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STABILITY OF GENERAL A-QUARTIC FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. The aim of this article is to study the Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability in non-Archimedean Intuitionistic fuzzy normed spaces. The paper introduces a new A-quartic functional equation and obtain solution for the same functional equation. Further, stability problem is investigated for the newly introduced A-quartic functional equation in non-Archimedean intuitionistic fuzzy normed spaces.

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1. Introduction

In 1940, at the University of Wisconsin, Ulam [21] raised the first stability problem. The problem was solved by a number of mathematicians including Hyers [6], Aoki [2], Rassias [19], Rassias [17], Găvruta [5] and Rassias [18]. The Ulam problem garnered world wide attention and led to several stability problems of functional equations such as the Hyers-Ulam-Rassias stability, Ulam-Găvruta-Rassias stability, generalized Hyers-Ulam-Rassias stability and J.M. Rassias stability problems. Several researchers have been investigating the Hyers-Ulam-Rassias stability in different spaces. Some of the recent publications relevant to this paper are Bodaghi et al. [1] and Xu et al. [23].

Starting with Katsaras[8], a lot of researchers such as Wu and Fang [22], Biswas [3], Cheng and Mordeson [4], Kramosil and Michalek[7] have dealt with the Hyers-Ulam-Rassias problem using the fuzzy concept.

The idea of intuitionistic fuzzy normed space was introduced in [20] and further studied in [10]–[16] and [25], to deal with some summability problems. Some definitions and preliminaries concerning our main results can also be obtained from Ehsan and Mursaleen [9] and Mohiuddine et al. [11].

In this paper, we obtain the general solution and investigate the generalized Hyers-Ulam-Rassias and Hyers-Ulam-Rassias stabilities of the new generalized A-quartic functional equation

$$\begin{aligned} f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay) \\ = 2a^2\{f(x+y)+f(x-y)\}+2(a^2-1)^2\{f(x)+f(y)\} \end{aligned} \tag{1}$$

in non-Archimedean intuitionistic fuzzy normed spaces, where $a \neq 0, \pm 1$.

The paper is organized as follows: In Section-2, we obtain the general solution of the functional equation (1), Section-3 discusses generalized Hyers-Ulam-Rassias and Hyers-Ulam-Rassias stabilities in non-Archimedean intuitionistic fuzzy normed spaces of the functional equation (1), and the conclusion is given in section-4.

2. General solution of A-quartic functional equation

In this section, we obtain a general solution of the functional equation (1). Throughout this section, let X and Y be vector spaces. Some basic facts on n-additive symmetric mappings can be found in [24].

Theorem 2.1. *A function $f : X \to Y$ is a solution of the functional equation (1) if and only if f is of the form $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : X^4 \to Y$.*

Proof. Assume f satisfies the functional equation (1). Letting (x, y) by $(0, 0)$ in (1), we get $f(0) = 0$. Setting $y = 0$ in (1), we obtain

$$f(ax) = a^4 f(x) \tag{2}$$

for all $x \in X$. Thus f is quartic. Putting $x = 0$ in (1) and using (2), we obtain $f(-y) = f(y)$ for all $y \in X$. Thus f is an even function. The functional equation (1) can rewrite in the form

$$\begin{aligned} f(x) &- \frac{1}{2(a^2-1)^2} f(ax+y) - \frac{1}{2(a^2-1)^2} f(ax-y) \\ &- \frac{1}{2(a^2-1)^2} f(x+ay) + \frac{2a^2}{2(a^2-1)^2} f(x+y) \\ &+ \frac{2a^2}{2(a^2-1)^2} f(x-y) - \frac{1}{2(a^2-1)^2} f(x-ay) + f(y) = 0 \end{aligned} \tag{3}$$

for all $x, y \in X$. By [24, Theorems 3.5 and 3.6], f is a generalized polynomial function of degree at most 4, that is, f is of the form

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{4}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3, 4$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and the function f is even. Thus $A^3(x) = A^1(x) = 0$. It follows that $f(x) = A^4(x) + A^2(x)$. By (2) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in Q$, we obtain $a^2A^2(x) = a^4A^2(x)$. Hence $A^2(x) = 0$ for all $x \in X$. Therefore $f(x) = A^4(x)$.

