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SOME PROPERTIES OF RESTRICTED DIVISOR FUNCTIONS II

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Abstract. We defined the $linear-abundant$, $linear-perfect$ and $linear-deficient$ in [4]. Then, we find some properties about them, and the results of this paper is an extension of [4]. In particular, 6 is the only $linear-perfect\ number$.

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1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ will be denoted the sets of positive integers, rational integers and complex numbers. For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $l \in \{0, 1\}$, various extensions of the standard divisor function are known in many areas of number theory:

$$\sigma_k(n) := \sum_{d|n} d^k, \quad \sigma_k^*(n) := \sum_{\substack{d|n \\ \frac{n}{d}\ odd}} d^k, \quad \sigma_{k,l}(n;2) := \sum_{\substack{d|n \\ d\equiv l(mod 2)}} d^k.$$

The history of studies of $perfect\ number$ can be found in P. Hagis [1]; R. P. Jerrard and N. Temperley [2]; C. Pomerance [5, 6]; P. Starni [7, 8].

In 1858, Liouville ([9]) stated the below identity. Let n be a positive integer and let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. Then

$$\begin{aligned} &\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (f(a-b) - f(a+b)) \\ &= f(0)(\sigma(n) - d(n)) + \sum_{\substack{d\in\mathbb{N} \\ d|n}} \Big(1 + \frac{2n}{d} - d\Big) f(d) - 2 \sum_{\substack{d\in\mathbb{N} \\ d|n}} \Big(\sum_{v=1}^{d} f(v)\Big). \end{aligned} \tag{1}$$

This formula gives a main idea in the proof of Theorem 1.1, and we can get special result in Theorem 1.2. By (1), we define the following divisor functions, which can be calculated by combinatoric convolution sums([3, 4]):

$$E^*(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} a, \quad E_0^*(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ a=b}} a.$$

A $perfect\ number$ in number theory is a positive integer that is equal to the sum of its proper positive divisors, that is, the sum of its positive divisors excluding the number itself. In this paper, a positive integer $n > 1$ is called $linear - abundant$(**l.a**), $linear - perfect$(**l.p**) and $linear - deficient$(**l.d**) according as $E_0^*(n) > 2n, = 2n, < 2n$. These notations can be connected with a kind of $perfect\ number$.

Let $\mathfrak{A} = \{a \mid a \text{ is } \mathbf{l.a}\}$, $\mathfrak{P} = \{p \mid p \text{ is } \mathbf{l.p}\}$ and $\mathfrak{D} = \{d \mid d \text{ is } \mathbf{l.d}\}$. These notations are in [4].

Then, we have the following properties in this paper as the main theorem, and we get a special theorem related to the number 6.

Theorem 1.1. *Let* p_i *be distinct primes and put* $n = \prod_{i=1}^{r} p_i^{e_i}$. *Let* $\Omega(n) = \sum_{i=1}^{r} e_i$ *where* $1 \leq i \leq r$. *If* $\Omega(n) \geq 3$, *then* $n \in \mathfrak{A}$.

By Theorem 1.1, we have the following property.

Theorem 1.2. 6 *is the only* ***l.p*** *number.*

2. Proof of Theorems

To prove Theorem 1.1 and 1.2, we introduce several properties of divisor functions as they are given in the following propositions.

Proposition 2.1. ([9]) *The followings hold true:*
(*i*) σ_k *is multiplicative. That is,*

$$\sigma_s(mn) = \sigma_s(m)\sigma_s(n) \text{ with } (m,n) = 1.$$

(*ii*) *Let* p *be a prime. Let* $k, n \in \mathbb{N}$. *Then,*

$$\sigma_k(pn) - (p^k + 1)\sigma_k(n) + p^k\sigma_k(n/p) = 0.$$

Let $f : \mathbb{Z} \to \mathbb{C}$ be an even function given by $f(n) = |n|$ in (1). Then we have the following Proposition 2.2.

