Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 2. pp. 305 – 325

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.2.305

ON METRIC REGULARITY OF COMPOSED MULTIMAPS

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Abstract. This paper concerns metric regularity of composition of set-valued maps between metric spaces. The study is focused on local properties using techniques based on several important tools like error bounds, the Ekeland variational principle and a new concept of local composition stability of multimaps. We give some new results and we provide an application to best proximity points.

2000 Mathematics Subject Classification. 47H04, 49J53, 90C30.

Keywords. Error bound, generalized equations, linear openness, metric regularity, pseudo-Lipschitz property.

1. Introduction

The theory of metric regularity has been significantly developed and has become a valuable concept in its own right one motivated by concrete applications in various research such as optimization, variational inequality theory, control theory and mathematical economics. Metric regularity has been extensively studied in varying degrees of generality by many authors leading, with appropriate tools techniques, to various and complementary results, such as implicit function and inversion theorems, stability and sensitivity analysis of generalized equations. The reader is referred to the works [1, 5, 4, 3, 8, 9, 11, 10, 12, 13, 25, 21, 19, 17, 15, 18, 16, 22, 23, 27, 28, 31, 32, 33, 34, 35, 36, 38, 37, 43, 44, 49, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60,], and the references given therein for many theoretical results on the metric regularity as well as its various applications.

Metric regularity is also a powerful framework for convergence of numerical methods for solving problems in nonlinear analysis. In fact, as explained in [21, p. 164], the aim of the metric regularity concept is to give an estimate for how far a point x is from being a solution to the generalized equation $y \in F(x)$ in terms of the residual $d(y, F(x))$. Namely, given $\bar{x}$ a solution of the inclusion $\bar{y} \in F(\bar{x})$ with $F : X \rightrightarrows Y$ metrically regular around $(\bar{x}, \bar{y})$ that is, for some constants $\delta, \rho, \tau > 0$, one has

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \tag{1}$$

for any (x, y) such that $d(x, \bar{x}) < \delta$ and $d(y, \bar{y}) < \rho$. Consider x_a and y_a be approximations to $\bar{x}$ and $\bar{y}$, respectively. Then from (1), the distance from x_a to the set of solutions of the inclusion $y_a \in F(x)$ is bounded by the constant τ times the residual $d(y_a, F(x_a))$. In applications, the residual is typically easy to compute or estimate, whereas finding a solution might be considerably more difficult. Metric regularity says that there exists a solution to the inclusion $y_a \in F(x)$ at distance from x_a proportional to the residual. In particular, if we know the rate of convergence of the residual to zero, then we will obtain the rate of convergence of approximate solutions to an exact one.

Recently, the study of the regularity and linear openness properties of the sum of set-valued maps on Banach space has been tackled (see, [2, 4, 6, 49, 23, 35, 36, 20, 18, 29, 24, 47, 48, 59, 60, 61]). The authors in [4], proved that if F is metrically regular and if we perturb F by a mapping $g : (x,p) \to g(x,p)$, Lipschitz with respect to x, uniformly in p, with a suitable Lipschitz constant, then the perturbed mapping $F(\cdot) + g(\cdot,p)$ is metrically regular for every p near a refferent point $\bar{p}$. And when we perturb a metrically regular multifunction F by a set-valued mapping G which is Lipschitz-like, the perturbed mapping $F+G$ fails in general to be metrically regular. However, if the so-called "sum-stable" property introduced in [2, 49, 23] holds, then metric regularity as well as the Aubin property of $F + G$ remains. Afterwards and since the sum of multimaps is a special case of composition of multimaps; besides, when dealing with the sum, it is implicitly assumed that the image space is a linear space, endowed with a shift-invariant metric; the attention has been drawn to the general setting of composed multimaps defined on metric spaces ([25, 23, 27, 28, 7, 62, 19, 20]). For instance in [23], the authors presented a result concerning the metric regularity of the composition of set-valued mappings in metric spaces. They obtained local results via techniques based on the theory of error bounds and they also gave some applications to generalized variational systems.

The purpose of this paper is to present some results concerning the metric regularity of the composition of multimaps acting between metric spaces. The approach we follow is focused on local properties and it is based on the Ekeland variational principle as in [23] (see also [49]) but with different assumptions.

The paper is organized as follows: In Section 2, we present useful notations and definitions related to hemi-continuity, metric regularity, openness and Aubin continuity concepts. In Section 3, we introduce a new property which plays an important role in our upcoming results we give sufficient conditions to ensure this property. And finally, Section 4 is devoted to the main result and we provide an application to best proximity points.

2. Preliminaries

In this section, we present some essential definitions and properties from set-valued analysis that will be used throughout this paper. Let X be a metric space. Given $\bar{x} \in X$, $r > 0$, we denote by $B(\bar{x},r)$ and $\bar{B}(\bar{x},r)$ the open and closed balls with center $\bar{x}$ and radius r, respectively. Given two subsets A, B of X, the excess of A over B is defined by the formula

$$e(A,B) := \sup_{a\in A} d(a,B)$$

with $d(a,B) := \inf_{b\in B} d(a,b)$ is the distance from a to B with the usual convention $e(\varnothing, B) = 0$ and $e(A,\varnothing) = +\infty$ if $A \neq \varnothing$. The gap between A and B is defined as

$$\text{gap}(A,B) := \inf_{(a,b)\in A\times B} d(a,b).$$

For any nonempty subset A of X, we define the neighborhood of A with radius $r > 0$, the set $B(A, r) := \{x \in X : d(x, A) < r\}$.

Let F be a multimap (or set-valued mapping or correspondence) from X into the subsets (possibly empty) of a metric space Y, indicated by $F : X \rightrightarrows Y$. We denote by $\mathrm{dom}F := \{x \in X : F(x) \neq \emptyset\}$ the domain of F and by $\mathrm{Gr}(F)$ the graph of F, that is $\mathrm{Gr}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$. When $\mathrm{Gr}(F)$ is closed, we say that F is a closed multimap. The inverse multimap of F is the set-valued mapping $F^{-1} : Y \rightrightarrows X$ such that $\mathrm{Gr}(F^{-1}) := \{(y, x) \in Y \times X : (x, y) \in \mathrm{Gr}(F)\}$.

Let us introduce the following definitions.

Definition 2.1. *A multimap $F : X \rightrightarrows Y$ between two metric spaces is said to be o.h.c or upper hemi-continuous or outer Hausdorff continuous at $\bar{x} \in X$ if $e(F(x), F(\bar{x})) \to 0$ as $x \to \bar{x}$ in domF.*

Hence F is o.h.c. at $\bar{x} \in \mathrm{dom}F$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta)$, $F(x) \subset B(F(\bar{x}), \varepsilon)$.

Definition 2.2. *A multimap $F : X \rightrightarrows Y$ between two metric spaces is said to be i.h.c or inner hemi-continuous or lower Hausdorff continuous at $\bar{x} \in X$ if gap$(F(x), F(\bar{x})) \to 0$ as $x \to \bar{x}$ in domF.*

Thus F is i.h.c. at $\bar{x} \in \mathrm{dom}F$ iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x) \cap B(F(\bar{x}), \varepsilon) \neq \emptyset$ for all $x \in B(\bar{x}, \delta)$.
When $\bar{x}$ is in the interior of the domain of F and F is o.h.c. at $\bar{x}$ then it is i.h.c. at $\bar{x}$. Note that if F is inner continuous at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$ in the sense that $d(\bar{y}, F(x)) \to 0$ as $x \to \bar{x}$, then F is obviously i.h.c. at $\bar{x}$.

A particular class of i.h.c. (resp. o.h.c.) multimaps is provided by Lipschitz multimaps defined as:

Definition 2.3. *A multimap $F : X \rightrightarrows Y$ between two metric spaces is said to be Lipschitz on $D \subset \mathrm{dom}F$ if there exists a Lipschitz constant $l \geq 0$ such that*

$$e(F(u_1), F(u_2)) \leq l d(u_1, u_2) \quad \forall u_1, u_2 \in D.$$

It is said to be pseudo-Lipschitz or Aubin continuous on a subset $U \times V$ of $X \times Y$, if there exists $l \geq 0$ such that, for every $u_1, u_2 \in U$,

$$e(F(u_1) \cap V, F(u_2)) \leq l d(u_1, u_2). \tag{2}$$

Recall that F is said pseudo-Lipschitz or Aubin continuous around a point $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$ if there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that F is pseudo-Lipschitz on $U \times V$. The Aubin property of a set valued mapping is closely tied with a property of its inverse, called metric regularity.

Definition 2.4. *A multimap $F : X \rightrightarrows Y$ between two metric spaces is said to be metrically regular around a point $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$ if there exist a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ and $\tau > 0$ such that*

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))$$

for all $(x, y) \in U \times V$.

This later concept is equivalent to the linear openness of F, that is,

Definition 2.5. *A multimap $F : X \rightrightarrows Y$ between two metric spaces is said to be open at a linear rate $\tau > 0$ (or $\tau-$linearly open) near $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$ if there exist a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ and $\varepsilon > 0$ such that for any $(x, y) \in \mathrm{Gr}(F) \cap (U \times V)$, one has*

$$B(y, \rho\tau) \subset F(B(x, \rho))$$

for every $\rho \in (0, \varepsilon)$.

