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Error estimation of Signals by Euler-Nörlund Operators

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Abstract

Mainly speaking, signals are treated as functions of one variable and images are represented by functions of two variables. Positive approximation processes play an important role in Approximation Theory and appear in a very natural way dealing with approximation of continuous functions, especially one, which requires further qualitative properties such as monotonicity, convexity and shape preservation and so on. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. In this paper, a new estimate for the error in approximation of a signal (function) by Euler-Nörlund operator of its Fourier series has been determined. Some corollaries have also been deduced from our main theorem and hence some results become particular cases in this direction.

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1 Introduction and Preliminaries

The theory of summability arises from the process of summation of series and the significance of the concept of summability has been strikingly demonstrated in various contexts (c.f. [23], [22]), e.g. in Analytic Continuation, Quantum Mechanics, Probability Theory, Fourier Analysis, Approximation Theory and Fixed Point Theory. The methods of almost summability and statistical summability have become an active area of research in recent years. The degree of approximation of functions belonging to $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$, $(r \geq 1)$ - classes by general summability matrices has been proved by various investigators like Govil [5], Khan [[9]-[10]], Mohapatra and Chandra [21], Mohapatra [20], Mishra and Mishra [14] and Mishra et al. [[17]-[18]] and many others [1, 2, 3, 12].
For a given 2π-periodic signal (function) $f \in L^p := L^p[0, 2\pi], p \geq 1$, let

$$s_n(f;x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos\, kx + b_k sin\, kx), \tag{1}$$

$n \in N$ with $s_0(f;x) = \frac{a_0}{2}$ denote the $(n+1)^{th}$ partial sums, called trigonometric polynomials of degree (order) n, of the Fourier series of f, and $N_0 = N \cup \{0\}, N = \{1, 2, 3, ...\}$.
A signal (function) $f \in Lip\ \alpha$, if

$$f(x+t) - f(x) = O(|t^{\alpha}|)\ for\ 0 < \alpha \leq 1, t > 0$$

and $f \in Lip(\alpha, r)$, $for\ a \leq x \leq b$, if

$$\left(\int_a^b |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} \leq M(|t|^{\alpha}),\ 0 < \alpha \leq 1,\ r \geq 1,\ t > 0,$$

where M is an absolute positive constant not necessarily the same at each occurrence (see McFadden [13]). If we take $r \to \infty$ then $Lip(\alpha, r) \equiv Lip\ \alpha$.
For a given positive increasing function $\xi(t)$, $f \in Lip(\xi(t), r)$ if

$$\|f(x+t) - f(x)\|_r = \left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\Big(\xi(t)\Big),\ r \geq 1,\ t > 0.$$

Given positive increasing function $\xi(t)$, an integer $r \geq 1$, a signal (function) f is said to belong to the generalized weighted Lipschitz $W^{'}(L_r, (\xi(t)))$-class

[11], if

$$\left\|[f(x+t)-f(x)]sin^{\beta}\Big(x/2\Big)\right\|_r = O\Big(\xi(t)\Big),\ \beta \geq 0,\ t>0. \qquad (2)$$

If $\beta = 0$, then the generalized weighted Lipschitz $W'(L_r, (\xi(t)))$ class coincides with the class $Lip(\xi(t), r)$, we observe that

$$W'(L_r, \xi(t)) \xrightarrow{\beta=0} Lip(\xi(t), r) \xrightarrow{\xi(t)=t^{\alpha}} Lip(\alpha, r) \xrightarrow{r\to\infty} Lip(\alpha)\ \ for\ 0<\alpha\leq 1,\ r\geq 1, t>0.$$

Let $\sum u_n := \sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its $(n+1)^t h$ partial sums $\{s_n\}$. The (E,q) transform is defined as the n^{th} partial sum of $(E,q),\ q>0$ summability and we denote it by E_n^q.
If

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k \to s, \text{ as } n\to\infty,\ \forall q>0,$$

then the series $\sum_{n=0}^{\infty} u_n$ is summable (E,q) to a definite number s [6]. Clearly (E,q) method is regular [6].
Let $\{p_n\}$ be a non-negative sequence of constants, real or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0\ \forall\, n\geq 0,\ p_{-1} = 0 = P_{-1} \text{ and } P_n \to \infty \text{ as } n\to\infty.$$