Proposition 2.2. ([3]) *Let* a, b, x, y *and* n *be positive integers. And if* $a = b$, *then*

$$E_0^*(n) = \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} a \;=\; n\sigma_0(n) - \sigma_1(n).$$

Let $n,\ m > 1$ and $(n,\ m) = 1$. By Proposition 2.1 and Proposition 2.2, it is easily checked that

$$E_0^*(nm) = nm\sigma_0(n)\sigma_0(m) - \sigma_1(n)\sigma_1(m).$$

Proposition 2.3. ([3, 4]) *The followings hold true:*
(*i*) *If* p *is a prime, then* $p \in \mathfrak{D}$.
(*ii*) $6 \in \mathfrak{P}$.
(*iii*) *If* p *is a prime, then* $p^2 \in \mathfrak{D}$.
(*iv*) *For any primes* p *and* q, *if* $p \neq q$, $pq \neq 6$ *and* $n = pq$, *then* $n \in \mathfrak{A}$.
(*v*) *If* p, q *and* r *are distinct primes and* $n = pqr$, *then* $n \in \mathfrak{A}$.
(*vi*) *If* p *is a prime, then* $p^3 \in \mathfrak{A}$.
(*vii*) *If* p *and* q *are distinct primes and* $n = p^2q$, *then* $n \in \mathfrak{A}$.
(*viii*) *If* $m,\ n > 1$ *where* $(m,n) = 1$ *and* $m,\ n \in \mathfrak{A}$, *then* $mn \in \mathfrak{A}$.

(ix) For any prime p, we have $p^4 \in \mathfrak{A}$.
(x) For any distinct primes p and q, we have $p^3 \cdot q \in \mathfrak{A}$.
(xi) For any distinct primes p and q, we have $p^2 \cdot q^2 \in \mathfrak{A}$.
(xii) For any distinct primes p, q and r, we have $p^2 \cdot q \cdot r \in \mathfrak{A}$.
(xiii) For any distinct primes p, q, r and s, we have $p \cdot q \cdot r \cdot s \in \mathfrak{A}$.
(xiv) Let $n = p_1 \cdot p_2 \cdots p_r$ and let p_i be distinct primes and $r \geq 3$. Then $n \in \mathfrak{A}$.
(xv) Let $n \geq 4$ be positive integers and let p be a prime. Then, $p^n \in \mathfrak{A}$.
(xvi) Let p_i be distinct primes and let $n_i \geq 4$ for $i = 1, \cdots, r$. Then, $N = \prod_{i=1}^{r} p_i^{n_i} \in \mathfrak{A}$.

Lemma 2.4. *Let p and q be distinct primes and $n \geq 4$ be a positive integer. Then $p^n q \in \mathfrak{A}$.*

Proof. Let p and q be distinct primes and $n \geq 4$ be a positive integers. Then

$$\begin{aligned} E_0^*(p^n q) &= p^n q \sigma_0(p^n)\sigma_0(q) - \sigma_1(p^n)\sigma_1(q) \\ &= 2(n+1)p^n q - (1 + p + p^2 + \cdots + p^n)(1+q). \end{aligned}$$

Assume $E_0^*(p^n q) \leq 2p^n q$. Then,

$$\begin{aligned} 2np^n q &\leq (1 + p + p^2 + \cdots + p^n)(1+q) \\ &= \Big(\frac{p^{n+1}-1}{p-1}\Big)(1+q). \end{aligned} \tag{2}$$

Multiplying $\frac{p-1}{q(p^{n+1}-1)}$ on both sides of (2), we get the following inequality.

$$2np^n \cdot \frac{p-1}{p^{n+1}-1} \leq \frac{1+q}{q} \leq \frac{3}{2}$$

and

$$\frac{np^n(p-1)}{p^{n+1}-1} \leq \frac{3}{4} \tag{3}$$

because $\frac{3}{2} \geq \frac{1+q}{q}$. On the other hand, since $n \geq 4$ and p is prime, we get the following inequality.

$$\frac{p^n(p-1)}{p^{n+1}-1} = \frac{p-1}{p-1/p^n} \geq \frac{1}{p-1/p^n} \geq \frac{1}{2-1/2^4} = \frac{16}{31}$$

and

$$\frac{np^n(p-1)}{p^{n+1}-1} \geq \frac{64}{31} > \frac{3}{4}.$$

Therefore, the equation (3) is a contradiction. Thus, $E_0^*(p^n q) > 2p^n q$. Therefore, $p^n q \in \mathfrak{A}$. □

Corollary 2.5. *Let p and q be distinct primes and $n \geq 2$. Then $p^n q \in \mathfrak{A}$.*

Proof. By Lemma 2.4 and (vii) and (x) of Proposition 2.3, it is obvious. □

Lemma 2.6. *Let p and q be distinct primes and $n \geq 3$. Then $p^n q^2 \in \mathfrak{A}$.*

