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The Eccentricity Extended Energy of a Graph

Mohammad Issa Sowaity[1], B. Sharada[2] and A. M. Naji[1]
(1) Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysuru - 570 006, INDIA
mohammad_d2007@hotmail.com, ama.mohsen78@gmail.com
(2) Department of Studies in Computer Science
University of Mysore, Manasagangotri
Mysuru - 570 006, INDIA
sharadab21@gmail.com

Abstract

Let e_i be the eccentricity of the vertex v_i of a graph G, where $e_i = max\{d(u, v) : u \in V(G)\}$. In this paper, we introduce the eccentricity extended matrix $A_{eex}(G)$, so that its (i,j)-entry is equal to $\frac{1}{2}(\frac{e_i}{e_j} + \frac{e_j}{e_i})$ for $v_iv_j \in E$ and 0 otherwise. Some properties of the eccentricity extended spectral radius μ_1 are obtained. The eccentricity extended energy $E_{eex}(G)$ of G is defined. Uper and lower bounds for $E_{eex}(G)$ are established.

Keywords: Eccentricity of a vertex, eccentricity extended matrix, eccentricity extended eigenvalues, eccentricity extended energy of a graph.
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1 INTRODUCTION

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, of order $|V(G)| = n$ and size $|E(G)| = m$. Let $A = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A are the eigenvalues of the graph G [3].
Since A is a symmetric matrix with zero trace, these eigenvalues are real with sum equal to zero. The energy of the graph G is defined as the sum of the absolute values of its eigenvalues [11, 15],

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

After the introduction of the graph energy concept by Gutman in 1978 [11], several other graph energies have been put forward and their mathematical properties have been extensively studied; for details see the recent monograph [9] and the survey [8].
The extended adjacency matrix of a graph G, denoted by $A_{ex} = A_{ex}(G)$, was put forward by Yang et al. [20], and is defined so that its (i, j)-entry is equal to $\frac{1}{2}(\frac{d_i}{d_j} + \frac{d_j}{d_i})$ if $v_iv_j \in E(G)$ and 0 otherwise.
Das et al. [4] studied on spectral radius η_1 and energy of extended adjacency matrix of graphs $E_{ex}(G)$ and they obtained lower bounds on η_1 and $E_{ex}(G)$. They also characterized the respective extremal graphs.
Adiga et al. [1] gave two upper bounds for $E_{ex}(G)$. They presented a pair of extended equienergetic graphs on n vertices for $n \equiv 0(mod8)$ starting with a pair of extended equienergetic nonsingular graphs on 8 vertices. Also they constructed a pair of extended equienergetic graphs on n vertices for all $n \geq 9$ starting with a pair of equienergetic regular graphs on 9 vertices.
Gutman [7] obtained a relation between energy and extended energy of a graph. He proved that the extended graph energy of a bipartite graph is not smaller than its ordinary, and posed the conjuncture that the result holds also for non-bipartite graphs.
Motivated by these works we define the eccentricity extended matrix $A_{eex}(G)$, and study some of its properties. Some upper and lower bounds for the spectral radius of $A_{eex}(G)$ are established. We derive the eccentricity extended energy $E_{eex}(G)$ and find some upper and lower bounds for it.

In this paper, all graphs are assumed to be finite connected simple graphs. The eccentricity of a vertex $v \in G$ is $e(v) = max\{d(u, v) : u \in V(G)\}$. The radius of G is $r = r(G) = min\{e(v) : v \in V(G)\}$ and the diameter of G is $D = D(G) = max\{e(v) : v \in V(G)\}$. Hence $r(G) \leq e(v) \leq D(G)$, for every $v \in V(G)$. A vertex v in a connected graph G is central vertex if $e(v) = r(G)$, while a vertex v in a connected graph G is peripheral vertex if $e(v) = D(G)$. A graph G is called a self-centered graph if $e(v) = r(G) = D(G)$. If G is a regular graph with $e(v) = r(G) = D(G)$, then G is a regular self centered graph. We denote the eccentricity of a vertex v_i by $e(v_i) = e_i$. For graph theoretic terminology we refer to [13].

