletion Process

We recall the following useful result:

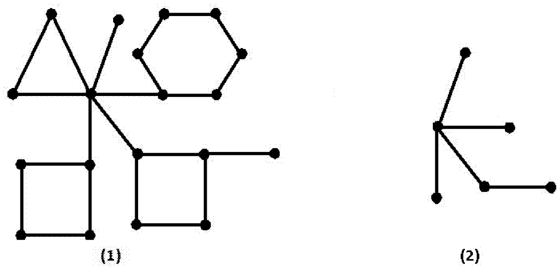

FIGURE 2. Shrinking Process

Lemma 2.1. [27] *For any graph G, we have*
$i-)$ $\eta(G) = \eta(G')$.
$ii-)$ $\nu(G) = \nu(G') + \nu(G - G')$.
$iii-)$ *To get G' from $G, \gamma(G)$ is equal to $\nu(G - G')$.*

We can prove the following fact about the matching numbers of a graph and a crucial subgraph of it:

Theorem 2.1. *Let G be a graph and G' be the crucial subgraph of G. Then*

$$\nu(G) - \nu(G') = \frac{k_G^{end}}{2} - \frac{k_{G'}^{end}}{2}.$$

Proof. Let G be a graph. By Lemma 2.1 (i), we have $\eta(G) = \eta(G')$. Let us rewrite the equation by means of k_G^{end}, $k_{G'}^{end}$ and as we know that number of vertices of G' is $n(G) - 2\gamma(G)$, we get $n(G) - k_G^{end} = (n(G) - 2\gamma(G)) - k_{G'}^{end}$. Hence, $\gamma(G) = \frac{k_G^{end} - k_{G'}^{end}}{2}$. By Lemma 2.1 (iii), we have $\nu(G - G') = \frac{k_G^{end} - k_{G'}^{end}}{2}$. Consequently, by Lemma 2.1 (ii), we obtain $\nu(G) - \nu(G') = \frac{k_G^{end}}{2} - \frac{k_{G'}^{end}}{2}$. □

Corollary 2.1.1. *Let G be a graph and G' be a crucial subgraph of G. Then*

$$\nu(G) = \frac{k_G^{end}}{2} \textit{ if and only if } \nu(G') = \frac{k_{G'}^{end}}{2}.$$

Proof. By Theorem 2.1, proof is clear. □

Theorem 2.2. *Let G be a graph and G' be a crucial subgraph of G. Then*

$$\nu(G') = \frac{k_{G'}^{even}}{2} \textit{ implies that } \nu(G) = \frac{k_G^{even}}{2}.$$

Proof. Let $\nu(G') = \frac{k_{G'}^{even}}{2}$. By Lemma 2.1 (ii) and (iii), we know that $\nu(G) - \nu(G') = \gamma(G)$. If we substitute the first equation into the second equation, then we get $\nu(G) = \frac{k_{G'}^{even}}{2} + \gamma(G)$ and by the proof of Theorem 2.1, we have $\nu(G) = \frac{k_{G'}^{even} - k_{G'}^{end}}{2} + \frac{k_G^{end}}{2}$. Since $\nu(G') = \frac{k_{G'}^{even}}{2}$, we have two cases to show. First one is $k_{G'}^{even} = k_{G'}^{end}$. In this case we get required result clearly. Second case is $k_{G'}^{end} = k_{G'}^{even} + 1$. In this case we have $\nu(G) = \frac{k_G^{end} - 1}{2}$. Consequently, we get $\nu(G) = \frac{k_G^{even}}{2}$. □

Corollary 2.2.1. *Let G be a graph and G' be a crucial subgraph of G. If $\nu(G') = \frac{k_{G'}^{even}}{2}$, then*

$$k_{G'}^{even} + 2\gamma(G) = k_G^{even}.$$

Proof. To prove the Corollary, we use Theorem 2.1 and 2.2. In the hypotesis we have $\nu(G') = \frac{k_{G'}^{even}}{2}$ so by Theorem 2.2, $\nu(G) = \frac{k_G^{even}}{2}$ and by Theorem 2.1, we get $\nu(G) = \frac{k_G^{end}}{2} - \frac{k_{G'}^{end}}{2} + \frac{k_{G'}^{even}}{2}$. By the proof of Theorem 2.1, we attain $\nu(G) = \gamma(G) + \frac{k_{G'}^{even}}{2}$. If we arrange the last equality, then we get $k_{G'}^{even} + 2\gamma(G) = k_G^{even}$. □