For a detailed account for the links between the previous notions and other new properties of the regularity concepts as well as various applications, one can referred to the works of many researchers; see, e.g., [25, 21, 19, 14, 20, 17, 15, 18, 16, 22, 17, 32, 33, 34, 35, 36, 54, 55] and the references therein.

Given a multimap $F : X \rightrightarrows Y$, the lower semicontinuons envelope associated to F is defined as $\varphi_F : X \times Y \to \mathbb{R} \cup \{+\infty\}$ such that

$$\varphi_F(x, y) := \liminf_{(u,v)\to(x,y)} d(v, F(u)).$$

or equivalently, $\varphi_F(x, y) = \liminf_{u\to x} d(y, F(u))$ (using the continuity of the map $x \to d(x, A)$). Since $0 \le \varphi_F(x, y) \le d(y, F(x))$ for all $(x, y) \in X \times Y$, one has

$$y \in \overline{F(x)} \iff d(y, F(x)) = 0 \Longrightarrow \varphi_F(x, y) = 0.$$

In fact, we have the following result given in [25, Lemma 1].

Proposition 2.6. *Let $F : X \rightrightarrows Y$ be a multimap between two metric spaces. Then, for every $y \in Y$,*

$$\varphi_F(x, y) = 0 \iff x \in (\mathrm{cl}F)^{-1}(y)$$

where $\mathrm{cl}F$ is the multimap whose graph is $\mathrm{Gr}\bar{(}F)$. In particular, if $\mathrm{Gr}(F)$ is closed,

$$F^{-1}(y) = \{x \in X : \varphi_F(x, y) = 0\}.$$

Given $\psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ an extended-real valued function on X a metric space and consider the set

$$S := \{x \in X : \psi(x) \le 0\},$$

which is a (possibly empty) subset of $\mathrm{dom}\psi := \{x \in X : \psi(x) < +\infty\}$ the domain of ψ.

For $\bar{x} \in X$ such that $\psi(\bar{x}) > 0$, let us set

$$\vartheta(\bar{x}) := \inf_{x\in\Lambda(\bar{x})} \sup_{x\neq u} \frac{\psi(x) - [\psi(u)]_+}{d(x, u)},$$

where $\Lambda(\bar{x}) := \{x \in X : d(x, \bar{x}) < d(\bar{x}, S),\ \psi(x) \le \psi(\bar{x})\}$ and $[\psi]_+ := \max(\psi, 0)$.

In the sequel, the following Theorems are useful. The first one gives an error bound estimation and the second concerns the Ekeland variational principle.

Theorem 2.7. [48] *Let X be a complete metric space, $\psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on X and let $\bar{x} \notin S$. Then*

$$\vartheta(\bar{x}) d(\bar{x}, S) \le \psi(\bar{x}).$$

Theorem 2.8 (Ekeland variational principle). *Let X be a complete metric space, $\psi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on X which is bounded from below. Let $\bar{x} \in \mathrm{dom}\psi$. Then for every $\varepsilon > 0$ there exists x_ε such that*

$$\psi(x_\varepsilon) + \varepsilon d(x_\varepsilon, \bar{x}) \leq \psi(\bar{x})$$

and

$$\psi(x_\varepsilon) < \psi(x) + \varepsilon d(x, x_\varepsilon) \qquad \forall x \in X, \quad x \neq x_\varepsilon.$$

3. Local stability of compositions

The concept of local composition stability was introduced in [23] in order to conserve the Aubin property for compositions of multifunctions. This concept allows to obtain regularity results of compositions around a reference point.

Given $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ two multimaps between metric spaces and consider the composition $H := G \circ F : X \rightrightarrows Z$ defined by $H(x) := \underset{y \in F(x)}{\cup} G(y)$ and the multimap $C : X \times Z \rightrightarrows Y$ such that $C(x,z) := F(x) \cap G^{-1}(z)$ for any $(x,z) \in X \times Z$.

Recall that, the pair (F,G) is said to be locally composition stable around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} \in G(\bar{y})$ and $\bar{y} \in F(\bar{x})$ if for every $\varepsilon > 0$, there exists $\rho > 0$, such that for any $x \in B(\bar{x}, \rho)$ and any $z \in H(x) \cap B(\bar{z}, \rho)$, there exists $y \in F(x) \cap B(\bar{y}, \varepsilon)$ such that $z \in G(y)$ (see [23]).

Let us now introduce the following Definition.

Definition 3.1. *Let $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ be two multimaps between metric spaces and $(\bar{x}, \bar{z}) \in X \times Z$ such that $\bar{z} \in H(\bar{x})$. We say that $G \circ F$ is coordinate locally stable around $(\bar{x}, \bar{z})$ if for every $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\bar{x}, \rho)$ and $z \in H(x) \cap B(\bar{z}, \rho)$, there exists $y \in F(x) \cap B(C(\bar{x}, \bar{z}), \varepsilon)$ such that $z \in G(y)$, i.e.,*

$$H(x) \cap B(\bar{z}, \rho) \subset G(F(x) \cap B(C(\bar{x}, \bar{z}), \varepsilon)). \tag{3}$$

Note that if $C(\bar{x}, \bar{z}) = \{\bar{y}\}$, we obtain the usual definition of local composition stability given in [23]. It is clear that if (F,G) is locally composition stable around $(\bar{x}, \bar{y}, \bar{z})$, $G \circ F$ is coordinate locally stable.

We give in the following some examples of pair of multimaps satisfying Definition 3.1 which is not locally composition stable.

Example 3.2. *Let $F, G : [0,1] \rightrightarrows \mathbb{R}$ such that $F(x) := [0,2]$, $G(x) := \{0\}$ if $x \in \left[0, \frac{1}{2}\right]$ otherwise $G(x) :=]0,2]$ so $H(x) = [0,2]$. Then for $(\bar{x}, \bar{z}) = (0,0)$ and $\varepsilon > 0$ sufficiently small, we have for any x, $F(x) \cap B(C(0,0), \varepsilon)) = \left[0, \frac{1}{2} + \varepsilon\right[$ (since $C(0,0) = \left[0, \frac{1}{2}\right]$) so that $G(\left[0, \frac{1}{2} + \varepsilon\right[) = [0,2]$. Hence (3) is satisfied since $H(x) \cap]-\rho, \rho[= [0, \rho[\subset [0,2]$ for any $\rho \in (0,2)$.*

On the other hand, take $\bar{y} := 0 \in F(0)$, $0 < \bar{\varepsilon} < \frac{1}{2}$ so that $G(F(0) \cap]-\bar{\varepsilon}, \bar{\varepsilon}[) = G([0, \bar{\varepsilon}[) = \{0\}$. And we have for an arbitrary $\rho > 0$, $H(0) \cap]-\rho, \rho[= [0, \rho[$ if $\rho < 2$ and $H(0) \cap]-\rho, \rho[= [0,2]$ for $\rho \geq 2$. Hence the pair (F,G) is not locally composition stable around $(0,0,0)$.

Example 3.3. *Let* $F : [0, +\infty) \rightrightarrows \mathbb{R}$ *and* $G : \mathbb{R} \rightrightarrows \mathbb{R}$ *be such that* $F(x) := [0, x]$ *for any* $x \geq 0$, $G(x) := \{1, x\}$ *if* $x < 1$ *and* $G(x) := [1, x]$ *if* $x \geq 1$. *Then for all* $x \geq 0$, $H(x) = [0, x]$. *Take* $\bar{x} := 2$ *so* $\bar{z} := 1 \in H(2)$ *and* $C(2, 1) = [0, 2]$. *Let* $\varepsilon \in \,]0, 1[$ *and* $\delta \in \left]0, \frac{\varepsilon}{2}\right[$. *Given* $x \in \,]2 - \delta, 2 + \delta[$ *and* $z \in H(x)$ *with* $z \in \,]1 - \delta, 1 + \delta[$. *Since* $B(C(2, 1), \varepsilon) = \,]-\varepsilon, 2 + \varepsilon[$ *hence* $F(x) \cap \,]-\varepsilon, 2 + \varepsilon[\, = [0, x]$ *so that* $z \in G([0, x]) = [0, x]$.
Take now $\bar{y} := 0 \in F(2)$, $\varepsilon := \dfrac{1}{2}$ *and let* $\delta > 0$ *be arbitrarily (we may assume* $\delta < 1$*). For* $x := \frac{7}{2}$, *we have* $G(F(x)) \cap \,]1 - \delta, 1 + \delta[\, = \,]1 - \delta, 1 + \delta[$ *and* $G(F(x) \cap \,]-\varepsilon, \varepsilon[) = G(]0, \varepsilon[) = \,]0, \varepsilon[\, \cup \{1\}$. *Consequently, the pair* (F, G) *is not locally composition stable around* $(2, 0, 1)$.

Let us provide now some conditions ensuring the property of coordinate local stability of $G \circ F$ introduced in Definition 3.1.

Proposition 3.4. *Let* X, Y, Z *be metric spaces,* $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ *be two multimaps and let* $(\bar{x}, \bar{z}) \in \mathrm{Gr}(H)$. *If the multimap* $C(\cdot, \cdot)$ *is o.h.c. at* $(\bar{x}, \bar{z})$, *then* $G \circ F$ *is coordinate locally stable around* $(\bar{x}, \bar{z})$.