In 2012 Ghorbani and Hosseinzadeh [6] introduced the second eccentric Zagreb index as an eccentric version of second Zagreb index of the molecular graph G and defined as

$$EM_2(G) = \sum_{v_iv_j \in E} e_ie_j.$$

Nilanjan De et al. [5] studied the total eccentricity index of some specific graphs, where the total eccentricity index is defined as

$$\zeta(G) = \sum_{i=1}^{n} e_i.$$

We use some previously known results which we state in the following.

Lemma 1.1. *[14] Let $B = (b_{ij})$ and $H = (h_{ij})$ be symmetric, non-negative matrices of order n. If $B \geq H$, i.e., $b_{ij} \geq h_{ij}$ for all i, j, then $\rho_1(B) \geq \rho_1(H)$, where ρ_1 is the largest eigenvalue.*

Lemma 1.2. *[2] Let G be a graph of order n with m edges. Then*

$$\lambda_1 \geq \frac{2m}{n}$$

with equality holding if and only if G is regular graph.

Lemma 1.3. *[2] If C is a symmetric $n \times n$ matrix with eigenvalues $\rho_1, \ \rho_2, \ \ldots, \rho_n$, then for any $\mathbf{x} \in \mathbb{R}^n$, such that $\mathbf{x} \neq 0$,*

$$\mathbf{x}^T C \mathbf{x} \leq \rho_1 \mathbf{x}^T \mathbf{x}.$$

Remark 1.4. *It is easy to note that $|e_i - e_j| \leq 1$, for any two adjacent vertices v_i and v_j in a graph G.*

2 ON THE SPECTRAL RADIUS OF THE ECCENTRICITY EXTENDED MATRIX

Motivated by the definition of extended adjacency matrix, we introduce the concept of eccentricity extended adjacency matrix.

Definition 2.1. *Let G be a connected graph with n vertices and m edges. The eccentricity extended matrix $A_{eex}(G) = (c_{ij})$ is the $n \times n$ matrix, whose entries are given by*

$$c_{ij} = \begin{cases} \frac{1}{2}\left(\frac{e_i}{e_j} + \frac{e_j}{e_i}\right), & if \ v_i v_j \in E, \\ 0, & otherwise. \end{cases}$$

The eigenvalues $\mu_1, \ \mu_2, \ \ldots, \ \mu_n$ of A_{eex} are called eccentricity extended eigenvalues of G. We assume throughout that $\mu_i \geq \mu_{i+1}$. The eccentricity extended energy $E_{eex}(G)$ of G is defined by

$$E_{eex}(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since A_{eex} is real symmetric, the eigenvalues $\mu_1, \ \mu_2, \ \ldots, \ \mu_n$ of A_{eex} are real with summation equal to zero.

Remark 2.2. *If G is a self-centered graph, then $A_{eex}(G) = A(G)$ and hence the eccentricity extended eigenvalues of G are the same as the eigenvalues of G.*

Theorem 2.3. *Let G be a graph with n vertices and m edges. Then*

$$A_{eex}(G) \geq A(G),$$

with equality holds if and only if G is a self-centered graph.

Proof. Let $v_iv_j \in E(G)$. Since c_{ij} is of the form $\frac{1}{2}(a + \frac{1}{a})$ where $a = \frac{e_i}{e_j}$, it follows that $c_{ij} \geq 1$ and equality holds if and only if $a = 1$.
Hence the result follows. □

Theorem 2.4. *Let G be a graph with n vertices and m edges. Then*

$$\mu_1 \geq \frac{2m}{n},$$

and equality holds if and only if G is a regular self centered graph.