The sequence to sequence transformation $t_n^N = \sum_{v=0}^{n} \frac{p_{n-v}s_v}{P_n}$ defines the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be N_p summable to the sum s if $\lim_{n\to\infty} t_n^N$ exists and is equal to a finite number s.
The conditions for regularity of Nörlund summability (N, p_n) are easily seen to be [6] i.e.

$$(i)\frac{p_k}{P_n} \to 0 \text{ as } n\to\infty,\quad (ii)\sum_{k=0}^{n}|p_k| = O(P_n) \text{ as } n\to\infty.$$

Further, the (E,q) transformation of the (N,p_n) transform of $\{s_n\}$ is defined by

$$E_n^q t_n^N = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} t_k^N = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\sum_{v=0}^{k}\frac{p_{k-v}s_v}{P_k}.$$

Thus if, $E_n^q t_n^N \to s$ as $n \to \infty$ then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,q)(N,p_n)$ method to a definite number 's'.

$$\begin{aligned} s_n \to s \;&\Rightarrow\; (N,p_n)(s_n) = t_n^N = \sum_{v=0}^{n} \frac{p_{n-v}s_v}{P_n} \to s, \text{ as } n \to \infty, (N,p_n) \text{ method is regular,} \\ &\Rightarrow\; ((E,q)(N,p_n)(s_n)) = E_n^q t_n^N \to s,\ as\ n \to \infty, (E,q) \text{ method is regular,} \\ &\Rightarrow\; (E,q)(N,p_n) \text{ method is regular.} \end{aligned}$$

We note that t_n^N and E_n^q are also trigonometric polynomials of degree (or order) n.
L_∞- norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \sup\{|f(x)| : x \in R\}$.
L_r- norm of a function is defined by $\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}$, $1 \leq r < \infty$.
The space $L_r[0,2\pi]$ with $r = \infty$ with includes the space $C_{2\pi}$ of all $2\pi-$ periodic continuous functions over $[0,2\pi]$, where $C_{2\pi}$ denote the Banach space of all $2\pi-$ periodic continuous functions defined on $[0,2\pi]$.
The degree of approximation of a function $f : R \to R$ by trigonometric polynomial t_n of order n under sup norm $\|\|_\infty$ is defined by [28].

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}.$$

and $E_n(f)$ of a function $f \in L_r$ is given by $E_n(f) = \min_{t_n} \|t_n - f\|_r$. This method of approximation is called Trigonometric Fourier approximation (TFA). We write throughout the paper
$\phi_x(t) = f(x+t) - 2f(x) + f(x-t)$,
$(EN)_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ q \end{array}\right) q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} \frac{\sin(v+1/2)t}{sin(t/2)} \right\}$.

2 Main Result

Approximation by trigonometric polynomials is at the heart of approximation theory. The most important trigonometric polynomials used in the approximation theory are obtained by linear summation methods of Fourier series of 2π-periodic signals (functions) on the real line R (i.e. Euler means, Nörlund means and Product Euler-Nörlund means, etc.). The product summability methods are more powerful than the individual summability methods and thus give an approximation for wider class of functions than the individual methods. The main aim of this paper is to determine the error in approximation by Euler-Nörlund operator of the Fourier series in various Lipschitz classes. We prove:

Theorem 2.1. *If f is a 2π-periodic signal (function) and integrable in the sense of Lebesgue over $[0, 2\pi]$. Then the error in approximation of $f \in W'(L_r, \xi(t))(r \geq 1)-$class with $0 \leq \beta \leq 1 - \frac{1}{r}$ by $(E, q)(N, p_n)$ operator of its Fourier series is given by*

$$\|\tau_n(f(t); x) - f(x)\|_r = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}, \ \forall n > 0, \tag{3}$$

provided that positive increasing function $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{is non-increasing in} `t', \tag{4}$$

$$\left\{\int_0^{\frac{\pi}{n+1}} \left(\frac{|\phi_x(t)|}{\xi(t)}\right)^r \sin^{\beta r}\left(\frac{t}{2}\right) dt\right\}^{1/r} = O(1), \tag{5}$$

and

$$\left(\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^r dt\right)^{1/r} = O((n+1)^{\delta}), \tag{6}$$

where δ is arbitrary number such that $(\beta - \delta)s - 1 > 0, r^{-1} + s^{-1} = 1, 1 \leq r \leq \infty$, conditions (5) and (6) hold uniformly in x.