Proof. Let $\varepsilon > 0$. Since $(x, z) \to C(x, z)$ is o.h.c. at $(\bar{x}, \bar{z})$, there exists $\rho > 0$ such that for all $(x, z) \in B(\bar{x}, \rho) \times B(\bar{z}, \rho)$, one has

$$e(C(x, z), C(\bar{x}, \bar{z})) < \varepsilon. \tag{4}$$

Given $x \in B(\bar{x}, \rho)$ and $z \in H(x) \cap B(\bar{z}, \rho)$ so that $z \in G(y)$ for some $y \in F(x)$ that is $y \in C(x, z)$. So by (4), $d(y, C(\bar{x}, \bar{z})) < \varepsilon$. Hence $z \in G(y)$ with $y \in F(x) \cap B(C(\bar{x}, \bar{z}), \varepsilon)$ and the conclusion follows. □

Observe that if F is o.h.c. at $\bar{x}$ and $C(\bar{x}, \bar{z}) = F(\bar{x})$ (in particular, if $F(\bar{x}) = \{\bar{y}\}$) then the multimap $(x, z) \to C(x, z)$ is o.h.c. at $(\bar{x}, \bar{z}) \in \mathrm{Gr}(G \circ F)$ since

$$e(C(x, z), C(\bar{x}, \bar{z})) \leq e(F(x), F(\bar{x}))$$

thus one gets the following Corollary.

Corollary 3.5. *Let* $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ *be two multimaps between metric spaces and let* $(\bar{x}, \bar{z}) \in \mathrm{Gr}(H)$. *If the multimap* F *is o.h.c. at* $\bar{x}$ *and if* $C(\bar{x}, \bar{z}) = F(\bar{x})$ *then* $G \circ F$ *is coordinate locally stable around* $(\bar{x}, \bar{z})$.

Proposition 3.6. *Let* $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ *be two multimaps between metric spaces. Let* $(\bar{x}, \bar{z}) \in \mathrm{Gr}(G \circ F)$ *and assume that*
(*i*) *the multimap* $C(\cdot, \cdot)$ *is i.h.c. at* $(\bar{x}, \bar{z})$,
(*ii*) *the multimap* G *is injective i.e.,* $G(y) \cap G(y') = \varnothing$ *whenever* $y \neq y'$.
Then $G \circ F$ *is coordinate locally stable around* $(\bar{x}, \bar{z})$.

Proof. Let $\varepsilon > 0$. Since $(x, z) \to C(x, z)$ is i.h.c. at $(\bar{x}, \bar{z})$, there exists $\rho > 0$ such that $C(x, z) \cap B(C(\bar{x}, \bar{z}), \varepsilon) \neq \varnothing$ for all $(x, z) \in B((\bar{x}, \bar{z}), \rho)$. Take $z \in G \circ F(x) \cap B(\bar{z}, \rho)$ with $x \in B(\bar{x}, \rho)$ that is, $z \in G(y)$ for some $y \in F(x)$.
On the other hand, there exists $y' \in F(x) \cap B(C(\bar{x}, \bar{z}), \varepsilon)$ such that $z \in G(y')$. Since G is injective, we get $y' = y$. Hence $z \in G(F(x) \cap B(C(\bar{x}, \bar{z}), \varepsilon))$ and the conclusion follows. □

In the next Proposition, we denote by $\Pi_i : Y_1 \times Y_2 \to Y_i$ the cartesian projection on Y_i ($\Pi_i(y_1, y_2) = y_i$ for $i = 1, 2$).

Proposition 3.7. *Let $F := (F_1, F_2) : X \rightrightarrows Y := Y_1 \times Y_2$ be a multimap and $g : Y \to Z$ be a single mapping between metric spaces. Let $(\bar{x}, \bar{z}) \in \mathrm{Gr}(g \circ F)$ and assume that*

(i) the multimap $(x, z) \to F_1(x) \cap \Pi_1(g^{-1}(z))$ is o.h.c. at $(\bar{x}, \bar{z})$,

(ii) for every $\varepsilon > 0$, there exists $\rho > 0$ such that

$$d(\bar{z}, g(y_1, F_2(\bar{x}) \cap \Pi_2(g^{-1}(\bar{z})))) < \varepsilon \quad \forall y_1 \in B(F_1(\bar{x}) \cap \Pi_1(g^{-1}(\bar{z})), \rho).$$

(iii) there exists $r > 0$ such that for all $y_1 \in B(F_1(\bar{x}) \cap \Pi_1(g^{-1}(\bar{z})), r)$, the mapping $g(y_1, \cdot)$ is an isometry.

Then $g \circ F$ is coordinate locally stable around $(\bar{x}, \bar{z})$.

Proof. Denote $C_i(x, z) := F_i(x) \cap \Pi_i(g^{-1}(z))$ $(i = 1, 2)$ for $(x, z) \in X \times Z$ and let $\varepsilon > 0$. From assumption (ii), there exists $\rho_1 > 0$ such that for any $y_1 \in B(C_1(\bar{x}, \bar{z}), \rho_1)$,

$$d(\bar{z}, g(y_1, C_2(\bar{x}, \bar{z}))) < \frac{\varepsilon}{2}. \tag{5}$$

As C_1 is o.h.c. at $(\bar{x}, \bar{z})$, there is $\rho_2 > 0$ such that for all $x \in B(\bar{x}, \rho_2)$ and $z \in B(\bar{z}, \rho_2)$,

$$e(C_1(x, z), C_1(\bar{x}, \bar{z})) < \theta \tag{6}$$

for $\theta := \min(r, \rho_1, \frac{\varepsilon}{2})$.

Take $\rho := \min(\rho_1, \rho_2, \frac{\varepsilon}{2})$ and let $z \in (g \circ F)(x) \cap B(\bar{z}, \rho)$ with $x \in B(\bar{x}, \rho)$. Hence $z = g(y_1, y_2) \in B(\bar{z}, \rho)$ with $(y_1, y_2) \in F(x) \cap g^{-1}(z)$, so that, by (6), $y_1 \in B(C_1(\bar{x}, \bar{z}), \theta)$ and since $\theta < \varepsilon$, $y_1 \in B(C_1(\bar{x}, \bar{z}), \varepsilon)$.

On the other hand, since $y_1 \in B(C_1(\bar{x}, \bar{z}), r)$ and $z = g(y_1, y_2)$, $d(z, g(y_1, v)) = d(y_2, v)$ for any v (by (iii)) and so

$$d(y_2, C_2(\bar{x}, \bar{z})) = \inf_{v \in C_2(\bar{x}, \bar{z})} d(y_2, v) = d(z, g(y_1, C_2(\bar{x}, \bar{z}))).$$

Now from inequality (5), we get

$$\begin{aligned} d(z, g(y_1, C_2(\bar{x}, \bar{z}))) &\le d(z, \bar{z}) + d(\bar{z}, g(y_1, C_2(\bar{x}, \bar{z}))) \\ &\le d(z, \bar{z}) + \frac{\varepsilon}{2} < \rho + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

so that $d(y_2, C_2(\bar{x}, \bar{z})) < \varepsilon$. Finally, we conclude that $z \in g(y)$ with

$$y = (y_1, y_2) \in F(x) \cap B(C(\bar{x}, \bar{z}), \varepsilon)$$

and the conclusion follows. □

4. Metric regularity of compositions

There are numerous studies in the literature that present results on metric regularity of composition of set-valued mappings, see for instance, [7, 37, 23, 29, 62]. In this section, we investigate metric regularity of composed multimaps using the concept of local stability as given in Definition 3.1. This concept is a useful tool to establish the following result:

Proposition 4.1. *Let X, Y, Z be metric spaces, $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ be two set-valued mappings and let $(\bar{x}, \bar{z}) \in \mathrm{Gr}(G \circ F)$. If $G \circ F$ is coordinate* *locally stable around* $(\bar{x}, \bar{z})$ *and there exist a neighborhood* $U \times W$ *of* $(\bar{x}, \bar{z})$ *and* $\tau, r > 0$ *such that*

$$(7)\quad d(x, (G \circ F)^{-1}(z)) \leq \tau d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) \quad \forall (x, z) \in U \times W,$$

then $H := G \circ F$ *is metrically regular around* $(\bar{x}, \bar{z})$ *with modulus* τ*.*

Proof. Suppose there exist $\rho_1, \rho_2 > 0$ such that for every $(x, z) \in B(\bar{x}, \rho_1) \times B(\bar{z}, \rho_2)$, the inequality (7) is fulfilled. As $G \circ F$ is coordinate locally stable around $(\bar{x}, \bar{z})$, there exists $\rho_3 > 0$ such that for any $x \in B(\bar{x}, \rho_3)$

$$(8)\qquad H(x) \cap B(\bar{z}, \rho_3) \subset G(F(x) \cap B(C(\bar{x}, \bar{z}), r)).$$