Proof. From Theorem 2.3, and using the Lemmas 1.1 and 1.2, we get the wanted result. To show the equality, if we assume that G is a regular graph with $e_i = e_j$ for all $j = 1, 2, ..., i-1, i+1, ..., n$. Then $c_{ij} = 1$ for all $v_iv_j \in E$.
Hence $A_{eex}(G) = A(G)$, and since the graph G is regular, then by using Lemma 1.2, we get

$$\mu_1 = \lambda_1 = \frac{2m}{n}.$$

On the other hand, let $\mu_1 = \frac{2m}{n}$, we consider the following cases.
Case 1. G is neither self-centered nor regular graph.
By Theorem 2.3 and Lemma 1.1 and 1.2

$$\mu_1 \geq \lambda_1 > \frac{2m}{n}.$$

Case 2. G is self centered not regular graph.
By Remark 2.2 and Lemma 1.2

$$\mu_1 = \lambda_1 > \frac{2m}{n}.$$

Case 3. G is self-centered and regular graph.
By using Remark 2.2 and Lemma 1.2

$$\mu_1 = \lambda_1 = \frac{2m}{n}.$$

□

In the following theorem we give another lower bound for μ_i.

Theorem 2.5. *Let G be a graph with n vertices and m edges. Then*

$$\mu_1 \geq \frac{2m}{n} + \frac{1}{nD^2}[\sum_{i=1}^{n}(e_i^2 d_i) - 2EM_2(G)].$$

The bound is sharp and any regular self-centered graph satisfies it.

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ be a unit vector. Then

$$\begin{aligned}
\mathbf{x}^T A_{eex}\mathbf{x} &= \sum_{v_iv_j \in E} [(\frac{e_i}{e_j} + \frac{e_j}{e_i})x_i x_j] \\
&= \sum_{v_iv_j \in E} \frac{e_i^2 + e_j^2}{e_i e_j} x_i x_j \\
&= \sum_{v_iv_j \in E} \frac{e_i^2 + e_j^2 - 2e_ie_j + 2e_ie_j}{e_i e_j} x_i x_j \\
&= \sum_{v_iv_j \in E} \frac{e_i^2 + e_j^2 - 2e_ie_j}{e_i e_j} x_i x_j + 2\sum_{v_iv_j \in E} x_i x_j \\
&\geq 2\sum_{v_iv_j \in E} x_i x_j + \sum_{v_iv_j \in E} \frac{e_i^2 + e_j^2 - 2e_ie_j}{D^2} x_i x_j.
\end{aligned}$$

If we put $\mathbf{x}^T = (\frac{1}{\sqrt{n}},\ \frac{1}{\sqrt{n}},\ ...,\ \frac{1}{\sqrt{n}})$, we get

$$\begin{aligned}
\mathbf{x}^T A_{eex}\mathbf{x} &\geq \frac{2m}{n} + \sum_{v_iv_j \in E} \frac{e_i^2 + e_j^2 - 2e_ie_j}{nD^2} \\
&= \frac{2m}{n} + \frac{1}{nD^2}[\sum_{v_iv_j \in E}(e_i^2 + e_j^2) - 2EM_2(G)] \\
&= \frac{2m}{n} + \frac{1}{nD^2}[\sum_{i=1}^{n}(e_i^2 d_i) - 2EM_2(G)].
\end{aligned}$$

Using Lemma 1.3, we can get the result. □

Corollary 2.6. *If G is a graph on n vertices and m edges, then*

$$\mu_1 \geq \frac{2m}{n} + \frac{1}{nD^2}[\delta\zeta(G) - 2EM_2(G)],$$

where $\zeta(G)$ is the total eccentricity index.

Proof. Since $d_i,\ e_i \geq 1$ for all $i = 1,\ 2,\ ...,\ n$,

$$\sum_{i=1}^{n} e_i^2 d_i \geq \sum_{i=1}^{n} e_i d_i.$$

Since $d_i \geq \delta$ for all $i = 1,\ 2,\ ...,\ n$,

$$\sum_{i=1}^{n} e_i d_i \geq \delta \sum_{i=1}^{n} e_i$$
$$= \delta\zeta(G).$$

Hence the result follows. □

Theorem 2.7. *Let G be a graph with n vertices and m edges. Then*

$$\mu_1 \leq \frac{5}{4}(n-1).$$

Proof. From Remark 1.4, $|e_i - e_j| \leq 1$, for all $v_i v_j \in E$.
Hence we consider the following cases.
Case 1. $e_i - e_j = 0$ for all $v_i v_j \in E$.
In this case $A_{eex}(G) = A(G)$; which leads to