Let us now define some notions for the next theorems. Let $k_{K_i'}^{even}$ and $k_{S_i'}^{even}$ be the maximum index of all even nonzero coefficients in the characteristic polynomial of components K_i' and S_i' of G', respectively. We define the number of components of G' that verify the conditions $\nu(K_i') > \frac{k_{K_i'}^{even}}{2}$ and $\frac{k_{S_i'}^{even}}{2} > \nu(S_i')$ by $\zeta(G)$ and $\vartheta(G)$, respectively. Similarly as stated earlier, let us define $k_{i'}^{end}$ and $k_{i'}^{even}$ the maximum index of all nonzero coefficients and the maximum index of all even nonzero coefficients in the characteristic polynomial of corresponding component of G', respectively. $P_{C_i'}(x)$ is the characteristic polynomial of corresponding component of G'. Also, we define $\sigma(G)$ as the number of even k that are in $P_{G'}(x)$ and greater than the sum of the $k_{i'}^{even}$ of every $P_{C_i'}(x)$ where the summation is over all components of G'. Also, let $\theta(G')$ be the number of components of G'. We can give the matching number of the graph G in terms of the matching numbers of the components of G', the number of even indices in $P_{G'}(x)$, and the indices of the coefficients of $P_G(x)$ and $P_{G'}(x)$:

Theorem 2.3. *Let G be a graph and G' be a crucial subgraph of G. Then*

$$\nu(G) = \frac{k_G^{end} - k_{G'}^{end}}{2} + \frac{k_{G'}^{even}}{2} + \left[\sum_{i=1}^{\zeta(G)} \nu(K_i') - \frac{k_{K_i'}^{even}}{2}\right] - \left[\sum_{i=1}^{\vartheta(G)} \frac{k_{S_i'}^{even}}{2} - \nu(S_i')\right] - \sigma(G).$$

Proof. Let G be any graph. By the definition of Sachs subgraphs, we know that $k_{G'}^{even}$ is constituted of elementary or spanning elementary subgraphs which are disjoint edges or disjoint cycles, see [3], [4], [15] . However, some $\frac{k_{G'}^{even}}{2}$ can be less than or more than $\nu(G')$ for some combination of elementary subgraphs, so the equality $\nu(G') = \frac{k_{G'}^{even}}{2}$ cannot be true all the time. Hence, for calculating $\nu(G')$, besides its equality to $\frac{k_{G'}^{even}}{2}$, we have two cases for examining. If for some component K_i' of G', $\nu(K_i')$ is greater than $\frac{k_{K_i'}^{even}}{2}$, then to calculate $\nu(G)$ we must add the positive difference of $\nu(K_i')$ and $\frac{k_{K_i'}^{even}}{2}$ to $\nu(G)$ for corresponding component. Similarly, in the second case where $\nu(S_i')$ is less than $\frac{k_{S_i'}^{even}}{2}$, we subtract the positive difference of $\nu(S_i')$ and $\frac{k_{S_i'}^{even}}{2}$ from $\nu(G)$ for corresponding component. Also, G' can be disconnected and observe that $\prod_{i=1}^{\theta(G')} k_{i'}^{end}$ is equal to $k_{G'}^{end}$. Hence, $\nu(G')$ can be more than $\frac{k_{G'}^{even}}{2}$ misleadingly, because of the the multiplication of some $k_{i'}^{end}$ of corresponding components that are odd. Thus, we must subtract $\sigma(G)$ from $\nu(G)$. by Lemma 2.1 (ii) and (iii), we get

$$\nu(G) = \gamma(G) + \frac{k_{G'}^{even}}{2} + \left[\sum_{i=1}^{\zeta(G)} \nu(K_i') - \frac{k_{K_i'}^{even}}{2}\right] - \left[\sum_{i=1}^{\vartheta(G)} \frac{k_{S_i'}^{even}}{2} - \nu(S_i')\right] - \sigma(G).$$