Taking $\rho := \min(\rho_1, \rho_2, \rho_3)$, $(x, z) \in B(\bar{x}, \frac{\rho}{2}) \times B(\bar{z}, \frac{\rho}{2})$ and let us assume that $d(z, H(x)) < \frac{\rho}{2}$. Consider $t > d(z, H(x))$ such that $t < \frac{\rho}{2}$. So one can pick some $w \in H(x)$ such that $d(z, w) < t$. Hence

$$d(w, \bar{z}) \leq d(w, z) + d(z, \bar{z}) < t + \frac{\rho}{2} < \rho$$

that is $w \in H(x) \cap B(\bar{z}, \rho) \subset H(x) \cap B(\bar{z}, \rho_3)$ and from (8), $w \in G(F(x) \cap B(C(\bar{x}, \bar{z}), r))$. Consequently,

$$d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) \leq d(z, w) < t.$$

Letting t decreasing to $d(z, H(x))$, we obtain

$$d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r)) \leq d(z, H(x))$$

and from (7), we get

$$d(x, H^{-1}(z)) \leq \tau d(z, H(x)).$$

And if $d(z, H(x)) \geq \frac{\rho}{2}$, choose $0 < \delta < \min(\tau\frac{\rho}{4}, \frac{\rho}{4})$ and let $(x, z) \in B(\bar{x}, \delta) \times B(\bar{z}, \delta)$. Clearly, if $d(\bar{x}, H^{-1}(z)) = 0$,

$$d(x, H^{-1}(z)) \leq d(\bar{x}, x) < \delta < \tau\frac{\rho}{4} < \frac{\tau}{2} d(z, H(x)).$$

Suppose now $d(\bar{x}, H^{-1}(z)) > 0$. So by (7), for any $\varepsilon > 0$, there exists $u \in H^{-1}(z)$ such that

$$\begin{aligned} d(\bar{x}, u) &< (1+\varepsilon) d(\bar{x}, H^{-1}(z)) \\ &< (1+\varepsilon)\tau d(z, G(F(\bar{x}) \cap B(C(\bar{x}, \bar{z}), r))) \\ &< (1+\varepsilon)\tau d(z, \bar{z}) < (1+\varepsilon)\tau\delta \\ &< (1+\varepsilon)\frac{\tau}{2} d(z, H(x)). \end{aligned}$$

Thus

$$\begin{aligned} d(x, u) &\leq d(x, \bar{x}) + d(\bar{x}, u) \\ &< \delta + (1+\varepsilon)\tau d(z, E(x)) \\ &< \frac{\tau}{2} d(z, H(x)) + (1+\varepsilon)\frac{\tau}{2} d(z, H(x)). \end{aligned}$$

Taking the limit as ε goes to 0, it follows that

$$d(x, H^{-1}(z)) \leq \tau d(z, H(x))$$

for all $(x, z) \in B(\bar{x}, \delta) \times B(\bar{z}, \delta)$ which leads to the conclusion. □

Our prupose now is getting sufficient conditions to give a rise to the property (7). For this aim, consider the set-valued map $R : X \times Y \rightrightarrows Z$ such that for all $(x, y) \in X \times Y$,

$$R(x,y) := \begin{cases} G(y) & \text{if } y \in F(x) \\ \varnothing & \text{otherwise} \end{cases}$$

and $\varphi_R : X \times Y \times Z \longrightarrow \mathbb{R} \cup \{+\infty\}$ the associated lower semicontinuous envelope defined as:

$$\varphi_R((x,y),z) := \liminf_{(u,v,w)\to(x,y,z)} d(w, R(u,v)) \quad \forall (u,v,w) \in X \times Y \times Z.$$

Here we endow $X \times Y$ with the box distance given by

$$d((x,y),(x',y')) := \max(d(x,x'), d(y,y')).$$

Proposition 4.2. *Let X, Y, Z be metric spaces, $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ be multimaps and $(\bar{x}, \bar{z}) \in \mathrm{Gr}(G \circ F)$. Let the following assertions:*

(i) there exist a neighborhood $U \times V$ of $(\bar{x}, \bar{z})$ and $\tau, r > 0$ such that for any $(x, z) \in U \times V$, and any $y \in B(C(\bar{x}, \bar{z}), r)$

$$d((x,y), R^{-1}(z)) \le \tau \varphi_R((x,y),z); \tag{9}$$

(ii) there exist a neighborhood $U \times V$ of $(\bar{x}, \bar{z})$ and $\tau, r > 0$ such that for all $(x, z) \in U \times V$,

$$d(x, H^{-1}(z)) \le \tau d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))); \tag{10}$$

(iii) there exist a neighborhood $U \times V$ of $(\bar{x}, \bar{z})$ and $\varepsilon, \tau, r > 0$ such that for every $(x, z) \in U \times V$ with $z \in G(F(x) \cap B(C(\bar{x}, \bar{z}), r))$ and for all $\rho \in]0, \varepsilon[$,

$$B(z, \rho\tau^{-1}) \subset H(B(x, \rho)). \tag{11}$$

Then $(i) \implies (ii) \implies (iii)$.

Moreover, if $H := G \circ F$ is closed and for some $r_0 > 0$, the multimap $G(F(\cdot) \cap B(C(\bar{x}, \bar{z}), r_0))$ inner continuous at $(\bar{x}, \bar{z})$ then $(iii) \implies (ii)$.

Proof. For $(i) \implies (ii)$. Let $r, \tau, \delta_1, \delta_2 > 0$ such that (9) is satisfies for every $(x, z) \in B(\bar{x}, \delta_1) \times B(\bar{z}, \delta_2)$ and $y \in B(C(\bar{x}, \bar{z}), r)$. Let $(x, z) \in B(\bar{x}, \delta_1) \times B(\bar{z}, \delta_2)$.

If $G(F(x) \cap B(C(\bar{x}, \bar{z}), r)) = \varnothing$, $d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) = +\infty$ then (10) is obvious. Let $w \in G(y)$ with $y \in F(x) \cap B(C(\bar{x}, \bar{z}), r))$.
If $d((x, y), R^{-1}(z)) = 0$ thus for any neighborhood $U \times V$ of (x, y) there exists $(x', y') \in U \times V$ such that $z \in R(x', y')$, that is $z \in H(x')$ with $x' \in U$. Thus $x \in \overline{H^{-1}(z)}$ which leads to (10). Assume now $d((x, y), R^{-1}(z)) > 0$ and let $\varepsilon > 0$. Using (9), there exists $(u, v) \in R^{-1}(z)$ which satisfies

$$d((x,y),(u,v)) < (1+\varepsilon)\tau\varphi_R((x,y),z).$$

As $\varphi_R((x, y), z) \le d(z, G(y))$, we get

$$d(x,u) < (1+\varepsilon)\tau d(z, G(y)).$$

Moreover, since $z \in G(v)$ with $v \in F(u)$, $u \in H^{-1}(z)$ and

$$d(x, H^{-1}(z)) \le d(x,u) < (1+\varepsilon)\tau d(z, G(y))$$

where y is arbitrary in $F(x) \cap B(C(\bar{x}, \bar{z}), r)$.

Consequently,

$$d(x, H^{-1}(z)) < (1+\varepsilon)\tau d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))).$$

Taking the limit as ε goes to 0, the conclusion follows.

For $(ii) \implies (iii)$. Let $\delta, \gamma, \tau, r > 0$ such that (10) is satisfies on $B(\bar{x}, \delta) \times B(\bar{z}, \gamma)$. Choose $\eta > 0$ and $0 < \varepsilon < r$ such that $\varepsilon\tau^{-1} + \eta < \gamma$ and let $(x, z) \in B(\bar{x}, \delta) \times B(\bar{z}, \eta)$ with $z \in G(F(x) \cap B(C(\bar{x}, \bar{z}), r))$. Take $w \in B(z, \rho\tau^{-1})$ with $\rho \in \,]0, \varepsilon[$. If $d(x, H^{-1}(w)) = 0$ so that $B(x, \rho) \cap H^{-1}(w) \neq \varnothing$. Thus there exists $x' \in B(x, \rho)$ such that $w \in H(x')$ that is, $w \in H(B(x, \rho))$. Assume now that $d(x, H^{-1}(w)) > 0$. As $z \in G(F(x) \cap B(C(\bar{x}, \bar{z}), r))$, we have

$$d(w, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) \leq d(w, z) < \rho\tau^{-1}$$

here $w \in B(\bar{z}, \gamma)$ since $d(w, \bar{z}) < \rho\tau^{-1} + \eta < \varepsilon\tau^{-1} + \eta < \gamma$. Hence, from (10), $d(x, H^{-1}(w)) < \rho$. Let t be such that $d(x, H^{-1}(w)) < t < \rho$ so there exists $u \in H^{-1}(w)$ such that $d(x, u) < t < \rho$ which leads to (11) since $w \in H(B(x, \rho))$.

Now, assume (iii) holds for $\varepsilon, \tau, r_1 > 0$ and the neighborhood $B(\bar{x}, \delta_1) \times B(\bar{z}, \gamma_1)$ with $\delta_1, \gamma_1 > 0$. Assume that $H := G \circ F$ is closed and for $r_0 > 0$, the multimap $G(F(\cdot) \cap B(C(\bar{x}, \bar{z}), r_0))$ is inner continuous at $(\bar{x}, \bar{z})$ or equivalently that the function $f : x \to d(\bar{z}, G(F(x) \cap B(C(\bar{x}, \bar{z}), r_0)))$ is upper semicontinuous at $\bar{x}$ (note that $f(\bar{x}) = 0$). Hence there exists $\delta_2 > 0$ such that for any $x \in B(\bar{x}, \delta_2)$, one has

$$d(\bar{z}, G(F(x) \cap B(C(\bar{x}, \bar{z})), r_0))) < \frac{\varepsilon'}{2}$$

with $\varepsilon' := \min(\frac{\gamma_1}{2}, \tau^{-1}\varepsilon)$.