Proof. Let p and q be distinct primes and $n \geq 3$ be a positive integer. Then

$$\begin{aligned} E_0^*(p^nq^2) &= p^nq^2\sigma_0(p^n)\sigma_0(q^2) - \sigma_1(p^n)\sigma_1(q^2) \\ &= 3(n+1)p^nq^2 - (1+p+p^2+\cdots+p^n)(1+q+q^2). \end{aligned}$$

Assume $E_0^*(p^nq^2) \leq 2p^nq^2$. Then,

$$3(n+1)p^nq^2 - (1+p+p^2+\cdots+p^n)(1+q+q^2) \leq 2p^nq^2$$

and

$$\begin{aligned} (3n+1)p^nq^2 &\leq (1+p+p^2+\cdots+p^n)(1+q+q^2) \\ &= \Big(\frac{p^{n+1}-1}{p-1}\Big)(1+q+q^2). \end{aligned} \tag{4}$$

Multiplying $\frac{1+q}{1+q+q^2} \cdot \frac{p-1}{p^{n+1}-1}$ on both sides of (4), we get the following;

$$(3n+1) \cdot p^n \cdot q^2 \cdot \frac{1+q}{1+q+q^2} \cdot \frac{p-1}{p^{n+1}-1} \leq 1+q \leq \frac{3}{2}q$$

because q is a prime number, that is $q \geq 2$. Simplifying the above equation, we obtain the following;

$$(3n+1) \cdot \frac{q+q^2}{1+q+q^2} \cdot \frac{p^n(p-1)}{p^{n+1}-1} \leq \frac{3}{2}. \tag{5}$$

On the other hand, since $n \geq 3$ and $p,\ q$ are primes, the following properties are obvious;

(i) $3n+1 \geq 10$,

(ii) $\frac{q+q^2}{1+q+q^2} \geq \frac{q^2}{1+q+q^2} = \frac{1}{1+1/q+1/q^2} \geq \frac{1}{1+1/2+1/2^2} = \frac{4}{7}$,

(iii) $\frac{p^n(p-1)}{p^{n+1}-1} = \frac{p-1}{p-1/p^n} \geq \frac{1}{2-1/2^3} = \frac{8}{15}$.

By (i), (ii) and (iii), the left side of the equation (5) is as follows.

$$(3n+1) \cdot \frac{q+q^2}{1+q+q^2} \cdot \frac{p^n(p-1)}{p^{n+1}-1} \geq 10 \cdot \frac{4}{7} \cdot \frac{8}{15} = \frac{64}{21}.$$

Therefore, equation (5) is a contradiction. Thus, $E_0^*(p^nq^2) > 2p^nq^2$. Therefore, $p^nq^2 \in \mathfrak{A}$. □

Corollary 2.7. *Let p and q be distinct primes and $n \geq 1$. Then $p^nq^2 \in \mathfrak{A}$.*

Proof. By Lemma 2.6 and (vii) and (xi) of Proposition 2.3, it is obvious. □

Lemma 2.8. *Let p, q and r be distinct primes and $n \geq 3$. Then $p^nqr \in \mathfrak{A}$.*

Proof. Let p, q and r be distinct primes and $n \geq 3$ be a positive integer. Then

$$\begin{aligned} E_0^*(p^nqr) &= p^nqr\sigma_0(p^n)\sigma_0(q)\sigma_0(r) - \sigma_1(p^n)\sigma_1(q)\sigma_1(r) \\ &= 4(n+1)p^nqr - (1+p+p^2+\cdots+p^n)(1+q)(1+r). \end{aligned}$$

Assume $E_0^*(p^nqr) \leq 2p^nqr$. Then,

$$4(n+1)p^nqr - (1+p+p^2+\cdots+p^n)(1+q)(1+r) \leq 2p^nqr$$

and

$$(6)\qquad \begin{aligned}(4n+2)p^nqr &\le (1+p+p^2+\cdots+p^n)(1+q)(1+r)\\ &= \Big(\frac{p^{n+1}-1}{p-1}\Big)(1+q)(1+r).\end{aligned}$$