Conversely, assume that $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : X^4 \to Y$. From

$$\begin{aligned} A^4(x+y) &= A^4(x) + 4A^{3,1}(x,y) + 6A^{2,2}(x,y) + 4A^{1,3}(x,y) + A^4(y), \\ A^4(rx) &= r^4A^4(x), \\ A^{3,1}(x,ry) &= rA^{3,1}(x,y),\ A^{3,1}(rx,y) = r^3A^{3,1}(x,y), \\ A^{2,2}(x,ry) &= r^2A^{2,2}(x,y),\ A^{2,2}(rx,y) = r^2A^{2,2}(x,y), \\ A^{1,3}(x,ry) &= r^3A^{1,3}(x,y),\ A^{1,3}(rx,y) = rA^{1,3}(x,y) \end{aligned}$$

for all $x, y \in X, r \in Q$, we see that f satisfies (1), which completes the proof of Theorem 2.1. □

3. Stability of A-quartic functional equation

In this section, we determine the generalized Hyers-Ulam-Rassias stability and Hyers-Ulam-Rassias stability concerning the A-quartic functional equation (1) in non-Archimedean intuitionistic fuzzy normed space.

Theorem 3.1. *Let K be a non-Archimedean field, X a vector space over K and (Z, τ', ν') be a non-Archimedean intuitionistic fuzzy normed space over K. Let $\psi : X \times X \to Z$ is a function such that for some $\alpha > 0$ and some positive integer $k, k \geq 2$ with $|a^k| < \alpha, |a| \neq 1$ and $a \neq 0$ such that*

$$\begin{aligned} &\tau'(\psi(a^{-k}x, a^{-k}y), t) \geq \tau'(\psi(x, y), \alpha t), \\ &\nu'(\psi(a^{-k}x, a^{-k}y), t) \leq \nu'(\psi(x, y), \alpha t), \end{aligned} \tag{5}$$

for all $x, y \in X$ and $t > 0$. Let (Y, τ, ν) be a non-Archimedean intuitionistic fuzzy Banach space over K and let $f : X \to Y$ be a $\psi -$ approximately quartic mapping in the sense that

$$\begin{aligned} &\tau\big(f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \\ &\quad - 2a^2\{f(x+y) + f(x-y)\} - 2(a^2-1)^2\{f(x) + f(y)\}, t\big) \\ &\qquad \geq \tau'(\psi(x, y), t), \\ &\nu\big(f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \\ &\quad - 2a^2\{f(x+y) + f(x-y)\} - 2(a^2-1)^2\{f(x) + f(y)\}, t\big) \\ &\qquad \leq \nu'(\psi(x, y), t), \end{aligned} \tag{6}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\begin{aligned} &\tau\big(f(x) - Q(x), t\big) \geq M(x, \alpha t), \\ &\nu\big(f(x) - Q(x), t\big) \leq N(x, \alpha t), \end{aligned} \tag{7}$$

for all $x, y \in X$ and $t > 0$, where

$$\begin{aligned} M(x,t) &= \tau'(\psi(x,0),2t) \star \tau'(\psi(ax,0),2t) \star \cdots \star \tau'(\psi(a^{k-1}x,0),2t), \\ N(x,t) &= \nu'(\psi(x,0),2t) \diamond \nu'(\psi(ax,0),2t) \diamond \cdots \diamond \nu'(\psi(a^{k-1}x,0),2t), \end{aligned} \tag{8}$$

for all $x, y \in X$ and $t > 0$.