Take $r := \max(r_0, r_1)$, $0 < \delta < \min(\delta_1, \delta_2)$, $0 < \gamma < \frac{\varepsilon'}{2}$ and let $(x, z) \in B(\bar{x}, \delta) \times B(\bar{z}, \gamma)$. If $d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) = 0$ so $d(z, H(x)) = 0$. Hence since H is closed, $x \in H^{-1}(z)$ which leads to $d(x, H^{-1}(z)) = 0$. Suppose now that $0 < d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) < +\infty$ (if $d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) = +\infty$, (ii) trivially holds) and let $\theta > 0$. So one can pick $v \in G(F(x) \cap B(C(\bar{x}, \bar{z}), r))$ such that

$$d(z, v) < (1+\theta)d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) := \tau^{-1}\rho$$

with $\rho := \tau(1+\theta)d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r)))$.

On the other hand, since

$$\begin{aligned} d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) &\leq d(z, \bar{z}) + d(\bar{z}, G(F(x) \cap B(C(\bar{x}, \bar{z}), r))) \\ &\leq \gamma + d(\bar{z}, G(F(x) \cap B(C(\bar{x}, \bar{z}), r_0))) \\ &< \gamma + \frac{\varepsilon'}{2} < \varepsilon', \end{aligned} \tag{12}$$

thus $\rho \in \left]0, \min(\frac{\tau\gamma_1}{2}, \varepsilon)\right[$ for $\theta > 0$ small enaugh (taking for instance, $\theta < \min(\frac{\gamma_1}{2\varepsilon'} - 1, \frac{\tau^{-1}\varepsilon}{\varepsilon'} - 1)$).

Here $v \in B(\bar{z}, \gamma_1)$ since

$$d(\bar{z}, v) \leq d(\bar{z}, z) + d(z, v) < \gamma + \tau^{-1}\rho < \gamma_1.$$

It follows from (11) that

$$z \in B(v, \tau^{-1}\rho) \subset H(B(x, \rho))$$

hence there is $u \in B(x, \rho)$ such that $z \in H(u)$, i.e., $u \in H^{-1}(z)$ so that

$$d(x, H^{-1}(z)) \le d(x, u) < \rho = \tau(1+\theta)d(z, G(F(x) \cap B(C(\bar{x}, \bar{z}), r)))$$

and by letting $\theta \to 0$, we get (10) and the conclusion follows. □

We present now a result on metric regularity of the composition $H := G \circ F$ with $F = (F_1, F_2)$ which should be compared to [23, Theorem 3.6]. Consider $F = (F_1, F_2) : X \rightrightarrows Y := Y_1 \times Y_2$, $G : Y \rightrightarrows Z$ and let $\bar{z} \in H(\bar{x})$. Denote $C_i(\bar{x}, \bar{z}) := F_i(\bar{x}) \cap \Pi_i(G^{-1}(\bar{z}))$ for $i = 1, 2$ and assume that for $\delta, \rho, r > 0$, we have

(i) F_1 is $m_{F_1}-$metrically regular on $B(\bar{x}, \delta) \times B(C_1(\bar{x}, \bar{z}), r)$ in the sense:

$$d(x, F_1^{-1}(y_1)) \le m_{F_1} d(y_1, F_1(x))$$

$x \in B(\bar{x}, \delta)$ and all $y_1 \in B(C_1(\bar{x}, \bar{z}), r)$.

(ii) F_2 is $l_{F_2}-$pseudo-Lipschitz on $B(\bar{x}, \delta) \times B(C_2(\bar{x}, \bar{z}), r)$.

(iii) $G_{y_2} : y_1 \to G(y_1, y_2)$ is m_G-metrically regular on $B(C_1(\bar{x}, \bar{z}), r) \times B(\bar{z}, \rho)$, uniformly for all $y_2 \in B(C_2(\bar{x}, \bar{z}), r)$ that is,

$$d(y_1, G_{y_2}^{-1}(z)) \le m_G d(z, G(y_1, y_2))$$

for all $z \in B(\bar{z}, \rho)$ and all $y_i \in B(C_i(\bar{x}, \bar{z}), r)$ $(i = 1, 2)$.

(iv) $G_{y_1} : y_2 \to G(y_1, y_2)$ is l_G-pseudo-Lipschitz on $B(C_2(\bar{x}, \bar{z}), r) \times B(\bar{z}, \rho)$, uniformly for all $y_1 \in B(C_1(\bar{x}, \bar{z}), r)$:

$$e(G(y_1, y_2) \cap B(\bar{z}, \rho), G(y_1, y_2')) \le l_G d(y_2, y_2')$$

for any $y_1 \in B(C_1(\bar{x}, \bar{z}), r)$ and $y_2, y_2' \in B(C_2(\bar{x}, \bar{z}), r)$.

Theorem 4.3. *Let X, Y_1, Y_2 be complete metric spaces, Z a metric space and let $F = (F_1, F_2) : X \rightrightarrows Y := Y_1 \times Y_2$, $G : Y \rightrightarrows Z$ two closed multimaps. Let $(\bar{x}, \bar{z}) \in X \times Z$ such that $\bar{z} \in H(\bar{x})$. Assume that the assumptions $(i)-(iv)$ are satisfied and that*

(v) $0 < l_{F_2} l_G m_{F_1} m_G < 1$.

Then, there exist a neighborhood $U \times W$ of $(\bar{x}, \bar{z})$ in $X \times Z$, $\tau, s > 0$ such that for any $(x, z) \in U \times W$ and $y \in B(C(\bar{x}, \bar{z}), s)$, one has

$$d((x, y), R^{-1}(z)) \le \tau \varphi_R((x, y), z).$$

Proof. Let $l, m, \mu, \eta > 0$ such that $ml\mu\eta < 1$ with

(13) $$m_{F_1} < m,\ l_{F_2} < l,\ m_G < \mu \text{ and } l_G < \eta.$$

and let $\tau_0 := \frac{m\mu}{1 - ml\mu\eta}$.

Consider the following metric on $X \times Y$,

$$d_0((x, y), (u, v)) := \max(d(x, u), md(y_1, v_1), l^{-1}d(y_2, v_2))$$

for any $x, u \in X$, $y := (y_1, y_2) \in Y$ and $v := (v_1, v_2) \in Y$. Then one can check that $(X \times Y, d_0)$ is a complete metric space.

From $(i) - (iv)$ and (13), for any $x, x' \in B(\bar{x}, \delta)$, $y_1 \in B(C_1(\bar{x}, \bar{z}), r)$, $y_2, y_2' \in B(C_2(\bar{x}, \bar{z}), r)$ and any $z \in B(\bar{z}, \rho)$, we have

(14) $$e(F_2(x) \cap B(C_2(\bar{x}, \bar{z}), r), F_2(x')) < ld(x, x')$$

$$e(G(y_1, y_2) \cap B(\bar{z}, \rho), G(y_1, y_2')) < \eta d(y_2, y_2') \tag{15}$$

$$d(x, F_1^{-1}(y_1)) < md(y_1, F_1(x)) \tag{16}$$

and

$$d(y_1, G_{y_2}^{-1}(z)) < \mu d(z, G(y_1, y_2)). \tag{17}$$

Take $0 < s < \min(\frac{\delta}{2}, \frac{r}{2}, \rho)$ and $\gamma > 0$ such that $\mu\gamma \leq \min(s, \frac{s}{m}, \frac{s}{ml})$. Let $x \in B(\bar{x}, s)$, $y = (y_1, y_2) \in B(C(\bar{x}, \bar{z}), s)$ and $z \in B(\bar{z}, s)$ such that $0 < \varphi_R((x, y), z) < \gamma$. So $z \notin G(y)$ and $y \in F(x) = F_1(x) \times F_2(x)$ since $\varphi_R(x, y, z)$ is finite.

Let us prove that for any $\theta > 0$, there exists $(u, v) \in X \times Y$ such that $u \in B(\bar{x}, s)$, $v \in F(u)$ such that $d(v, C(\bar{x}, \bar{z})) < s$ and

$$0 < d_0((x, y), (u, v)) < (\tau + \theta)(\varphi_R((x, y), z) - \varphi_R((u, v), z)). \tag{18}$$

Take $\theta > 0$ and consider a sequence $((x_n, y_n))_n \subset X \times Y$ with $y_n := (y_{1n}, y_{2n})$ such that (x_n, y_n) converges to (x, y) and

$$\varphi_R((x, y), z) = \lim_{n \to +\infty} d(z, R(x_n, y_n)).$$

Hence for n large enaugh, $0 < d(z, R(x_n, y_n)) < \gamma$ (since $0 < \varphi_R((x, y), z) < \gamma$) so $y_n \in F(x_n)$, $z \notin G(y_n)$ and $d(z, G(y_n)) < \gamma$.

