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Independent metric dimension of Möbius ladders

B. Suganya and S. Arumugam*
National Centre for Advanced Research in Discrete Mathematics
Kalasalingam University
Anand Nagar, Krishnankoil-626 126, Tamil Nadu, India.
e-mail: suganyaptj@gmail.com, s.arumugam.klu@gmail.com

Abstract

Let $G = (V, E)$ be a connected graph. The distance $d(u, v)$ between two vertices $u, v \in V$ is the length of a shortest u-v path. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a subset of V with an order imposed on it. For $v \in V$, the vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$ is called the metric representation of v with respect to W. If $r(v|W) \neq r(u|W)$ for any two distinct vertices $u, v \in V$, then W is called a resolving set of G. The minimum cardinality of a resolving set of G is called the metric dimension of G and is denoted by $dim(G)$. A subset W of V is called an independent resolving set for G if W is both independent and resolving. The minimum cardinality $indim(G)$ of an independent resolving set in G is called the independent metric dimension of G. In this paper we show that for Möbius ladders, the independent metric dimension and the metric dimension are equal.

Keywords : Resolving set, Metric dimension, Independent resolving set, Independent metric dimension, Möbius ladders.

*Also at Department of Computer Science, Liverpool Hope University, Liverpool, UK; Department of Computer Science, Ball State University, USA; Department of Mathematics, Bits-Pilani K.K. Birla Goa Campus, Goa

1 Introduction

By a graph $G = (V, E)$ we mean a finite, connected and undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest path between u and v. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a subset of V with an order imposed on it and let $v \in V$. The representation $r(v \mid W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is called a resolving set for G if for any two distinct vertices $u, v \in V$, we have $r(u \mid W) \neq r(v \mid W)$. A resolving set of minimum cardinality is called a basis for G. The number of vertices in a basis for G is called the metric dimension of G and is denoted by $dim(G)$. Slater [12] introduced these ideas and used the term locating set and location number instead of resolving set and metric dimension. Harary and Melter [6] independently introduced these concepts and used the term metric dimension. For a survey of results in metric dimension we refer to Chartrand and Zhang [4].

Several types of resolving sets have been investigated by imposing conditions on the subgraph induced by a resolving set. Saenpholphat and Zhang [8, 9, 10] introduced the concept of connected resolvability [11]. Chitra and Arumugam [5] introduced the concept of resolving sets without isolated vertices.

Chartrand et.al [3] introduced the concept of independent resolving number of a graph.

Definition 1.1. [3] *A subset $W \subseteq V$ in a connected graph G which is both a resolving set and an independent set is called an independent resolving set of G. The minimum cardinality of an independent resolving set of G is called the independent resolving number of G and is denoted by $indim(G)$.*

Definition 1.2. [1] *Let $n \geq 4$ be an even integer. The Möbius ladder M_n is the graph obtained from the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ by adding the edges $v_i v_{i+\frac{n}{2}}$ where $1 \leq i \leq n/2$.*

Ali et al. [1] attempted to prove that for any even positive integer $n \geq 4$, $dim(M_n) = 3$ when $n \equiv 2 (\text{mod } 8)$ and $3 \leq dim(M_n) \leq 4$

otherwise. Munir et al. [7] pointed out a mistake in their proof and proved the following theorem.

Theorem 1.3. [7] *Let $n \geq 8$ be an even integer.The metric dimension of Möbius ladders M_n is given by*

$$dim(M_n) = \begin{cases} 3 & if\ n \equiv 0\ or\ 4 \pmod 8 \\ 4 & if\ n \equiv 2\ or\ 6 \pmod 8 \end{cases}$$

It was observed in [3] that there are several families of graphs such as $K_n, n \geq 3, K_{m,n}$, where $m, n \geq 2$ and C_4 which do not have independent resolving sets. They have also determined the value of $indim(G)$ for several families of graphs, such as complete multipartite graphs, cycles, trees, unicyclic graphs and wheels. Thus there are infinite families of graphs which do not have independent resolving sets and also several infinite families of graphs which have independent resolving sets. Further characterization of graphs which admit an independent resolving set is an open problem. The resolving sets for Möebius ladders given in the proof of Theorem 1.3 are not independent. Hence the following problem arises naturally.

Problem 1.4. *Does there exist an independent resolving set for Möbius ladders M_n ?*

In this paper we investigate the above problem.

2 Main Results

Theorem 2.1. *Let $n \geq 8$ be an even integer. The independent resolving number of Möbius ladder M_n is given by*

$$indim(M_n) = \begin{cases} 3 & if\ n \equiv 0\ or\ 4 \pmod 8 \\ 4 & if\ n \equiv 2\ or\ 6 \pmod 8 \end{cases}$$

Proof. **Case 1:** $n = 8$.

Let $W = \{v_1, v_3, v_6\}$. Clearly W is independent. Now $r(v_2 \mid W) = (1,1,1), r(v_4 \mid W) = (2,1,2), r(v_5 \mid W) = (1,2,1), r(v_7 \mid W) = (2,1,1)$ and $r(v_8 \mid W) = (1,2,2)$. Hence it follows that W is an independent resolving set and $indim(M_8) = dim(M_8) = 3$.

Case 2: $n = 10$.

Let $W = \{v_1, v_3, v_5, v_7\}$. Clearly W is an independent set. Now $r(v_2 \mid W) = (1,1,3,1), r(v_4 \mid W) = (3,1,1,3), r(v_6 \mid W) = (1,3,1,1)$, $r(v_8 \mid W) = (3,1,3,1), r(v_9 \mid W) = (2,2,2,2)$ and $r(v_{10} \mid W) = (1,3,1,3)$. Hence it follows that W is an independent resolving set and $indim(M_{10}) = dim(M_{10}) = 4$.

Case 3: $n = 12$.

Let $W = \{v_1, v_3, v_8\}$. Clearly W is an independent set. Now $r(v_2 \mid W) = (1,1,1), r(v_4 \mid W) = (3,1,3), r(v_5 \mid W) = (3,2,3), r(v_6 \mid W) = (2,3,2), r(v_7 \mid W) = (1,3,1), r(v_9 \mid W) = (3,1,1), r(v_{10} \mid W) = (3,2,2), r(v_{11} \mid W) = (2,3,3)$ and $r(v_{12} \mid W) = (1,3,3)$. Hence it follows that W is an independent resolving set and $indim(M_{12}) = dim(M_{12}) = 3$.

Case 4: $n = 14$.

Let $W = \{v_1, v_3, v_7, v_9\}$. Clearly W is an independent set. Now $r(v_2 \mid W) = (1,1,3,1), r(v_4 \mid W) = (3,1,3,3), r(v_5 \mid W) = (4,2,2,4)$, $r(v_6 \mid W) = (3,3,1,3), r(v_8 \mid W) = (1,3,1,1), r(v_{10} \mid W) = (3,1,3,1)$, $r(v_{11} \mid W) = (4,2,4,2), r(v_{12} \mid W) = (3,3,3,3), r(v_{13} \mid W) = (2,4,2,4)$ and $r(v_{14} \mid W) = (1,3,1,3)$. Hence it follows that W is an independent resolving set and $indim(M_{14}) = dim(M_{14}) = 4$.

Case 5: $n \equiv 0$ or $4 (\text{mod } 8)$ and $n \neq 8, 12$.