Proof. First, we show that by induction on j for all $x \in X, t > 0$ and $j \geq 1$.

$$\begin{aligned} \tau(f(a^jx) - a^{4j}f(x), t) &\geq M_j(x, t) \\ &= \tau'(\psi(x,0),2t) \star \cdots \star \tau'(\psi(a^{j-1}x,0),2t), \\ \nu(f(a^jx) - a^{4j}f(x), t) &\leq N_j(x, t) \\ &= \nu'(\psi(x,0),2t) \diamond \cdots \diamond \nu'(\psi(a^{j-1}x,0),2t), \end{aligned} \tag{9}$$

for all $x, y \in X$ and $t > 0$. Putting $y = 0$ in (6), we obtain

$$\begin{aligned} &\tau(f(ax) - a^4f(x), t) \geq \tau'(\psi(x,0),2t), \\ &\nu(f(ax) - a^4f(x), t) \leq \nu'(\psi(x,0),2t), \end{aligned} \tag{10}$$

for all $x \in X, t > 0$. This proves (9) for $j = 1$. Let (9) hold for some $j > 1$. Replacing y by 0 and x by $a^j x$ in (6), we get

$$\begin{aligned} \tau(f(a^{j+1}x) - a^4 f(a^j x), t) \geq \tau'(\psi(a^j x, 0), 2t), \\ \nu(f(a^{j+1}x) - a^4 f(a^j x), t) \leq \nu'(\psi(a^j x, 0), 2t), \end{aligned} \tag{11}$$

for all $x \in X, t > 0$. Since $|a| \leq 1$, it follows that

$$\begin{aligned} &\tau\left(f(a^{j+1}x) - a^{4(j+1)}f(x), t\right) \\ &\quad = \tau\left(f(a^{j+1}x) - a^4 f(a^j x) + a^4 f(a^j x) - a^{4(j+1)}x, t\right) \\ &\quad = \tau\left(f(a^{j+1}x) - a^4 f(a^j x), t\right) \star \tau\left(a^4(f(a^j x) - a^{4j}f(x)), t\right) \\ &\quad \geq \tau'(\psi(a^j x, 0), 2t) \star \tau\left(f(a^j x) - a^{4j}f(x), \frac{t}{|a|^4}\right) \\ &\quad \geq \tau'(\psi(a^j x, 0), 2t) \star \tau\left(f(a^j x) - a^{4j}f(x), t\right) \\ &\quad \geq \tau'(\psi(a^j x, 0), 2t) \star M_j(x, t) = M_{j+1}(x, t), \\ &\nu\left(f(a^{j+1}x) - a^{4(j+1)}f(x), t\right) \\ &\quad = \nu\left(f(a^{j+1}x) - a^4 f(a^j x) + a^4 f(a^j x) - a^{4(j+1)}x, t\right) \\ &\quad = \nu\left(f(a^{j+1}x) - a^4 f(a^j x), t\right) \diamond \nu\left(a^4(f(a^j x) - a^{4j}f(x)), t\right) \\ &\quad \leq \nu'(\psi(a^j x, 0), 2t) \diamond \nu\left(f(a^j x) - a^{4j}f(x), \frac{t}{|a|^4}\right) \\ &\quad \leq \nu'(\psi(a^j x, 0), 2t) \diamond \nu\left(f(a^j x) - a^{4j}f(x), t\right) \\ &\quad \leq \nu'(\psi(a^j x, 0), 2t) \diamond M_j(x, t) = M_{j+1}(x, t), \end{aligned} \tag{12}$$

for all $x \in X$ and $t > 0$. Thus (9) holds for all $j \geq 1$. In particular, we have

$$\begin{aligned} \tau(f(a^k x) - a^{4k}f(x), t) \geq M(x, t), \\ \nu(f(a^k x) - a^{4k}f(x), t) \leq N(x, t), \end{aligned} \tag{13}$$

for all $x \in X$ and $t > 0$. Replacing x by $a^{-(kn+k)}x$ in (13) and using inequality (5), we obtain