As $y = (y_1, y_2) \in B(C(\bar{x}, \bar{z}), s)$, $y \in B(\bar{y}, s)$ for some $\bar{y} \in C(\bar{x}, \bar{z}) = F(\bar{x}) \cap G^{-1}(\bar{z})$ (indeed, if $d(y, C(\bar{x}, \bar{z})) = 0$ then $\bar{y} := y \in C(\bar{x}, \bar{z})$ and if $0 < d(y, C(\bar{x}, \bar{z})) < s$, there is $\bar{y} \in C(\bar{x}, \bar{z})$ with $d(y, \bar{y}) < t$ for any t such that $d(y, C(\bar{x}, \bar{z})) < t < s$).

Now since $((x_n, y_n))$ converges to (x, y) with $x \in B(\bar{x}, s)$ and $y \in B(\bar{y}, s)$ so for n sufficiently large, $x_n \in B(\bar{x}, s)$ and $d(y_n, \bar{y}) < s$ that is $y_{in} \in B(C_i(\bar{x}, \bar{z}), s)$. Whence from (17), one gets

$$d(y_{1n}, G_{y_{2n}}^{-1}(z)) < \mu d(z, G(y_n)) < \mu\gamma < s \tag{19}$$

which ensures that $d(y_{1n}, G_{y_{2n}}^{-1}(z)) \neq \infty$ i.e., $G_{y_{2n}}^{-1}(z) \neq \emptyset$. Moreover since G is closed and $z \notin G(y_n)$, $d(y_{1n}, G_{y_{2n}}^{-1}(z)) > 0$ for n large enaugh, thus one can pick for any $\varepsilon > 0$, $v_{1n} \in G_{y_{2n}}^{-1}(z)$ such that

$$d(y_{1n}, v_{1n}) < \left(1 + \frac{\varepsilon}{2\mu}\right) d(y_{1n}, G_{y_{2n}}^{-1}(z)) < \left(\mu + \frac{\varepsilon}{2}\right) d(z, G(y_n)) \tag{20}$$

or equivalently

$$d(y_{1n}, v_{1n}) < \left(\mu + \frac{\varepsilon}{2}\right) [d(z, G(y_n)) - d(z, G(v_{1n}, y_{2n}))] \tag{21}$$

as $d(z, G(v_{1n}, y_{2n})) = 0$. We deduce from (20) and (19) that the sequence $(d(y_{1n}, v_{1n}))$ is bounded so we may suppose without loosing the generality that $\lim_{n \to \infty} d(v_{1n}, y_{1n})$ exists with $\lim_{n \to +\infty} d(v_{1n}, y_1) > 0$, because $\mathrm{Gr}(G)$ is closed, $y_{1n} \notin G_{y_{2n}}^{-1}(z)$ and

$$0 < d(y_{1n}, G_{y_{2n}}^{-1}(z)) \leq d(y_{1n}, v_{1n}).$$

Similarly, for $\varepsilon > 0$ arbitrary such that $\left(1 + \frac{\varepsilon}{2\mu}\right) d(z, G(y_n)) < \gamma$,

$$d(y_{1n}, v_{1n}) < \mu\gamma < s$$

from (20).

On the other hand, as for n large enaugh, $y_n \in B(C(\bar{x},\bar{z}),s)$ so $d(y_n,\bar{y}_n) < s$ for some $\bar{y}_n$ in $C(\bar{x},\bar{z})$ and hence

$$d(v_{1n},\bar{y}_{1n}) \le d(v_{1n},y_{1n}) + d(y_{1n},\bar{y}_{1n}) < 2s < r$$

that is $v_{1n} \in B(C_1(\bar{x},\bar{z}),s)$ for n sufficiently large.

Now, we have $x_n \in B(\bar{x},s)$ with $s < \delta$ and $v_{1n} \in B(C_1(\bar{x},\bar{z}),r)$ so by (16) and by the fact that $y_{1n} \in F_1(x_n)$, it yields

$$d(x_n,F_1^{-1}(v_{1n})) < md(v_{1n},y_{1n}) < m\mu\gamma < s.$$

Whence there exists $u_n \in F_1^{-1}(v_{1n})$ such that

$$d(x_n,u_n) < md(v_{1n},y_{1n}) < s \tag{22}$$

with $u_n \in B(\bar{x},\delta)$ since

$$d(u_n,\bar{x}) \le d(u_n,x_n) + d(x_n,\bar{x}) \le 2s < \delta.$$

By (14) as $y_{2n} \in F_2(x_n) \cap B(C_2(\bar{x},\bar{z}),s)$ with $s < r$,

$$d(y_{2n},F_2(u_n)) < ld(x_n,u_n)$$

and again, one can pick $v_{2n} \in F_2(u_n)$ such that

$$d(y_{2n},v_{2n}) < ld(x_n,u_n) < lmd(v_{1n},y_{1n}) < lm\mu\gamma < s. \tag{23}$$

Moreover, from (23),

$$d(v_{2n},\bar{y}_{2n}) \le d(v_{2n},y_{2n}) + d(y_{2n},\bar{y}_{2n}) \le 2s < r$$

that is $v_{2n} \in B(C_2(\bar{x},\bar{z}),r)$ so we get by (15),

$$d(z,G(v_{1n},v_{2n})) \le e(G(v_{1n},y_{2n}) \cap B(\bar{z},\rho),G(v_{1n},v_{2n})) < \eta d(y_{2n},v_{2n}). \tag{24}$$

Observe now that

$$l^{-1}d(y_{2n},v_{2n}) < d(x_n,u_n) < md(v_{1n},y_{1n})$$

thus

$$d_0((x_n,y_n),(u_n,v_n)) = md(v_{1n},y_{1n}) \tag{25}$$

where $y_n = (y_{1n},y_{2n})$ and $v_n = (v_{1n},v_{2n})$. Thus by (21), (25) and (24), we find

$$\begin{aligned}\frac{d(z,R(x_n,y_n)) - d(z,R(u_n,v_n))}{d_0((x_n,y_n),(u_n,v_n))} &= \frac{d(z,G(y_n)) - d(z,G(v_{1n},y_{2n})) - d(z,G(v_n))}{md(y_{1n},v_{1n})}\\ &\ge \frac{d(z,G(y_n)) - d(z,G(v_{1n},y_{2n})) - \eta d(y_{2n},v_{2n})}{md(y_{1n},v_{1n})}\\ &\ge \frac{d(z,G(y_n)) - d(z,G(v_{1n},y_{2n}))}{md(y_{1n},v_{1n})} - l\eta\\ &\ge \frac{1}{m(\mu+\varepsilon/2)} - l\eta. \qquad (26)\end{aligned}$$

Now by the choice of τ_0 and the fact that $lm\mu\eta < 1$, one can check that

$$m\mu < \frac{\tau_0 + \theta/2}{1 + l\eta(\tau_0 + \theta/2)}$$

and so, one can assume that $\varepsilon > 0$ is taken sufficiently small such that

$$\frac{1}{m(\mu+\varepsilon/2)} - l\eta > \frac{1}{\tau+\theta/2}. \tag{27}$$

Hence using inequalities (26) and (27), one gets

$$\limsup_{n\to+\infty} \frac{d(z,R(x_n,y_n)) - d(z,R(u_n,v_n))}{d_0((x_n,y_n),(u_n,v_n))} > \frac{1}{\tau_0+\theta/2}.$$

Pick $r \in (0,L)$ with $L := \lim_{n\to+\infty} d_0((u_n,v_n),(x_n,y_n)) > 0$ since

$$L = \lim_{n\to+\infty} md(v_{1n},y_{1n}) = \lim_{n\to+\infty} md(v_{1n},y_1) > 0.$$

Given $\varrho \in (0,1)$ such that $(1+\varrho)(\tau_0+\frac{\theta}{2}) < (1-\varrho)(\tau_0+\theta)$,

$$d_0((u_n,v_n),(x_n,y_n)) \geq r, \quad 0 < d_0((x_n,y_n),(x,y)) < \varrho r$$

$$d(z,R(x_n,y_n) < \varphi_R(x,y,z) + \frac{\varrho}{\tau_0+\theta/2} d_0((u_n,v_n),(x_n,y_n)), \tag{28}$$

and

$$d_0((u_n,v_n),(x_n,y_n)) < \left(\tau_0+\frac{\theta}{2}\right)[d(z,R(x_n,y_n)) - d(z,R(u_n,v_n))] \tag{29}$$

for n large enaugh. Now since $d_0((u_n,v_n),(x,y)) \leq (1+\varrho)d_0((u_n,v_n),(x_n,y_n))$, we obtain from (28) and (29),

$$\begin{aligned} d_0((u_n,v_n),(x,y)) &< \frac{1+\varrho}{1-\varrho}(\tau_0+\frac{\theta}{2})[\varphi_R((x,y),z) - \varphi_R((u_n,v_n),z)] \\ &< (\tau_0+\theta)[\varphi_R((x,y),z) - \varphi_R((u_n,v_n),z)] \end{aligned}$$

which leads to (18).

Fix $t \in (0,\min(\tau_0,\theta))$ and $s' > 0$ such that $s' < \min\left\{s, \frac{s}{6\tau_0}, \frac{sm}{6\tau_0}, \frac{s}{6l\tau_0}, \frac{\gamma}{4}\right\}$. For any $z \in B(\bar{z},s')$ and any $\bar{y} \in C(\bar{x},\bar{z})$, we have

$$\varphi_R((\bar{x},\bar{y}),z) \leq d(z,R(\bar{x},\bar{y})) \leq d(z,\bar{z}) < s',$$

$$\varphi_R((\bar{x},\bar{y}),z) < \inf_{(x,y)\in X\times Y} \varphi_R((x,y),z) + s'.$$

By applying now the Ekeland variational principle to the lower semicontinuous function $(x,y) \to \varphi_R((x,y),z)$ with respect to the distance d_0, one can find $(u,v) \in X \times Y$ such that

$$\varphi_R((u,v),z) \leq \varphi_R((\bar{x},\bar{y}),z) < s', \tag{30}$$

$$d_0((u,v),(\bar{x},\bar{y})) \leq s'(\tau_0+t) < 2\tau s', \tag{31}$$

$$\varphi_R((u,v),z) \leq \varphi_R((u',v'),z) + \frac{1}{\tau_0+t} d_0((u,v),(u',v')) \tag{32}$$

for any $(u',v') \neq (u,v)$.

From (31), $(u,v) \in B((\bar{x},\bar{y}),2\tau s')$ (for the distance d_0) hence one can check that $u \in B(\bar{x},s)$ and $v = (v_1,v_2) \in B(C_1(\bar{x},\bar{z}),s) \times B(C_2(\bar{x},\bar{z}),s)$. Let us prove now that $\varphi_R((u,v),z) = 0$. If $0 < \varphi_R((u,v),z) < s' < \gamma$ so from (18) and (32), there exists $(u',v')\in X\times Y$ such that $u'\in B(\bar{x},s)$ and $v=(v_1,v_2)\in F(u')\cap B(C(\bar{x},\bar{z}),s)$ so that

$$0<d_0((u,v),(u',v'))<(\tau_0{+}t)\left[\varphi_R((u,v),z)-\varphi_R((u',v'),z)\right]\le d_0((u,v),(u',v'))$$

which leads to a contradiction and thus $\varphi_R((u,v),z)=0$ i.e., $(u,v)\in R^{-1}(z)$.

Let $\bar{y}\in F(\bar{x})$ such that $\bar{z}\in G(\bar{y})$ and let $(x,y)\in B((\bar{x},\bar{y}),2\tau s')$. If $\varphi_R((x,y),z)\ge\gamma$,

$$\begin{aligned}d_0((x,y),R^{-1}(z))&\le d_0((x,y),(u,v))\\&\le d_0((x,y),(\bar{x},\bar{y}))+d_0((\bar{x},\bar{y}),(u,v))\\&<4\tau s'<\gamma\tau\le\tau\varphi_R((x,y),z).\end{aligned}$$

Suppose now that $\varphi_R(x,y,z)\in(0,\gamma)$ (the case $\varphi_R(u,v,z)=0$ is obvious). Consider

$$\vartheta(x,y):=\inf_{(\zeta,\xi)\in\Lambda(x,y)}\left(\sup_{X\times Y\ni(u',v')\neq(\zeta,\xi)}\frac{\varphi_R((\zeta,\xi),z)-\varphi_R((u',v'),z)}{d_0((\zeta,\xi),(u',v'))}\right),$$

where

$$\Lambda(x,y):=\big\{(\zeta,\xi)\in X\times Y: d_0((x,y),(\zeta,\xi))<d_0((x,y),R^{-1}(z))\text{ and }\varphi_R((\zeta,\xi),z)\le\varphi_R((x,y),z)$$

Hence by Theorem 2.7, it yields

$$\vartheta(x,y)d_0((x,y),R^{-1}(z))\le\varphi_R((x,y),z)\tag{33}$$

since $\varphi_R(\cdot,\cdot,z)$ is lower semicontinuous w.r.t. d_0 and

$$R^{-1}(z)=\{(a,b)\in X\times Y: z\in R(a,b)\}=\{(a,b)\in X\times Y:\varphi_R((a,b),z)\le 0\}.$$

Consequently, to obtain the desired result, we check that $\vartheta(x,y)\ge\dfrac{1}{\tau_0+\theta}$ which is obvious if $\Lambda(x,y)=\varnothing$.

Consider a point $(\zeta,\xi)\in\Lambda(x,y)$, thus

$$\begin{aligned}d_0((\zeta,\xi),(\bar{x},\bar{y}))&\le d_0((\zeta,\xi),(x,y))+d_0((x,y),(\bar{x},\bar{y}))\\&<d_0((x,y),R^{-1}(z))+d_0((x,y),(\bar{x},\bar{y}))\\&\le d_0((\bar{x},\bar{y}),R^{-1}(z))+2d_0((x,y),(\bar{x},\bar{y}))\\&\le d_0((\bar{x},\bar{y}),(u,v))+2d_0((x,y),(\bar{x},\bar{y}))\le 6\tau_0 s'.\end{aligned}$$

Hence $\zeta\in B(\bar{x},s)$ and $\xi\in B(C(\bar{x},\bar{z}),s)$ since $d(\xi_1,\bar{y}_1)<6\tau_0 s'm^{-1}<s$ and $d(\xi_2,\bar{y}_2)<6\tau_0 s'l<s$ so from (18), one can pick $(u'',v'')\neq(\zeta,\xi)$ such that $v''\in F(u'')\cap B(C(\bar{x},\bar{z}),s)$ and

$$\frac{\varphi_R((\zeta,\xi),z)-\varphi_R((u'',v''),z)}{d_0((\zeta,\xi),(u'',v''))}>\frac{1}{\tau_0+\theta}$$

which leads to $\vartheta(x,y)\ge\dfrac{1}{\tau_0+\theta}$. From (33) and by getting $\theta\to 0$, one has

$$d_0((x,y),R^{-1}(z))\le\tau_0\varphi_R((x,y),z).$$

Now as

$$d((x,y),(u,v))\le\max(1,l,m^{-1})d_0((x,y),(u,v))$$

we obtain the desired result with $\tau:=\max(1,l,m^{-1})\tau_0$. □

Consequently, we have the following Corollary.

Corollary 4.4. *Assume all assumptions of Theorem 4.3 satisfied. If, in addition, $G \circ F$ (with $F = (F_1, F_2)$) is coordinate locally stable around $(\bar{x}, \bar{z}) \in \mathrm{Gr}(G \circ F)$ then $G \circ F$ is metrically regular around $(\bar{x}, \bar{z})$.*

In the case where $Y_1 = Y_2 := Y$ is a linear space, endowed with a shift-invariant metric, the following corollary holds:

Corollary 4.5. *Let X, Y be complete metric spaces where Y is a linear space endowed with a shift-invariant metric. Asumme that the multimaps $F_1 : X \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$ are closed and.satisfy for $\bar{y} \in (F_1 + F_2)(\bar{x})$ the assumptions:*

(i) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta)$ and any $y \in (F_1 + F_2)(x) \cap B(\bar{y}, \delta)$ there exist $y_1 \in F_1(x) \cap B(F_1(\bar{x}), \varepsilon)$ and $y_2 \in F_1(x) \cap B(F_2(\bar{x}), \varepsilon)$ such that $y = y_1 + y_2$.

(ii) there exist $r_1, r_2 > 0$ such that F_1 is $m-$metrically regular on $B(\bar{x}, r) \times B(F_1(\bar{x}), r_1)$ and F_2 is $l-$pseudo-Lipschitz on $B(\bar{x}, r) \times B(F_2(\bar{x}), r_2)$

(iii) $0 < ml < 1, .$

Then $F_1 + F_2$ is metrically regular around $(\bar{x}, \bar{y})$.

Proof. Take $Z := Y$ and G the single-valued map defined by $G(y_1, y_2) = \{y_1 + y_2\}$. It is easy to see that G satisfies assumptions (iii)-(iv) of Theorem 4.3 with $m_G = l_G = 1$. Note also that since

$$G^{-1}(\bar{z}) = \{(y_1, y_2) \in Y \times Y : y_1 + y_2 = \bar{z}\},$$

$C_i(\bar{x}, \bar{z}) = F_i(\bar{x})$ for $i = 1, 2$. So all assumptions of Theorem 4.3 are satisfy. And from condition (i), $G \circ (F_1, F_2)$ is coordinate locally stable around $(\bar{x}, \bar{y})$. Then $G \circ (F_1, F_2) = F_1 + F_2$.is metrically regular around $(\bar{x}, \bar{y})$.