Note 1. Using condition (4), we get the inequality:
$\xi\left(\frac{\pi}{n+1}\right) \leq \pi\xi\left(\frac{1}{n+1}\right)$, for $\left(\frac{\pi}{n+1}\right) \geq \left(\frac{1}{n+1}\right)$.

Note 2. The product transform $(E, q)(N, p_n)$ plays an important role in signal theory as a double digital filter ([15]-[16]) and the theory of Machines in Mechanical Engineering [[15], Note 3].

3 Lemmas

We need the following new type of lemmas for the proof of our Theorem.

Lemma 3.1. $|(EN)_n(t)| = O(n)$ *for* $0 \leq t \leq \frac{\pi}{(n+1)}$.

Proof. For $0 \le t \le \frac{\pi}{(n+1)}$, $|\sin nt| \le n|\sin nt|$, we have

$$\begin{aligned}
|(EN)_n(t)| &\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\frac{\sin(v+1/2)t}{\sin(t/2)}\right\}\right| \\
&\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\frac{(2v+1)\sin t/2}{\sin t/2}\right\}\right| \\
&\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}(2k+1)\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\right\}\right| \\
&\le \frac{(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\right| = O(n).
\end{aligned}$$

□

Lemma 3.2. *For $\pi/(n+1) < t \le \pi$, we have $|(EN)_n(t)| = O(1/t)$.*

Proof. For $\pi/(n+1) \le \pi$, $(\sin t/2)^{-1} \le \pi/t$ for $0 < t \le \pi$, $|\sin nt| \le 1$, we have

$$\begin{aligned}
|(EN)_n(t)| &\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\frac{\sin(v+1/2)t}{\sin(t/2)}\right\}\right| \\
&\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\frac{\pi}{t}\right\}\right| \\
&\le \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\right\}\right| \\
&\le \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{q}q^{n-k}\right| = O(1/t).
\end{aligned}$$

□

4 Proof of Theorem 2.1

Using Riemann-Lebesgue theorem, following Titchmarsh [27], the partial sums $s_n(f(t);x)$ of the Fourier series is given by

$$s_n(f(t);x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\frac{\sin(n+1/2)t}{\sin(t/2)}dt.$$

The (N,p_n) transform of $s_n(f(t);x)$ is given by

$$t_n^N(f(t);x)-f(x)=\frac{1}{2\pi P_n}\int_0^\pi \phi_x(t)\sum_{k=0}^{n}p_{n-k}\frac{\sin(n+1/2)t}{\sin(t/2)}dt.$$

Representing $(E,q)(N,p_n)$ means of $\{s_n(f(t);x)\}$ by $\tau_n(f(t);x)$, we have

$$\begin{aligned}\|\tau_n(f(t);x)-f(x)\| &= \frac{1}{2\pi(1+q)^n}\int_0^\pi \phi_x(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_{k-v}\frac{\sin(v+1/2)t}{\sin(t/2)}\right\}dt\\ &= \int_0^\pi \phi_x(t)(EN)_n(t)\,dt\\ &= \left[\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right]\phi_x(t)(EN)_n(t)\,dt\\ &= I_1+I_2, \text{say}. \qquad (7)\end{aligned}$$

Using Hölder's inequality, $\phi_x(t)\in W'(L_r,\xi(t))$, condition (5), Lemma 3.1, Note 1, $(\sin(t/2))^{-1}\le\frac{\pi}{t}$, for $0<t\le\pi$ and Second Mean Value Theorem for integrals, we get

$$\begin{aligned}|I_1| &= \frac{1}{2\pi(1+q)^n}\left|\int_0^{\pi/(n+1)}\frac{\phi_x(t)\,\sin^\beta(t/2)}{\xi(t)}\frac{\xi(t)(EN)_n(t)}{\sin^\beta(t/2)}dt\right|\\ &\le \left\{\int_0^{\pi/(n+1)}\left(\frac{\phi_x(t)\,\sin^\beta(t/2)}{\xi(t)}\right)^r dt\right\}^{1/r}\left\{\lim_{\epsilon\to0}\int_\epsilon^{\pi/(n+1)}\left(\frac{\xi(t)|(EN)_n(t)|}{\sin^\beta(t/2)}\right)^s dt\right\}^{1/s}\\ &= O(1)\left\{\lim_{\epsilon\to0}\int_\epsilon^{\pi/(n+1)}\left(\frac{\xi(t)n}{\sin^\beta(t/2)}\right)^s dt\right\}^{1/s}\\ &= O\left\{n\xi\left(\frac{\pi}{(n+1)}\right)\right\}\left\{\lim_{\epsilon\to0}\int_\epsilon^{\pi/(n+1)}t^{-\beta s}dt\right\}^{1/s}\\ &= O\left\{n\xi\left(\frac{\pi}{n+1}\right)\right\}O\{(n+1)^{\beta-\frac{1}{s}}\}\\ &= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}, r^{-1}+s^{-1}=1. \qquad (8)\end{aligned}$$

