Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 2. pp. 327 – 330

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.2.327

NOT NECESSARILY CONTINUOUS LOCALLY BOUNDED FINITE-DIMENSIONAL INDECOMPOSABLE PSEUDOREPRESENTATIONS OF CONNECTED LIE GROUPS

A. I. Shtern

Abstract. We describe the structure of indecomposable locally bounded finite-dimensional pseudorepresentations of connected Lie groups.

§ 1. Introduction

The structure of irreducible locally bounded finite-dimensional pseudorepresentations of connected Lie groups was described in [1]. In the present paper, using results of [2–4], we discuss the structure of indecomposable locally bounded finite-dimensional pseudorepresentations of connected Lie groups.

§ 2. Preliminaries

The finite-dimensional quasirepresentations of groups have the following general structure.

Theorem 1 [4]**.** *Let G be a group and let T be a quasirepresentation of the group G in a finite-dimensional vector space E_T. Let E_T^* be the space dual to E_T. Let L be the set of vectors $\xi \in E_T$ such that the orbit $\{T(g)\xi \mid g \in G\}$ is bounded in E; let M be the set of functionals $f \in E_T^*$ such that the orbit $\{T(g)^* f \mid g \in G\}$ is bounded in E_T^*. In this case, both the set L and the annihilator $M^\perp$ of M are T-invariant vector subspaces of E_T. Consider the*

2010 *Mathematics Subject Classification.* Primary 22A99 Secondary 22A25, 22A10.
Key words and phrases. Connected Lie group, locally bounded, representation, pseudorepresentation, continuity conditions.

ascending family of subspaces $\{0\}$, $L \cap M^{\perp}$, $M^{\perp}$, $L + M^{\perp}$, *and* $E = E_T$ *and write out the matrix* $t(g)$ *of the operator* $T(g)$, $g \in G$, *in the block form corresponding to the decomposition of the space* E *into the direct sum of the subspaces* $L \cap M^{\perp}$, $M^{\perp} \setminus (L \cap M^{\perp})$, $L \setminus (L \cap M^{\perp})$, *and* $E \setminus (L + M^{\perp})$, *where the symbol* "$\setminus$" *means the passage to a complementary subspace,*

$$(1) \qquad t(g) = \begin{pmatrix} \alpha(g) & \varphi(g) & \sigma(g) & \tau(g) \\ 0 & \beta(g) & 0 & \rho(g) \\ 0 & 0 & \gamma(g) & \chi(g) \\ 0 & 0 & 0 & \delta(g) \end{pmatrix}, \qquad g \in G.$$

(*Here we have* $t_{23}(g) = 0$ *because* L *is* T*-invariant.*) *Then the following assertions hold*:

(1) *the mappings* α, δ, γ, σ, *and* χ *are bounded*;

(2) *the matrix-valued mappings* t_1 *and* t_2 *defined by the relations*

$$t_1(g) = \begin{pmatrix} \alpha(g) & \varphi(g) \\ 0 & \beta(g) \end{pmatrix}$$

and

$$t_2(g) = \begin{pmatrix} \beta(g) & \rho(g) \\ 0 & \delta(g) \end{pmatrix}$$

are representations of G;

(3) *the mapping* τ *is a quasicocycle with respect to the representations* t_1 *and* t_2, *i.e., the mapping*

$$(g, h) \mapsto \tau(gh) - \alpha(g)\tau(h) - \varphi(g)\rho(h) - \tau(g)\delta(h), \qquad g, h \in G,$$

is bounded.

§ 3. Main results

Theorem 2. *Let* G *be a connected Lie group, let* R *be the radical of* G, *and let* L *be a Levi subgroup of* G. *Let* π *be an indecomposable locally bounded pseudorepresentation of* G *in a finite-dimensional complex vector space* E. *Then the following cases are possible*:

(1) π *is a product of a continuous unitary finite-dimensional irreducible representation of* L *which is trivial on the noncompact part of* L *by a* (*not necessarily continuous*) *central character* χ *of* R (*i.e.,*

$$\chi(grg^{-1}) = \chi(r)$$

for every $r \in R$ *and* $g \in G$);

(2) π *is an ordinary indecomposable* (*ordinary*) *representation of* G *having no nonzero bounded orbits in* E *and the dual space of* E;

(3) π *is a product of a continuous unitary finite-dimensional irreducible representation of* L *which is trivial on the noncompact part of* L *by a* (*not necessarily continuous*) *central character of* R *and by a nontrivial one-dimensional pseudorepresentation of* L (*provided that the center of* L *is infinite*); *this pseudorepresentation is of the form*

$$g \mapsto \exp(i\chi(g)), \qquad g \in G,$$

where χ *is a nontrivial Guichardet–Wigner pseudocharacter on* G (*see* [4], *Definitions* 2.5.12 *and* 3.3.13).

Proof. By Theorem 1, since π is indecomposable, it follows that π coincides with one of the pseudorepresentations α, β, γ, and δ. Since α and δ are ordinary bounded representations it follows that these representations are trivial on the noncompact part of L. Therefore, both the representations can be regarded as representations of a compact extension of L. This extension is amenable; therefore, we may assume that α and δ are unitary. Therefore, these representations are direct sums of irreducible representations. Since they are indecomposable, it follows that they are irreducible, and the result follows from the theorem in [1].

If π coincides with the unbounded representation β with unbounded dual, then the assertion is immediate.

By Theorem 3.3.17 of [4], part 2, if the group G (or L) has a nontrivial Hermitian symmetric quotient group, then the mapping γ is a direct sum Γ of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G, some one-dimensional Guichardet–Wigner pseudorepresentations (i.e., mappings of the form

$$g \to \exp(i\chi(g)), \qquad g \in G,$$

where χ stands for a Guichardet–Wigner pseudocharacter on G), and some G-central unitary characters of the group R and, if the group G has no nontrivial Hermitian symmetric quotient groups, then the mapping γ is the direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G and some G-central unitary characters of the group R. This completes the proof of the last case of the theorem, and thus of the whole theorem.

§ 4. Comments

It would be of interest to find the structure of indecomposable locally bounded finite-dimensional pseudorepresentations for a more general class of groups.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN), theme NIR 0065-2018-0004.

References

1. A. I. Shtern, *Irreducible locally bounded finite-dimensional pseudorepresentations of connected locally compact groups*, Russ. J. Math. Phys. **25** (2018 issue2), 239–240.
2. A. I. Shtern, *Locally bounded finally precontinuous finite-dimensional quasirepresentations of connected locally compact groups*, Mat. Sb. **208** (2017), no. 10, 149–170; English transl., Sb. Math. **208** (2017), no. 10, 1557–1576.
3. A. I. Shtern, *Continuity Conditions for Finite-Dimensional Locally Bounded Representations of Connected Locally Compact Groups*, Russ. J. Math. Phys. **25** (2018), no. 5, 345–382.
4. A. I. Shtern, *Finite-Dimensional Quasirepresentations of Connected Lie Groups and Mishchenko's Conjecture*, Fundam. Prikl. Mat. **13** (2007), no. 7, 85–225; J. Math. Sci. (N.Y.) **159** (2009), no. 5, 653–751.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis (FGU FNTs NIISI RAN),
Russian Academy of Sciences,
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru