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SOME IDENTITIES OF FULLY DEGENERATE BELL POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS

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Abstract. Recently, degenerate, partly degenerate and fully degenerate Bell polynomials are studied. In this paper, we study the differential equations on fully degenerate polynomials. From this differential equations, we derive some identities of the fully degenerate Bell polynomials.

1. Introduction

In combinatorial mathematics, the *Bell polynomials*, denoted by $B_n(x)$, are used in the study of set partitions. The Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

where $B_n(1) = B_n$ are called n-th Bell numbers. As is well known, the B_n is equal to the number of partitions of n-set(see [1, 2, 4, 5, 9, 10, 18]).

The Stirling numbers of second kind are defined by the generating function, for nonnegative integer k,

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}(\ [14]).$$

where $S_2(n, k)$ are called the Stirling numbers of the second kind.

Recently, many mathematicians have studied $(1 + \lambda t)^{\frac{1}{\lambda}}$, which is called the degenerate exponential function(see [3, 7, 10–17]). If λ goes to zero, then $(1 + \lambda t)^{\frac{1}{\lambda}}$ approaches to e^t. Throughout in this article, we write $e_\lambda(t)$ for the degenerate exponential function.

From the degenerate exponential function, the degenerate Stirling numbers of the second kind, denoted by $S_{2,\lambda}(n, k)$, are defined by

$$\frac{(e_\lambda(t)-1)^k}{k!} = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}.$$

It is well known that $S_{2,\lambda}(n, k)$ converges to $S_2(n, k)$ if λ goes to 0. The degenerate Stirling numbers of the second kind can be found in the [11, 14].

In [10], the degenerate Bell numbers are introduced and studied as follows.

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$$(1+\lambda)^{\frac{x}{\lambda}(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}.$$

In [12], T. Kim, D. S. Kim and D. V. Dolgy introduced the partly degenerate Bell polynomials

$$e^{x(e_\lambda(t)-1)} = e^{x((1+\lambda t)^{\frac{1}{\lambda}}-1)} = \sum_{n=0}^{\infty} bel_{n,\lambda}(x)\frac{t^n}{n!}.$$

From the definition of partly degenerate Bell numbers, they gave some identities between several special numbers and those polynomials. In [17], they studied the degenerate Daehee numbers, and obtained identities arising from differential equations.

In this paper, we study the differential equations on fully degenerate polynomials. From this differential equations, we derive some identities of the fully degenerate Bell polynomials.

2. Identities for the fully degenerate Bell polynomials

Throughout in this article, for positive integer n, we denote $(x)_{n,\lambda}$ for the generalized falling factorial $x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$. We use $\binom{\lambda}{n} = \frac{(\lambda)_n}{n!}$ for generalized binomial coefficient. This give us that

$$(1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}.$$

In [7], D. Dolgy et al. defined the *fully degenerate Bell polynomials*, denoted by $B_{n,\lambda}(x)$, by the generating function to be

$$\begin{aligned} e_\lambda(x(e_\lambda(t)-1)) &= (1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1))^{\frac{1}{\lambda}} \\ &= \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

The definition of fully degenerate Bell polynomials (2.1) shows the following identity easily

$$\lim_{\lambda\to 0}(1+\lambda x((1+\lambda t)^{\frac{1}{\lambda}}-1))^{\frac{1}{\lambda}} = e^{x(e^t-1)}. \tag{2.2}$$

The equation (2.2) says that $\lim_{\lambda\to 0} B_{n,\lambda}(x) = B_n(x)$.

From the definition of fully degenerate Bell polynomials, they gave the identities (see [7])

$$\begin{aligned}B_{n,\lambda}(x) &= \sum_{l=0}^{n}(1)_{l,\lambda}S_{2,\lambda}(n,l)x^l \\ &= \sum_{l=0}^{n}\sum_{m=0}^{l} S_1(l,m)S_{2,\lambda}(n,l)\lambda^{l-m}x^l \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1)_{l,\lambda}(m)_{n,\lambda}\frac{x^l}{l!}.\end{aligned} \tag{2.3}$$