Now, again by Hölder's inequality, we find

$$|I_2|\le\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\phi_x(t)|\sin^\beta(t/2)}{\xi(t)}\right)^r dt\right]^{1/r}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta}sin^\beta(t/2)}|(EN)_n(t)|\right)^s dt\right]^{1/s}.$$

Using $(\sin t/2)^{-1} \leq \pi/t$, for $0 < t \leq \pi$, $|\sin t/2| \leq 1$, conditions (4), (6), Note 1, Lemma 3.2, Second Mean Value Theorem for integrals, we get

$$\begin{aligned} |I_2| &= O(n^\delta)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^s dt\right]^{1/s} \\ &= O(n^\delta)\left[\int_{1/\pi}^{(n+1)/\pi}\left(\frac{\xi(1/y)}{y^{\delta-\beta-1}}\right)^s \frac{dy}{y^2}\right]^{1/s} \\ &= O\left(n^\delta\frac{(n+1)}{\pi}\xi\left[\frac{\pi}{(n+1)}\right]\right)\left(\int_{\epsilon_1}^{\frac{(n+1)}{\pi}} y^{(\beta-\delta)s-2}dy\right)^{1/s}, \text{ for some } (1/\pi) < \epsilon_1 < (n+1)/\pi \\ &= O\left(n^{\delta+1}\xi\left[\frac{1}{(n+1)}\right]\right)\left(\frac{(n+1)^{(\beta-\delta)s-1} - (\epsilon_1)^{(\beta-\delta)s-1}}{(\beta-\delta)s-1}\right)^{1/s} \\ &= O\left(n^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right)O\{(n+1)^{\beta-\delta-1/s}\} \\ &= O\left\{(n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right\}. \end{aligned} \tag{9}$$

On collecting (7)-(9), we have

$$|\tau_n(f(t);x) - f(x)| = O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right). \tag{10}$$

Now, using the L_r-norm of function, we get

$$\begin{aligned} \|\tau_n(f(t);x) - f(t)\|_r &= \left\{\int_0^{2\pi}|\tau_n(f(t);x) - f(t)|^r dx\right\}^{1/r} \\ &= O\left(\int_0^{2\pi}\left\{(n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right)^{1/r} \\ &= O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\left(\int_0^{2\pi}dx\right)^{1/r}\right) \\ &= O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{\sqrt{n+1}}\right)\right). \end{aligned}$$

This completes the proof of Theorem 2.1.

5 Applications

Fourier series and operators based on it are of great importance in both theoretical and applied mathematics because they can be considered as representation of a signal or function. Hardy and Littlewood [7] have specified without proof that the class of functions $Lip(\alpha, p)$ is identical with the class of functions approximable in the L_r-norm with degree of approximation $O(n^{-\alpha})$, by trigonometric polynomials of order n. Trigonometric Fourier approximation, originated from a well known theorem of Approximation theory i.e. Weierstrass Approximation Theorem, which become an exciting interdisciplinary field of study for the last 130 years. These approximations have assumed important new dimensions due to their wide range of applications in signal theory [24], in general and digital signal processing [25] in particular, in view of classical Shannon sampling theorem [18]. With the intention of applications, sharper evaluates of infinite matrices [4] are capable to get bounds for the lattice norms (which take place in solid state physics) of matrix valued functions and enables to investigate perturbations of matrix valued functions and then compare them. The following corollaries can be derived from our main Theorem 2.1.