Multiplying $\frac{p-1}{qr(p^{n+1}-1)}$ on both sides of (6), we get the following;

$$(7)\qquad (4n+2)\cdot\frac{p^n(p-1)}{p^{n+1}-1} \le \frac{1+q}{q}\cdot\frac{1+r}{r} \le \Big(\frac{3}{2}\Big)^2$$

because $\frac{3}{2}\ge\frac{1+q}{q}$ and $\frac{3}{2}>\frac{1+r}{r}$.
On the other hand, since $n\ge 3$ and p is prime, the following properties are obvious;
(i) $4n+2\ge 14$, and
(ii) $\frac{p^n(p-1)}{p^{n+1}-1}\ge\frac{8}{15}$.
By (i) and (ii), the left side of the equation (7) is as follows.

$$(4n+2)\cdot\frac{p^n(p-1)}{p^{n+1}-1}\ge\frac{112}{15}.$$

Therefore, equation (7) is a contradiction. Thus, $E_0^*(p^nqr)>2p^nqr$. Therefore, $p^nqr\in\mathfrak{A}$. □

Corollary 2.9. *Let p, q and r be distinct primes and $n\ge 1$. Then $p^nqr\in\mathfrak{A}$.*

Proof. By Lemma 2.8 and (v) and (xii) of Proposition 2.3, it is obvious. □

Lemma 2.10. *Let p, q and r be distinct primes. Then $p^2q^2r\in\mathfrak{A}$.*

Proof. Let p, q and r be distinct primes. Then

$$\begin{aligned}E_0^*(p^2q^2r) &= p^2q^2r\sigma_0(p^2)\sigma_0(q^2)\sigma_0(r)-\sigma_1(p^2)\sigma_1(q^2)\sigma_1(r)\\ &= 18p^2q^2r-(1+p+p^2)(1+q+q^2)(1+r).\end{aligned}$$

Assume $E_0^*(p^2q^2r)\le 2p^2q^2r$. Then,

$$16p^2q^2r\le(1+p+p^2)(1+q+q^2)(1+r)$$

and

$$(8)\qquad 16\cdot\frac{p^2}{(1+p+p^2)}\cdot\frac{q^2r}{(1+q+q^2)}\le\frac{(1+r)}{r}.$$

On the other hand, since $\frac{3}{2}\ge\frac{1+r}{r}$, $\frac{p^2}{1+p+p^2}\ge\frac{4}{7}$ and $\frac{q^2}{1+q+q^2}\ge\frac{4}{7}$, the left side of the equation (8) is expressed follows;

$$(9)\qquad 16\cdot\frac{p^2}{(1+p+p^2)}\cdot\frac{q^2r}{(1+q+q^2)}\ge 16\cdot\frac{4}{7}\cdot\frac{4}{7}=\frac{256}{49}.$$

Therefore, equation (9) is a contradiction. Thus, $E_0^*(p^2q^2r)>2p^2q^2r$. Therefore, $p^2q^2r\in\mathfrak{A}$. □

Lemma 2.11. *Let p, q, r and s be distinct primes. Then we can obtain the following properties: (i) $p^2qrs\in\mathfrak{A}$,*
(ii) $p^2q^2r^2\in\mathfrak{A}$ and
(iii) $p^2q^2rs\in\mathfrak{A}$.

Proof. (i) Let p, q, r and s be distinct primes. Then

$$\begin{aligned} E_0^*(p^2qrs) &= p^2qrs\sigma_0(p^2)\sigma_0(q)\sigma_0(r)\sigma_0(s) - \sigma_1(p^2)\sigma_1(q)\sigma_1(r)\sigma_1(s) \\ &= 24p^2qrs - (1+p+p^2)(1+q)(1+r)(1+s). \end{aligned}$$

Assume $E_0^*(p^2qrs) \le 2p^2qrs$. Then,

$$22p^2qrs \le (1+p+p^2)(1+q)(1+r)(1+s)$$

and

$$22 \cdot \frac{p^2}{(1+p+p^2)} \le \frac{(1+q)}{q} \cdot \frac{(1+r)}{r} \cdot \frac{(1+s)}{s}. \tag{10}$$

On the other hand, since $\frac{p^2}{1+p+p^2} \ge \frac{4}{7}$, $\frac{3}{2} \ge \frac{1+q}{q}$, $\frac{3}{2} \ge \frac{1+r}{r}$ and $\frac{3}{2} \ge \frac{1+s}{s}$, it follows from (10) that

$$\frac{88}{7} \le 22 \cdot \frac{p^2}{(1+p+p^2)} \le \frac{(1+q)}{q} \cdot \frac{(1+r)}{r} \cdot \frac{(1+s)}{s} \le \frac{27}{8}. \tag{11}$$

Therefore, equation (11) is a contradiction. Thus, $E_0^*(p^2qrs) > 2p^2qrs$. Therefore, $p^2qrs \in \mathfrak{A}$.