The following can be obtained by differentiating the equation (2.1). Let $F(t,x) = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}$. Then k-th differentiation gives us the following

$$\begin{aligned}\frac{\partial^k}{\partial t^k}F(t,x) &= \frac{\partial^k}{\partial t^k}\left(\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}\right) \\ &= \sum_{n=k}^{\infty} B_{n,\lambda}(x)\frac{t^{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{\infty} B_{n+k,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Let us observe that

$$\begin{aligned}\frac{1}{1+\lambda x(e_\lambda(t)-1)} &= \sum_{j=0}^{\infty}(-1)^j\lambda^j j!x^j\frac{(e_\lambda(t)-1)^j}{j!} \\ &= \sum_{j=0}^{\infty}(-1)^j\lambda^j j!x^j\sum_{m=j}^{\infty} S_{2,\lambda}(m,j)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^j\lambda^j j!x^j S_{2,\lambda}(m,j)\frac{t^m}{m!}\end{aligned} \tag{2.5}$$

and we easily get the identity

$$(1+\lambda t)^{\frac{1-\lambda}{\lambda}} = \sum_{k=0}^{\infty}(1)_{k+1,\lambda}\frac{t^k}{k!}. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}\frac{\partial F}{\partial t} &= \frac{(1+\lambda x(e_\lambda(t)-1))^{\frac{1}{\lambda}}}{1+\lambda x(e_\lambda(t)-1)}x(1+\lambda t)^{\frac{1}{\lambda}-1}\\ &= \sum_{l=0}^{\infty} B_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^j\lambda^j j!x^j S_{2,\lambda}(m,j)\frac{t^m}{m!}x\sum_{k=0}^{\infty}(1)_{k+1,\lambda}\frac{t^k}{k!}\\ &= \sum_{l=0}^{\infty} B_{l,\lambda}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{m}{k}(-1)^j\lambda^j j!x^{j+1}S_{2,\lambda}(k,j)(1)_{m-k+1,\lambda}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{k}B_{n-m,\lambda}(x)(-1)^j\lambda^j j!x^{j+1}S_{2,\lambda}(k,j)(1)_{m-k+1,\lambda}\Bigg)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

From the equations (2.5) and (2.7), we have the following theorem.

Theorem 2.1. *For any real λ and nonnegative integer n,*

$$\begin{aligned}&B_{n+1,\lambda}(x)\\ &= \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{k}B_{n-m,\lambda}(x)(-1)^j\lambda^j j!x^{j+1}S_{2,\lambda}(k,j)(1)_{m-k+1,\lambda}.\end{aligned} \tag{2.8}$$

Let λ tend to zero in the equation (2.8), then, only $j=0$ is possible from λ^j in the equation (2.8). And we get $k=0$ from $S_{2,\lambda}(k-j,j)$. Finally, we get the following identity

$$B_{n+1}(x) = x\sum_{m=0}^{n}\binom{n}{m}B_m(x).$$

3. Differential equation using the fully degenerate Bell polynomials

In order to write this paper briefly, let T denote $((1+\lambda t)^{\frac{1}{\lambda}}-1)$ and $F=F(t,x)$ the fully degenerate Bell polynomials. That is,

$$F = F(t,x) = (1+\lambda Tx))^{\frac{1}{\lambda}}. \tag{3.1}$$

We observe that

$$\frac{\partial F}{\partial x} = T(1+\lambda Tx)^{\frac{1-\lambda}{\lambda}}. \tag{3.2}$$

By induction we see that

$$\frac{\partial^N F}{\partial x^N} = (1)_{N,\lambda}T^N(1+\lambda Tx)^{\frac{1-N\lambda}{\lambda}}. \tag{3.3}$$

Since

$$\begin{aligned}(1+\lambda x(e_\lambda(t)-1))^{\frac{1-N\lambda}{\lambda}} &= \sum_{l=0}^{\infty}\frac{\left(\frac{1-N\lambda}{\lambda}\right)_l}{l!}\lambda^l x^l (e_\lambda(t)-1)^l \\ &= \sum_{l=0}^{\infty}\left(\frac{1-N\lambda}{\lambda}\right)_l \lambda^l x^l \sum_{m=l}^{\infty} S_{2,\lambda}(m,l)\frac{t^m}{m!} \\ &= \frac{1}{(1)_{N,\lambda}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}(1)_{N+l,\lambda}x^l S_{2,\lambda}(n,l)\frac{t^n}{n!},\end{aligned} \tag{3.4}$$

and

$$T^N = ((1+\lambda t)^{\frac{1}{\lambda}}-1)^N = N!\sum_{i=N}^{\infty} S_{2,\lambda}(i,N)\frac{t^i}{i!}. \tag{3.5}$$