$$\mu_1 = \lambda_1 \leq \frac{5}{4}(n-1).$$

Case 2. $|e_i - e_j| = 1$ for some $v_i v_j \in E$.
Assume, without loss of generality, that $e_i = e_j + 1$, which implies that the entries of $A_{eex}(G) = (c_{ij})$ must be

$$c_{ij} = \frac{1}{2}\left(\frac{e_i}{e_j} + \frac{e_j}{e_i}\right) = \frac{e_i^2 + e_j^2}{2e_i e_j}$$
$$= \frac{(e_j+1)^2 + e_j^2}{2e_j(e_j+1)}$$
$$= \frac{e_j^2 + 2e_j + 1 + e_j^2}{2e_j(e_j+1)}$$
$$= \frac{2e_j(e_j+1)+1}{2e_j(e_j+1)}.$$

Let $f(x) = \frac{2x(x+1)+1}{2x(x+1)}$. Easy calculations give $f(x)$ is decreasing for $x \geq \frac{1}{2}$ and since we are concern on discrete numbers, we get that the maximum c_{ij} holds when $e_j = 1$ and $e_i = 2$ for all $v_i v_j \in E$. Thus

$$c_{ij} \leq \frac{5}{4}. \quad (1)$$

Hence $1 \leq c_{ij} \leq \frac{5}{4}$ *for all* $v_i v_j \in E$. Using Lemma 1.1, we get

$$\mu_1 \leq \rho_1(B),$$

where B is the matrix that has all its entries $\frac{5}{4}$ other than the diagonal which are all zeros. Since $\rho_1(B) = \frac{5}{4}(n-1)$ we get the wanted result.

□

3 THE ECCENTRICITY EXTENDED ENERGY OF A GRAPH AND SOME OF ITS BOUNDS

The eccentricity extended energy of a graph $E_{eex}(G) = E_{eex}$, is defined to be the summation of the absolute values of the eigenvalues of the eccentricity extended matrix $A_{eex}(G)$. We present here some results and bounds related to E_{eex}.

Theorem 3.1. *Let G be a graph with n vertices and m edges. Then*

$$E_{eex}(G) \leq (\frac{D}{r})^2\sqrt{2mn},$$

where $D = diam(G),\ r = rad(G)$ are the diameter and radius of G.

Proof. Since $r \leq e_i \leq D$, it follows that

$$c_{ij} = \frac{e_i^2 + e_j^2}{2e_ie_j} \leq \frac{D^2}{r^2}.$$

We have

$$\sum_{i=1}^{n} \mu_i^2 = trace(A_{eex}^2(G)) \leq 2m(\frac{D^2}{r^2})^2.$$

Using Cauchy Schwartz Inequality, we get

$$\begin{aligned}\sum_{i=1}^{n} |\mu_i| &\leq \sqrt{n\sum_{i=1}^{n} \mu_i^2} \\ &\leq \sqrt{2mn(\frac{D^2}{r^2})^2} \\ &= \frac{D^2}{r^2}\sqrt{2mn}.\end{aligned}$$

Hence

$$E_{eex}(G) \leq \frac{D^2}{r^2}\sqrt{2mn}.$$

□

Corollary 3.2. *If G is a self centered graph, then*

$$E_{eex}(G) = \mathcal{E}(G) \leq \sqrt{2mn}.$$

Theorem 3.3. *Let G be a graph with m edges. Then*

$$\sum_{v_iv_j \in E} c_{ij} \leq \frac{5m}{2},$$

and equality holds if and only if $G = K_{1,n-1}$.