Note that condition (i) is related to the property of local sum-stability of the multimaps (F_1, F_2) at $(\bar{x}, \bar{y}_1, \bar{y}_2)$ with $\bar{y}_1 \in F_1(\bar{x})$ and $\bar{y}_2 \in F_2(\bar{x})$ which is defined in [49, 24] by: the pair (F_1, F_2) local sum-stable at $(\bar{x}, \bar{y}_1, \bar{y}_2)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta)$ and any $y \in (F_1 + F_2)(x) \cap B(\bar{y}_1 + \bar{y}_2, \delta)$ there are $y_1 \in F_1(x) \cap B(\bar{y}_1, \varepsilon)$ and $y_2 \in F_1(x) \cap B(\bar{y}_2, \varepsilon)$ such that $y = y_1 + y_2$. ☐

4.1. **Application to best proximity points.** Let A, B be nonempty subsets of a metric space and $T : A \rightrightarrows B$ be a multimap. If $A \cap B = \emptyset$, the inclusion $x \in T(x)$ has no solution. In this case, one can look for a point $x \in A$ such that $d(x, T(x))$ is minimum. Since $d(x, T(x))$ is at least $\mathrm{gap}(A, B)$, the point x is a solution of the equation

$$d(x, Tx) = \mathrm{gap}(A, B) := \inf_{(x,y) \in A \times B} d(x, y).$$

This point is called a best proximity point of T. Denote by $\mathrm{Best}(T) := \{x \in A : d(x, Tx) = \mathrm{gap}(A, B)\}$ the best proximity points set of T. Note that if $A \cap B \neq \varnothing$ and T is closed valued then

$$\mathrm{Best}(T) = \mathrm{Fix}(T) := \{x \in A : x \in Tx\}$$

that is, best proximity points coincide with fixed points of T.

Consider the following sets

$$A_0 := \{a \in A : d(a, b) = \mathrm{gap}(A, B) \text{ for some } b \in B\}$$
$$B_0 := \{b \in B : d(a, b) = \mathrm{gap}(A, B) \text{ for some } a \in A\}$$

it is clear that if $A_0 \neq \varnothing$ then so is B_0 and conversely. Note also that there are some sufficient conditions which guarantee the nonemptiness of A_0 and B_0. One such simple condition is that A is compact and B is approximatively compact with respect to A in the sense that for every sequence (x_n) of B such that $d(y, x_n) \to d(y, B)$ for some y in A admits a convergent subsequence.

We give now a simple existence result for a best proximity point. We refer the reader to [39, 57, 30] and the references therein for others results on best proximity points.

Theorem 4.6. *Let A, B be nonempty and closed subsets of a complete metric space such that $A \cap B = \varnothing$. Let $T : A \rightrightarrows B$ be a set-valued mapping with nonempty and closed values. Assume that A_0 is nonempty and closed with $T(A_0) \subset B_0$. Let $\bar{x} \in A_0$ and assume that there exist $r, \rho > 0$ such that the function $f_y : x \to d(y, T(x))$ satisfies*

$$d(x, f_y^{-1}(\omega)) \leq m\,|\omega - f_y(x)|$$

for all $x, y \in B(\bar{x}, r)$ and $\omega \in B(\bar{\omega}, \rho)$ with $\bar{\omega} := d(\bar{x}, T(\bar{x}))$ and $0 < m < 1$.

Then, $\mathrm{Best}(T) \neq \varnothing$ *and there exist a neighborhood U of $\bar{x}$ and $\tau > 0$ such that for any $x \in U$, one has*

$$d(x, \mathrm{Best}(T)) \leq \tau(d(x, T(x)) - \mathrm{gap}(A, B)).$$

Proof. Take $X = Y_1 = Y_2 := A_0$ and let $\alpha \in (0, 1)$. Define the map $G : Y_1 \times Y_2 \to \mathbb{R}^+$ such that

$$G(y_1, y_2) := \alpha(d(y_2, T(y_1)) - \mathrm{gap}(A, B))$$

and $R : X \times Y_1 \times Y_2 \rightrightarrows \mathbb{R}^+$ by

$$R(x, y_1, y_2) = \begin{cases} G(y_1, y_2) & \text{if } (y_1, y_2) \in F(x); \\ \varnothing & \text{otherwise} \end{cases}$$

with $F = (F_1, F_2) : X \rightrightarrows Y_1 \times Y_2$ such that $F_1(x) = F_2(x) := \{x\}$.

Hence

$$R(x, y_1, y_2) = \begin{cases} \alpha(d(x, T(x)) - \mathrm{gap}(A, B)) & \text{if } y_1 = y_2 = x; \\ \varnothing & \text{otherwise} \end{cases}$$

and observe

$$0 \in R(x, y_1, y_2) \iff y_1 = y_2 = x \text{ and } d(x, T(x)) = \mathrm{gap}(A, B)$$

so

$$(x, x, x) \in H^{-1}(0) \iff x \in \mathrm{Best}(T).$$

Given $\bar{z} := \alpha(d(\bar{x}, T(\bar{x})) - \mathrm{gap}(A, B)) \in H(\bar{x})$ so that $C(\bar{x}, \bar{z}) = F(\bar{x}) \cap G^{-1}(\bar{z}) = F(\bar{x}) = \{(\bar{x}, \bar{x})\}$. Assume $\bar{z} > 0$ (otherwise $\bar{x}$ is a best proximity point of T and there is nothing to prove).

Clearly, F_1 and F_2 satisfy hypothesis (i) and (ii) of Theorem 4.3 with $l_{F_2} = m_{F_1} = 1$. From the assumption, we have

$$d(x, f_y^{-1}(\omega)) \leq m\,|\omega - f_y(x)|$$

for all $\omega \in B(\bar{\omega}, \rho)$ and all $x, y \in B(\bar{x}, r)$.

Hence $G_{y_2} : y_1 \to G(y_1, y_2) = \alpha(d(y_2, T(y_1)) - \text{gap}(A, B))$ is m_G-metrically regular on $B(\bar{x}, r) \times B(\bar{z}, \alpha\rho)$, uniformly for all $y_2 \in B(\bar{x}, r)$ with $m_G := \alpha^{-1}m$. Indeed,

$$\begin{aligned} G_{y_2}^{-1}(z) &:= \{x \in A_0 : z = \alpha(d(y_2, T(x)) - \text{gap}(A, B))\} \\ &= \left\{x \in A_0 : f_{y_2}(x) = \alpha^{-1}z + \text{gap}(A, B)\right\} = f_{y_2}^{-1}(\omega) \end{aligned}$$

with $\omega := \alpha^{-1}z + \text{gap}(A, B) \in B(\bar{\omega}, \rho)$ and so

$$\begin{aligned} d(x, G_{y_2}^{-1}(z)) &\leq m\,|\omega - f_{y_2}(x)| = m\left|\alpha^{-1}z + \text{gap}(A, B) - d(y_2, T(x))\right| \\ &\leq \alpha^{-1}m\,|z - G_{y_2}(x)| \end{aligned}$$

On the other hand, as the function $d(\cdot, K)$ is $1-$Lipschitz for any nonempty subset K,

$$\left|d(v, T(u)) - d(v', T(u))\right| \leq d(v, v')$$

for all $u, v, v' \in A_0$. Hence $G_{y_1} : y_2 \to G(y_1, y_2) = \alpha(d(y_2, T(y_1)) - \text{gap}(A, B))$ is l_G-Lipschitz on $B(\bar{x}, r) \times B(\bar{z}, \alpha\rho)$, uniformly for all $y_1 \in B(\bar{x}, r)$ with $l_G := \alpha$. Finally, condition (v) is obviously satisfied.

Now from Theorem 4.3, we deduce that there exists a neighborhood $U \times W$ of $(\bar{x}, \bar{z})$ and $\tau > 0$ such that for every $(x, z) \in U \times W$, we have

$$d((x, x, x), R^{-1}(z)) \leq \tau\varphi_R((x, x, x), z) \leq \tau d(z, R(x, x, x)).$$

Since for α sufficiently small, $0 \in W$ and thus for any $x \in U$, one has

$$\begin{aligned} d(x, \text{Best}(T)) &= d((x, x, x), R^{-1}(0)) \leq \tau\varphi_R((x, x, x), 0) \\ &\leq \tau d(0, R(x, x, x)) = \tau\alpha(d(x, T(x)) - \text{gap}(A, B)) \end{aligned}$$

and since $T(x) \neq \varnothing$ for any $x \in X$,

$$d(x, \text{Best}(T)) < \infty$$

consequently, $\text{Best}(T)$ is nonempty.

Following the idea of Durea et al. in [23], one can also define the multimap $S : X \rightrightarrows Y_1 \times Y_2$ such that

$$S(x) := \{y \in A_0 : d(y, T(x)) = \text{gap}(A, B)\}.$$

Observe that $S(x)$ is nonempty and closed for all $x \in A_0$ and

$$\text{Fix}(S) \subset \text{Best}(T).$$

Hence it suffices to prove that the set $\text{Fix}(S)$ is nonempty. For this aim, consider the map $G : Y_1 \times Y_2 \to \mathbb{R}^+$ such that

$$G(y_1, y_2) := d(y_2, S(y_1))$$

and let the composition $H : X \to \mathbb{R}^+$:

$$H(x) := G \circ F(x) = G(x, x).$$

Hence $H^{-1}(0) = \text{Fix}(S)$ since

$$0 \in H(x) \iff d(x, S(x)) = 0.$$

Now from Theorem 4.3 and Proposition 4.2, we deduce that there exist a neighborhood U of $\bar{x}$ and $\tau > 0$ such that for every $x \in U$, we have

$$d(x, H^{-1}(0)) \leq \tau d(0, G(x, x)) = \tau d(x, S(x)) < \infty$$

since $S(x) \neq \varnothing$ for any $x \in X$ thus $H^{-1}(0) \neq \varnothing$ and so is $\mathrm{Fix}(S)$. Consequently, $\mathrm{Best}(T)$ is nonempty. $\square$

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