From the equations (3.4) and (3.5), equation (3.3) becomes

$$\begin{aligned}&(1)_{N,\lambda}T^N(1+\lambda Tx)^{\frac{1-N\lambda}{\lambda}} \\ &= (1)_{N,\lambda}N!\sum_{i=N}^{\infty}S_{2,\lambda}(i,N)\frac{t^i}{i!}\frac{1}{(1)_{N,\lambda}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}(1)_{N+l,\lambda}x^l S_{2,\lambda}(n,l)\frac{t^n}{n!} \\ &= N!\sum_{n=N}^{\infty}\sum_{i=0}^{n-N}\sum_{l=0}^{i}\binom{n}{i}(1)_{N+l,\lambda}S_{2,\lambda}(n-i,N)S_{2,\lambda}(i,l)x^l\frac{t^n}{n!}.\end{aligned} \tag{3.6}$$

We get the following by n differentiations of the equation F with respect to x.

$$\begin{aligned}\frac{\partial^N F}{\partial x^N} &= \frac{\partial^N}{\partial x^N}\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!} \\ &= \sum_{n=N}^{\infty}\left(\frac{\partial^N}{\partial x^N}B_{n,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.7}$$

From the equation (3.6) and (3.7), we have the following identity.

Theorem 3.1. *For any real λ and nonnegative integers n and N, with $n \geq N$,*

$$\frac{\partial^N}{\partial x^N}B_{n,\lambda}(x) = N!\sum_{i=0}^{n-N}\sum_{l=0}^{i}\binom{n}{i}(1)_{N+l,\lambda}S_{2,\lambda}(n-i,N)S_{2,\lambda}(i,l)x^l. \tag{3.8}$$

From the equation (2.3), we have

$$\begin{aligned}\frac{\partial^N}{\partial x^N}B_{n,\lambda}(x) &= \sum_{l=0}^{n-N}(1)_{l+N,\lambda}S_{2,\lambda}(n,l+N)(l+N)_N x^l \\ &= \sum_{l=0}^{n-N}\sum_{m=0}^{l+N}S_1(l+N,m)S_{2,\lambda}(n,l+N)\lambda^{l+N-m}(l+N)_N x^l \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{l+N}\binom{l+N}{m}(-1)^{l+N-m}(1)_{l+N,\lambda}(m)_{n,\lambda}\frac{x^l}{l!}.\end{aligned} \tag{3.9}$$

The equations (3.8) and (3.9) give the following identities.

Theorem 3.2. *For any real λ and nonnegative integers n and N,*

$$\begin{aligned} &N!\sum_{i=0}^{n-N}\sum_{l=0}^{i}\binom{n}{i}(1)_{N+l,\lambda}S_{2,\lambda}(n-i,N)S_{2,\lambda}(i,l)x^l. \\ &=\sum_{l=0}^{n-N}(1)_{l+N,\lambda}S_{2,\lambda}(n,l+N)(l+N)_N x^l \\ &=\sum_{l=0}^{n-N}\sum_{m=0}^{l+N}S_1(l+N,m)S_{2,\lambda}(n,l+N)\lambda^{l+N-m}(l+N)_N x^l \\ &=\sum_{l=0}^{\infty}\sum_{m=0}^{l+N}\binom{l+N}{m}(-1)^{l+N-m}(1)_{l+N,\lambda}(m)_{n,\lambda}\frac{x^l}{l!}. \end{aligned}$$

Let, for brevity, G be a differentiable function with respect to t and $F=F(t)=G^{\frac{1}{\lambda}}$. Then we get the following differential equation

$$\lambda G F^{(1)} = G^{(1)}F. \tag{3.10}$$

The following equation is obtained by taking the derivative to both sides of the above equation (3.10).

$$\lambda G F^{(2)} + \lambda G^{(1)}F^{(1)} = G^{(1)}F^{(1)} + G^{(2)}F$$

After taking N-th derivative of both sides of the equation (3.10), we have

$$\lambda\sum_{k=0}^{N}\binom{N}{k}G^{(N-k)}F^{(k+1)} = \sum_{k=0}^{N}\binom{N}{k}G^{(N-k+1)}F^{(k)}. \tag{3.11}$$

The equation (3.11) give us a following differential equation.

Theorem 3.3. *Let G be any differentiable function with respect to t and $F=F(t)=G^{\frac{1}{\lambda}}$, then F is a solution of the following differential equation.*

$$\lambda\sum_{k=0}^{N}\binom{N}{k}G^{(N-k)}F^{(k+1)} = \sum_{k=0}^{N}\binom{N}{k}G^{(N-k+1)}F^{(k)}. \tag{3.12}$$

where $F^{(0)}=F$ and $F^{(k)}=\frac{\partial^k F}{\partial t^k}$ for any nonnegative integer k.

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