Finally, by proof of Theorem 2.1, we get

$$\nu(G) = \frac{k_G^{end} - k_{G'}^{end}}{2} + \frac{k_{G'}^{even}}{2} + \left[\sum_{i=1}^{\zeta(G)} \nu(K_i') - \frac{k_{K_i'}^{even}}{2}\right] - \left[\sum_{i=1}^{\vartheta(G)} \frac{k_{S_i'}^{even}}{2} - \nu(S_i')\right] - \sigma(G).$$

□

Let C_1 and C_2 be two cycles in G. If there is a path between a vertex of C_1 and a vertex of C_2 with the property that the internal vertices of the path do not belong to any cycle, we call the cycles C_1 and C_2 as adjacent cycles. Let us consider the case that G is cycle-vertex-disjoint graph and there is at least one quasi-pendant vertex on the shortest path that is between every pairwise adjacent cycles in G. In this case we give next theorems. We denote the number of $4t-$ cycles, odd-cycles and $(4t+2)-$cycles in G' by $\alpha(G')$, $\beta(G)$, $\delta(G)$, respectively.

Theorem 2.4. *Let G be a cycle-vertex-disjoint graph and G' be a crucial subgraph of G. In G, there is at least one quasi-pendant vertex on the shortest path that is between every pairwise adjacent cycles. Then matching number of*

G is equal to

$$\nu(G) = \frac{k_G^{even}}{2} + \alpha(G') - \lfloor\frac{\beta(G)}{2}\rfloor.$$

Proof. Let us consider a cycle-vertex-disjoint graph G and a crucial subgraph G'. Observe that since G is a cycle-vertex-disjoint graph and there is at least a quasi-pendant vertex on the shortest path that is between every pairwise adjacent cycles in G, G' constitutes of components which are only $2-$regular graphs or null graphs. Considering these components, let us denote $r_{i'}$ as the number of vertices of cycle components C_i', i.e., $2-$regular components, and we set matching number of every cycle component C_i' as the following:

$$\nu(C_i') = \begin{cases} \frac{k_{i'}^{end}-1}{2}, & 1 \equiv r_{i'}(mod4) \text{ or } 3 \equiv r_{i'}(mod4) \\ \frac{k_{i'}^{end}}{2} = \frac{k_{i'}^{even}}{2}, & 2 \equiv r_{i'}(mod4) \\ \frac{k_{i'}^{end}+2}{2} = \frac{k_{i'}^{even}+2}{2}, & 0 \equiv r_{i'}(mod4). \end{cases}$$

Moreover, we know that matching number of null components are zero, so we get that $\nu(G')$ is equal to the sum of the matching numbers of cycle components of G'. As a result, we have

$$\nu(G') = \sum_{i=1}^{\beta(G')} \frac{k_{i'}^{end} - 1}{2} + \sum_{i=1}^{\delta(G')} \frac{k_{i'}^{end}}{2} + \sum_{i=1}^{\alpha(G')} \frac{k_{i'}^{end} + 2}{2}.$$

By Lemma 2.1 (ii) and (iii), we obtain

$$\nu(G) = \sum_{i=1}^{\beta(G')} \frac{k_{i'}^{end} - 1}{2} + \sum_{i=1}^{\delta(G')} \frac{k_{i'}^{end}}{2} + \sum_{i=1}^{\alpha(G')} \frac{k_{i'}^{end} + 2}{2} + \gamma(G).$$

Consequently, by the above equation we reach

$$\nu(G) = \frac{k_G^{even}}{2} + \alpha(G') - \lfloor\frac{\beta(G')}{2}\rfloor.$$

□

Corollary 2.4.1. *Let G be a cycle-vertex-disjoint graph with vertex set $V(G)$ and edge set $E(G)$. Let there be at least one quasi-pendant vertex on the shortest path between every pairwise adjacent cycles in G. G' be a crucial subgraph of G. Then*

$$\nu(G') = \frac{k_G^{even}}{2} - \frac{k_G^{end}}{2} + \frac{k_{G'}^{end}}{2} + \alpha(G') - \lfloor\frac{\beta(G')}{2}\rfloor.$$

Moreover, if $\lfloor\frac{\beta(G')}{2}\rfloor$ *is even, then*

$$\nu(G') = \frac{k_{G'}^{end}}{2} + \alpha(G') - \lfloor\frac{\beta(G')}{2}\rfloor.$$

Proof. It is clear by Lemma 2.1 (ii), (iii), by proof of Theorem 2.1 and by Theorem 2.4. □

We need some new symbols for the next theorem. Let us define a_i, b_j as the number of vertices of corresponding ith component that is an even cycle and jth component that is an odd cycle, respectively.