Proof. Let G be a graph with m edges. Then by using Equation 1, we get

$$\sum_{v_iv_j \in E} c_{ij} \leq \frac{5}{4}(2m) = \frac{5m}{2}.$$

To show the equality, let $G = K_{1,n-1}$, then the entries $c_{ij} = \frac{5}{4}$ for all $v_iv_j \in E$, and zero for the nonadjacent vertices and the diagonal of $A_{eex}(G)$. Thus the result follows.
On the other hand, let $\sum_{v_iv_j \in E} c_{ij} = \frac{5m}{2}$. Then $c_{ij} = \frac{5}{4}$, for all $v_iv_j \in E$. Assume without loss of generality, that $e_i > e_j$. Then $e_i = 2$ and $e_j = 1$. Otherwise, if $e_i \geq 3$, then $c_{ij} < \frac{5}{4}$ for some $v_iv_j \in E$, a contradiction.
Let G be a connected graph such that $e_i = 2,\ e_j = 1$ for all $v_iv_j \in E$, no two vertices have the same eccentricity are adjacent. Let

$$V_1 = \{v_i \in V \ : \ e_i = 1\}$$

$$V_2 = \{v_i \in V \ : \ e_i = 2\}.$$

We claim that $|V_1| \leq 1$.
To show the claim we let $|V_1| = n_0 \geq 2$, with $G[V_1] = \overline{K}_{n_0}$ and $G[V_2] = \overline{K}_{n-n_0}$.
Take $u, v \in V_1$, since G is connected, then there is a path between u and v. Since $uv \notin E$, then $d(u, v) \geq 2$.
So the eccentricity $e(u) \geq 2$, which contradict that $u \in V_1$. Hence $|V_1| \leq 1$.
So G must be a graph with $V_1 = \{u\}$ for some $u \in V$, and V_2 be all other vertices in G such that $G[V_2] = \overline{K}_{n-1}$.
Since G is connected, then all the vertices in V_2 are adjacent to u.
Hence G is a star. □

Theorem 3.4. *Let G be a graph with m edges. Then*

$$E_{eex}(G) \leq \frac{5m}{2}.$$

Proof. The Girshgourin disc theorem gives

$$E_{eex}(G) \leq \sum_{v_iv_j \in E} c_{ij}.$$

Thus by using Theorem 3.3, we get the result. □

Theorem 3.5. *Let G be a nonsingular graph with n vertices and m edges. Then*

$$E_{eex}(G) \geq \frac{2m}{n} + n - 1 + ln|det(A_{eex})| - ln\frac{2m}{n}.$$

Proof. Since G is nonsingular graph and $|\mu_i| > 0$ for all $i = 1, 2, ..., n$.
If we consider the function $f(x) = x - 1 - ln\ x$. Then easy calculations give $f(x)$ is decreasing on $0 < x \leq 1$ and is increasing when $x > 1$. Also we have $f(1) = 0$, so

$$f(x) \geq 0,\ for\ x > 0.$$

Applying $f(x)$ on E_{eex}, we have

$$\begin{aligned}
E_{eex}(G) &= \sum_{i=1}^{n} |\mu_i| \\
&= \mu_1 + \sum_{i=2}^{n} |\mu_i| \\
&\geq \mu_1 + \sum_{i=2}^{n} (1 + ln\ |\mu_i|) \\
&= \mu_1 + n - 1 + \sum_{i=2}^{n} ln\ |\mu_i| \\
&= \mu_1 + n - 1 + ln|\prod_{i=2}^{n} \mu_i| \\
&= \mu_1 + n - 1 + ln|\frac{\prod_{i=1}^{n} \mu_i}{\mu_1}| \\
&= \mu_1 + n - 1 + ln\ |\prod_{i=1}^{n} \mu_i| - ln\ \mu_1 \\
&= \mu_1 + n - 1 + ln\ |det(A_{eex}(G)| - ln\ \mu_1.
\end{aligned}$$

Letting $g(x) = x + n - 1 + ln\ |A_{eex}(G)| - ln\ x$. Easily we can get that $g(x)$ is increasing function for $x > 0$.
Using Theorem 3.4, we get

$$g(\mu_1) \geq g(\frac{2m}{n}).$$

Hence

$$E_{eex}(G) \geq \frac{2m}{n} + n - 1 + ln|det(A_{eex})| - ln\frac{2m}{n}.$$

□

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