Let $k = \frac{n}{8}$ if $n \equiv 0 (\text{mod } 8)$ and $k = \frac{n+4}{8}$ if $n \equiv 4 (\text{mod } 8)$. Let $W = \{v_1, v_3, v_{\frac{n}{2}+2}\}$. Clearly W is an independent set. The metric representation for the vertices of M_n are given below.

$$r(v_i \mid W) = \begin{cases} (0,2,2) & \text{if } i = 1 \\ (1,1,1) & \text{if } i = 2 \\ (i-1, i-3, i-1) & \text{if } 3 \leq i \leq 2k+1 \\ (\frac{n}{2} - i + 2, i-3, \frac{n}{2} - i + 2) & \text{if } 2k+2 \leq i \leq 2k+3 \\ (\frac{n}{2} - i + 2, \frac{n}{2} - i + 4, \frac{n}{2} - i + 2) & \text{if } 2k+4 \leq i \leq \frac{n}{2} + 1 \\ (2,2,0) & \text{if } i = \frac{n}{2} + 2 \\ (i - \frac{n}{2}, i - \frac{n}{2} - 2, i - \frac{n}{2} - 2) & \text{if } \frac{n}{2} + 3 \leq i \leq \frac{3n}{4} \\ (n - i + 1, i - \frac{n}{2} - 2, i - \frac{n}{2} - 2) & \text{if } \frac{3n}{4} + 1 \leq i \leq \frac{3n}{4} + 2 \\ (n - i + 1, n - i + 3, n - i + 3) & \text{if } \frac{3n}{4} + 3 \leq i \leq n \end{cases}$$

Clearly no two vertices of M_n have the same metric representation. Hence $indim(M_n) \leq 3$. Also it follows from Theorem 1.3 that $indim(M_n) \geq dim(M_n) = 3$. Thus $indim(M_n) = 3$.

Case 6: $n \equiv 2$ or $6 (\text{mod } 8)$ and $n \neq 10, 14$.

Let $k = \frac{n-2}{8}$ if $n \equiv 2 (\text{mod } 8)$ and $k = \frac{n+2}{8}$ if $n \equiv 6 (\text{mod } 8)$. Let $W = \{v_1, v_3, v_{\frac{n}{2}}, v_{\frac{n}{2}+2}\}$. Clearly W is an independent set . The metric representation for the vertices of M_n are given below.

$$r(v_i \mid W) = \begin{cases} (0,2,2,2) & \text{if } i=1 \\ (1,1,3,1) & \text{if } i=2 \\ (i-1,i-3,i+1,i-1) & \text{if } 3 \le i \le 2k \\ (i-1,i-3,\frac{n}{2}-i,i-1) & \text{if } 2k+1 \le i \le 2k+2 \\ (\frac{n}{2}-i+2,i-3,\frac{n}{2}-i,\frac{n}{2}-i+2) & \text{if } 2k+3 \le i \le 2k+4 \\ (\frac{n}{2}-i+2,\frac{n}{2}-i+4,\frac{n}{2}-i,\frac{n}{2}-i+2) & \text{if } 2k+5 \le i \le \frac{n}{2} \\ (i-\frac{n}{2},\frac{n}{2}-i+4,i-\frac{n}{2},\frac{n}{2}-i+2) & \text{if } \frac{n}{2}+1 \le i \le \frac{n}{2}+2 \\ (i-\frac{n}{2},i-2-\frac{n}{2},i-\frac{n}{2},i-2-\frac{n}{2}) & \text{if } \frac{n}{2}+3 \le i \le \lceil\frac{3n}{4}\rceil \\ (n-i+1,i-\frac{n}{2}-2,n-i+1,i-\frac{n}{2}-2) & \text{if } \lceil\frac{3n}{4}\rceil+1 \le i \le \lceil\frac{3n}{4}\rceil+2 \\ (n-i+1,n-i+3,n-i+1,n-i+3) & \text{if } \lceil\frac{3n}{4}\rceil+3 \le i \le n \end{cases}$$

Clearly no two vertices of M_n have the same metric representation. Hence $indim(M_n) \le 4$ and also it follows from Theorem 1.3 that $indim(M_n) \ge dim(M_n) = 4$. Thus $indim(M_n) = 4$. □

3 Conclusion and Scope

Since there exist infinite families of graphs such as K_n where $n \ge 3$ and $K_{r,s}$ where $2 \le r \le s$, which do not admit independent resolving sets, the following problem arises naturally.

Problem 3.1. *Characterize the class of graphs which admit an independent resolving set.*

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