Corollary 2.4.2. *Let G be a cycle-vertex-disjoint graph and G' be a crucial subgraph of G. Let there be at least one quasi-pendant vertex on the shortest path between all pairwise adjacent cycles in G. Then*

$$\frac{k_G^{even}}{2} + \alpha(G') = \sum_{i=1}^{\alpha(G')+\delta(G')} \frac{a_i}{2} + \left(\sum_{j=1}^{\beta(G')} \lfloor\frac{b_j}{2}\rfloor\right) + \lfloor\frac{\beta(G')}{2}\rfloor + \frac{k_G^{end}}{2} - \frac{k_{G'}^{end}}{2}.$$

Proof. The proof is clear by Lemma 2.1 (ii), (iii), by Theorem 2.4 and by proof of Theorem 2.1. □

Theorem 2.5. [22] *Let G be a graph. G satisfies the minimal nullity condition if and only if the following conditions are all satisfied:*
$i-$) All cycles of G are cycle-vertex-disjoint.
$ii-$) The length of each cycle of G is odd.
$iii-$) $\nu(T_G) = \nu(T_G - W_G)$.

Theorem 2.6. [26] *Let G be a graph.G satisfies the maximal nullity condition if and only if the following conditions are all satisfied:*
$i-$) All cycles of G are cycle-vertex-disjoint.
$ii-$) The length of each cycle of G is congruent to 0 modulo 4.
$iii-$) $\nu(T_G) = \nu(T_G - W_G)$.

Next two theorems show the conservations of minimal nullity and maximal nullity conditions after appyling K_2 deletion process, respectively.

Theorem 2.7. *Let G be a graph and G' be a crucial subgraph of G. Then*

$$\eta(G) = n(G) - 2\nu(G) - c(G) \text{ implies that } \eta(G') = n(G') - 2\nu(G') - c(G').$$

Proof. Let G be a graph that satisfies the minimal nullity condition. By Theorem 2.5 $(i), (ii)$ and (iii), all cycles of G are cycle-vertex-disjoint, odd and $\nu(T_G) = \nu(T_G - W_G)$. Thus, all cycles of G' are cycle-vertex-disjoint and odd. Now, we need to prove that $\nu(T_{G'}) = \nu(T_{G'} - W_{G'})$. We have $\nu(T_G) = \nu(T_G - W_G)$ so we can find maximum matching without the edge that connects each cycle to the related acyclic part of G. Also, we know that $\nu(G) = \nu(G') + \gamma(G)$ by Lemma 2.1 $(ii),(iii)$. Therefore, each edge that connects each cycle to the related acyclic part of G cannot be counted in $\nu(G)$. As a result, we get $\nu(T_{G'}) = \nu(T_{G'} - W_{G'})$. Hence, since all cycles of G' are cycle-vertex-disjoint, odd and $\nu(T_{G'})$ is equal to the $\nu(T_{G'} - W_{G'})$, G' satisfies the minimal nullity condition $\eta(G') = n(G') - 2\nu(G') - c(G')$, by Theorem 2.5. □

Theorem 2.8. *Let G be a graph G' be a crucial subgraph of G. Then*

$$\eta(G) = n(G) - 2\nu(G) + 2c(G) \text{ implies that } \eta(G') = n(G') - 2\nu(G') + 2c(G').$$

Proof. Let G be a graph. Let us consider the case that G satisfies the maximal nullity condition. By Theorem 2.6, all cycles of G are cycle-vertex-disjoint and the length of each cycle of G is congruent to 0 modulo 4. Hence, all cycles of G' are cycle-vertex-disjoint and the length of each cycle of G' is congruent to 0 modulo 4. Also, since G satisfies the maximal nullity condition, we have $\nu(T_G) = \nu(T_G - W_G)$. By the proof of previous theorem, we get $\nu(T_{G'}) = \nu(T_{G'} - W_{G'})$. Consequently, by Theorem 2.6, G' satisfies the maximal nullity condition $\eta(G') = n(G') - 2\nu(G') + 2c(G')$. □