Corollary 5.1. *If $\beta = 0$, then the generalized weighted Lipschitz $W^{'}(L_r, \xi(t))(r \geq 1)-$ class reduces to the class $Lip(\xi(t), r)$ and the degree of approximation of a function f belonging to the class $Lip(\xi(t), r)$ is given by*

$$|\tau_n(f(t); x) - f(t)| = O((n+1)^{\frac{1}{r}}\xi(1/(n+1))). \tag{11}$$

Proof. The result follows by setting $\beta = 0$ in equation (3), we have for $r \geq 1$

$$\|\tau_n(f(t); x) - f(t)\|_r = \left\{ \int_0^{2\pi} |\tau_n(f(t); x) - f(t)|^r dx \right\}^{1/r} = O\left((n+1)^{1/r}\xi\left(\frac{1}{(n+1)}\right)\right).$$

Thus, we get

$$|\tau_n(f(t); x) - f(t)| = \{\int_0^{2\pi} |\tau_n(f(t); x) - f(t)|^r dx\}^{1/r} = O((n+1)^{1/r}\xi(\tfrac{1}{(n+1)})).$$

□

Corollary 5.2. *If $\beta = 0, \xi(t) = t^{\alpha}, 0 < \alpha \leq 1$, then the generalized weighted Lipschitz $W^{'}(L_r, \xi(t))(r \geq 1)-$ class reduces to the class $Lip(\alpha, r)$, $(1/r) < \alpha < 1$ and then the error in approximation of a function f belonging to the class $Lip(\alpha, r)$ is given by*

$$|\tau_n(f(t); x) - f(t)| = O\left(\frac{1}{(n+1)^{\alpha - 1/r}}\right).$$

Proof. Putting $\beta = 0, \xi(t) = t^\alpha, 0 < \alpha \le 1$ in Theorem 2.1, we have
$\|\tau_n(f(t);x) - f(t)\|_r = \left\{ \int_0^{2\pi} |\tau_n(f(t);x) - f(t)|^r dx \right\}^{1/r}$
or, $O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right) = \left\{ \int_0^{2\pi} |\tau_n(f(t);x) - f(t)|^r dx \right\}^{1/r}$
or, $O(1) = \left\{ \int_0^{2\pi} |\tau_n(f(t);x) - f(t)|^r dx \right\}^{1/r} O\left\{ \frac{1}{(n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)} \right\}$,
since otherwise the right hand side of the above equation will not be $O(1)$. Hence,

$$|\tau_n(f(t);x) - f(t)| = O\left(\left(\frac{1}{n+1}\right)^\alpha (n+1)^{1/r}\right) = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right).$$

□

Corollary 5.3. *If* $\beta = 0, \xi(t) = t^\alpha$ *for* $0 < \alpha < 1$ *and* $r \to \infty$ *in (3), then* $f \in Lip\alpha$*. In this case, the degree of approximation of a function* f *belonging to the class* $Lip\alpha\,(0 < \alpha < 1)$ *is given by*

$$|\tau_n(f(t);x) - f(t)| = O((n+1)^{-\alpha}).$$

Proof. For $r \to \infty$ in Corollary 5.2, we get
$\|\tau_n(f(t);x) - f(t)\|_\infty = \sup_{0 \le x \le 2\pi} |\tau_n(f(t);x) - f(t)| = O((n+1)^{-\alpha})$.
Thus, we have

$$\begin{aligned} |\tau_n(f(t);x) - f(t)| &\le \|\tau_n(f(t);x) - f(t)\|_\infty \\ &= \sup_{0 \le x \le 2\pi} |\tau_n(f(t);x) - f(t)| = O((n+1)^{-\alpha}), \end{aligned}$$

which in turn generalizes the result of Sarangi et al. [26] and Misra et al. [19]. □

6 Conclusion

Several fascinating results regarding to the error in approximation of periodic signals (functions) belonging to the different Lipschitz classes by product transforms have been reviewed. Further, a suitable set of constraints have been discussed to rectify the errors in this direction. Some fascinating applications of the approximation of signals used in this manuscript have also been highlighted. The results of our theorem and corollaries are more general rather than the results of any other previous proved lemmas and theorems, which will be enrich the literate of Applications of summability analysis in the theory of approximations. The researchers and professionals working or intend to work in areas of analysis and its applications will find this research article to be quite useful. Consequently, the results so established may be found useful in several interesting situations appearing in the literature on Mathematical Analysis, Applied Mathematics and Mathematical Physics etc.

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