(ii) Let p, q and r be distinct primes. Then

$$\begin{aligned} E_0^*(p^2q^2r^2) &= p^2q^2r^2\sigma_0(p^2)\sigma_0(q^2)\sigma_0(r^2) - \sigma_1(p^2)\sigma_1(q^2)\sigma_1(r^2) \\ &= 27p^2q^2r^2 - (1+p+p^2)(1+q+q^2)(1+r+r^2). \end{aligned}$$

Assume $E_0^*(p^2q^2r^2) \le 2p^2q^2r^2$. Then,

$$25p^2q^2r^2 \le (1+p+p^2)(1+q+q^2)(1+r+r^2)$$

and

$$\frac{p^2}{(1+p+p^2)} \cdot \frac{q^2}{(1+q+q^2)} \cdot \frac{r^2}{(1+r+r^2)} \le \frac{1}{25}. \tag{12}$$

On the other hand, since $\frac{p^2}{1+p+p^2} \ge \frac{4}{7}$, $\frac{q^2}{1+q+q^2} > \frac{4}{7}$ and $\frac{r^2}{1+r+r^2} > \frac{4}{7}$, the equation (12) is rearranged as follows;

$$\frac{64}{343} \le \frac{p^2}{(1+p+p^2)} \cdot \frac{q^2}{(1+q+q^2)} \cdot \frac{r^2}{(1+r+r^2)} \le \frac{1}{25}. \tag{13}$$

Therefore, equation (13) is a contradiction. Thus, $E_0^*(p^2q^2r^2) > 2p^2q^2r^2$. Therefore, $p^2q^2r^2 \in \mathfrak{A}$.

(iii) Let p, q, r and s be distinct primes. Then

$$\begin{aligned} E_0^*(p^2q^2rs) &= p^2q^2rs\sigma_0(p^2)\sigma_0(q^2)\sigma_0(r)\sigma_0(s) - \sigma_1(p^2)\sigma_1(q^2)\sigma_1(r)\sigma_1(s) \\ &= 36p^2q^2rs - (1+p+p^2)(1+q+q^2)(1+r)(1+s). \end{aligned}$$

Assume $E_0^*(p^2q^2rs) \le 2p^2q^2rs$. Then,

$$34p^2q^2rs \le (1+p+p^2)(1+q+q^2)(1+r)(1+s)$$

and

$$34 \cdot \frac{p^2}{(1+p+p^2)} \cdot \frac{q^2}{(1+q+q^2)} \le \frac{1+r}{r} \cdot \frac{1+s}{s}. \tag{14}$$

On the other hand, since $\frac{p^2}{1+p+p^2} \geq \frac{4}{7}$, $\frac{q^2}{1+q+q^2} > \frac{4}{7}$, $\frac{1+r}{r} \leq \frac{3}{2}$ and $\frac{1+s}{s} \leq \frac{3}{2}$, the equation (14) is expressed as follows;

$$\frac{544}{49} \leq 34 \cdot \frac{p^2}{(1+p+p^2)} \cdot \frac{q^2}{(1+q+q^2)} \leq \frac{1+r}{r} \cdot \frac{1+s}{s} \leq \frac{9}{4}. \tag{15}$$

Therefore, equation (15) is a contradiction. Thus, $E_0^*(p^2q^2rs) > 2p^2q^2rs$. Therefore, $p^2q^2rs \in \mathfrak{A}$. □

Proof of Theorem 1.1. It is obtained by Proposition 2.3 $(vi) \sim (xvi)$, Corollary 2.5, Corollary 2.7, Corollary 2.9, Lemma 2.10 and Lemma 2.11. □

Proof of Theorem 1.2. We know that $6 \in \mathfrak{P}$ by Proposition 2.3. By Theorem 1.1, all numbers n satisfy $\Omega(n) \geq 3$ are elements of $\mathfrak{A}$. Also, if all numbers n satisfy that $\Omega(n) = 1$, then these numbers belong to $\mathfrak{A}$. Consequently, all numbers n belong to $\mathfrak{A}$ by (iii) and (iv) of Proposition 2.3 if $n \neq 6$ and $\Omega(n) = 2$. Thus, 6 is the only **l.p** number. □

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