Corollary 2.8.1. *Let G be a graph satisfying the maximal nullity or minimal nullity condition and G' be a crucial subgraph of G. Then $c(G) = c(G')$.*

Proof. Let G be a graph. We make the proof for maximal nullity condition and minimal nullity condition can be proved similarly. Let G satisfy the maximal nullity condition. We know that $n(G') = n(G) - 2\gamma(G)$ and by Lemma 2.1, $\nu(G) = \nu(G') + \gamma(G)$. Thus, we have $\eta(G) = n(G') + 2\gamma(G) - 2(\nu(G') + \gamma(G)) + 2c(G)$ that is equal to $\eta(G) = n(G') - 2\nu(G') + 2c(G)$. Also, we know that $\eta(G) = \eta(G')$ by Lemma 2.1. Then by previous theorem, G' satisfies the maximal nullity condition since G satisfies the maximal nullity condition. As a result, we get the equality $\eta(G) = n(G') - 2\nu(G') + 2c(G) = n(G') - 2\nu(G') + 2c(G')$. Hence, we get $c(G) = c(G')$. □

Theorem 2.9. *Let G be a graph satisfying the maximal nullity condition and G' be a crucial subgraph of G. Then $\nu(G) = \frac{k^{even}_{G'}}{2} + c(G') + \frac{n(G)-n(G')}{2}$.*

Proof. Let G be a graph which satisfies the equality $\eta(G) = n(G) - 2\nu(G) + 2c(G)$. By the proof of Theorem 2.8, since $\nu(T_{G'}) = \nu(T_{G'} - W_{G'})$, reader can observe that G' consists of completely $c(G)$ cycles and null graphs. We know that the lengths of all $c(G)$ cycles of G' are congruent to 0 modulo 4 by the proof of Theorem 2.8 so that since for every cycle, we found that $\frac{k_{G'}^{even}}{2}+1$ is equal to $\nu(G')$, and hence we get $\nu(G')=\frac{k_{G'}^{even}}{2}+c(G)$. By Lemma 2.1 we know that $\nu(G)=\nu(G')+\gamma(G)$. Then we get $\nu(G)=\frac{k_{G'}^{even}}{2}+c(G)+\gamma(G)$. Since G satisfies the maximal nullity condition, if we write the last equation in the maximal nullity condition, we get $\eta(G)=n(G)-2(\frac{k_{G'}^{even}}{2}+c(G)+\gamma(G))+2c(G)$. Since by Lemma 2.1 $\eta(G)=\eta(G')$ and $\eta(G')=n(G')-k_{G'}^{end}$, we get $k_{G'}^{even}-k_{G'}^{end}+2\gamma(G)=n(G)-n(G')$. Finally G' consists of completely $c(G)$ cycles and null graphs, $k_{G'}^{even}=k_{G'}^{end}$ so $\gamma(G)=\frac{n(G)-n(G')}{2}$, we get $\nu(G)=\frac{k_{G'}^{even}}{2}+c(G)+\frac{n(G)-n(G')}{2}$. By Corollary 2.8.1, $c(G)=c(G')$, and as a conclusion, we get $\nu(G)=\frac{k_{G'}^{even}}{2}+c(G')+\frac{n(G)-n(G')}{2}$. □

Theorem 2.10. *Let G be a graph satisfying the minimal nullity condition and G' be a crucial subgraph of G. Then*

$$\nu(G)=\begin{cases}\frac{k_{G'}^{even}}{2}-\frac{c(G')}{2}+\frac{n(G)-n(G')}{2}, & if\ c(G)\ is\ even\\ \frac{k_{G'}^{even}}{2}-\frac{c(G')-1}{2}+\frac{n(G)-n(G')}{2}, & if\ c(G)\ is\ odd.\end{cases}$$

Proof. Proof can be made similarly to previous theorem. □

It is worthy to note that when G satisfies the maximal or minimal nullity condition, G' consists of exactly $c(G)=c(G')$ disjoint cycles and null graphs by the proof of Theorem 2.7 and 2.8. Therefore, in Theorem 2.4, Corollary 2.4.1 and Corollary 2.4.2, we can write the condition that G satisfies the maximal or minimal nullity condition instead of the condition that G is cycle-vertex-disjoint and at least one quasi-pendant vertex exists on the shortest path between all pairwise adjacent cycles in G.

Theorem 2.11. *Let G be a cycle-vertex-disjoint graph. If $\nu(T_G)=\nu(T_G-W_G)$ and $\nu(G)=\frac{k_G^{even}}{2}-\lfloor\frac{\beta(G')}{2}\rfloor$, then G satisfies the minimal nullity condition and the equality*

$$\nu(G)=\begin{cases}\frac{k_G^{even}-c(G)}{2}, & if\ \beta(G')\ is\ even\\ \frac{k_G^{even}-c(G)+1}{2}, & if\ \beta(G')\ is\ odd\end{cases}$$

is satisfied.

Proof. Let G be a cycle-vertex-disjoint graph. If $\nu(T_G)=\nu(T_G-W_G)$, then we need to show that the lengths of all cycles of G are odd by Theorem 2.5. By the proof of Theorem 2.7, since $\nu(T_G)=\nu(T_G-W_G)$, there exists a maximum matching without the edge that connects each cycle to the relative acyclic part of G. Moreover, we know that $\nu(G) = \nu(G') + \gamma(G)$ by Lemma 2.1 $(ii), (iii)$ so that each edge that connects each cycle to the related acyclic part of G cannot be counted in $\nu(G)$. In brief, it means that $c(G) = c(G')$. By the proof of Theorem 2.7 and Corollary 2.8.1, reader can observe that G' consists of only cycles and null graphs. So by Theorem 2.4 and hypotesis $\nu(G) = \frac{k_G^{even}}{2} + \alpha(G') - \lfloor \frac{\beta(G')}{2} \rfloor = \frac{k_G^{even}}{2} - \lfloor \frac{\beta(G')}{2} \rfloor$, we get the result that all cycles of G' are odd. Since $c(G) = c(G')$, all cycles of G are odd, so G satisfies the minimal nullity condition. Now, we know that $\eta(G) = n(G) - 2\nu(G) - c(G)$ and we arrange the equation as $\eta(G) = n(G) - 2(\frac{k_G^{even}}{2} - \lfloor \frac{\beta(G')}{2} \rfloor) - c(G)$. It is clear that $\beta(G') = c(G)$. Thus, if $\beta(G')$ is even, then we get $\eta(G) = n(G) - k_G^{even}$ and if $\beta(G')$ is odd, then we get $\eta(G) = n(G) - k_G^{even} - 1$. As a result, the equality

$$\eta(G) = n(G) - 2\nu(G) - c(G) = \begin{cases} \eta(G) = n(G) - k_G^{even}, & if\ \beta(G')\ is\ even \\ \eta(G) = n(G) - k_G^{even} - 1, & if\ \beta(G')\ is\ odd \end{cases}$$

is satisfied. Hence we get

$$\nu(G) = \begin{cases} \frac{k_G^{even} - c(G)}{2}, & if\ \beta(G')\ is\ even \\ \frac{k_G^{even} - c(G) + 1}{2}, & if\ \beta(G')\ is\ odd. \end{cases}$$

□

Theorem 2.12. *Let G be a cycle-vertex-disjoint graph. If $\nu(T_G) = \nu(T_G - W_G)$ and $\nu(G) = \frac{k_G^{even}}{2} + \alpha(G')$, then G satisfies the maximal nullity condition and the equality $\nu(G) = \frac{k_G^{even} + 2c(G)}{2}$ is satisfied.*

Proof. Proof can be completed analogously to the proof of Theorem 2.11. □

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Department of Mathematics, Faculty of Engineering and Natural Sciences, Bursa Technical University, Bursa-Turkey
Email address: sinan.oz@btu.edu.tr

Department of Mathematics, Bursa Uludag University, 16059 Bursa-Turkey
Email address: cangul@uludag.edu.tr