$$\begin{aligned} &\tau\left(f\left(\frac{x}{a^{kn}}\right) - a^{4k}f\left(\frac{x}{a^{kn+k}}\right), t\right) \geq M\left(\frac{x}{a^{kn+k}}, t\right) \geq M\left(x, \alpha^{n+1}t\right) \\ &\nu\left(f\left(\frac{x}{a^{kn}}\right) - a^{4k}f\left(\frac{x}{a^{kn+k}}\right), t\right) \leq N\left(\frac{x}{a^{kn+k}}, t\right) \leq N\left(x, \alpha^{n+1}t\right) \end{aligned} \tag{14}$$

for all $x \in X$, $t > 0$, $n \geq 0$, and so

$$\begin{aligned}
&\tau\left((a^{4k})^n f\left(\frac{x}{(a^k)^n}\right) - \left(a^{4k}\right)^{n+1} f\left(\frac{x}{\left(a^k\right)^{n+1}}\right), t\right) \\
&\qquad \geq M\left(x, \frac{\alpha^{n+1}}{\left|\left(a^{4k}\right)^n\right|} t\right) \geq M\left(x, \frac{\alpha^{n+1}}{\left|\left(a^{k}\right)^n\right|} t\right) \\
&\nu\left((a^{4k})^n f\left(\frac{x}{(a^k)^n}\right) - \left(a^{4k}\right)^{n+1} f\left(\frac{x}{\left(a^k\right)^{n+1}}\right), t\right) \\
&\qquad \leq N\left(x, \frac{\alpha^{n+1}}{\left|\left(a^{4k}\right)^n\right|} t\right) \leq N\left(x, \frac{\alpha^{n+1}}{\left|\left(a^{k}\right)^n\right|} t\right)
\end{aligned} \tag{15}$$

for all $x \in X$, $t > 0$, $n \geq 0$. Since

$$\lim_{m\to\infty} M\left(x, \frac{\alpha^{m+1}}{|(a^k)^m|} t\right) = 1, \lim_{m\to\infty} N\left(x, \frac{\alpha^{m+1}}{|(a^k)^m|} t\right) = 0. \tag{16}$$

Therefore (15) shows that $\left\{\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right)\right\}$ is a Cauchy sequence in non-Archimedean intuitionistic fuzzy Banach space (Y, τ, ν). Hence, we can define a mapping $Q : X \to Y$ by $Q(x) = (\tau, \nu) - \lim_{n\to\infty} \left\{\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right)\right\}$. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \tau\left((a^{4k})^n f\left(\frac{x}{(a^k)^n}\right) - Q(x), t\right) = 1, \\
&\lim_{n\to\infty} \nu\left((a^{4k})^n f\left(\frac{x}{(a^k)^n}\right) - Q(x), t\right) = 0,
\end{aligned} \tag{17}$$

for all $x \in X$ and $t > 0$. Next, for all $n \geq 1, x \in X$ and $t > 0$, we have

$$\begin{aligned}
&\tau\left(f(x) - \left(a^{4k}\right)^n f\left(\frac{x}{\left(a^k\right)^n}\right), t\right) \\
&\quad = \tau\left(\sum_{i=0}^{n-1} \left(a^{4k}\right)^j f\left(\frac{x}{(a^k)^i}\right) - \left(a^{4k}\right)^{i+1} f\left(\frac{x}{(a^k)^{i+1}}\right), t\right) \\
&\quad \geq \prod_{i=0}^{n-1} \tau\left(\left(a^{4k}\right)^i f\left(\frac{x}{(a^k)^i}\right) - \left(a^{4k}\right)^{i+1} f\left(\frac{x}{(a^k)^{i+1}}\right), t\right) \\
&\quad \geq M(x, \alpha t), \\
&\nu\left(f(x) - \left(a^{4k}\right)^n f\left(\frac{x}{\left(a^k\right)^n}\right), t\right) \\
&\quad = \nu\left(\sum_{i=0}^{n-1} \left(a^{4k}\right)^j f\left(\frac{x}{(a^k)^i}\right) - \left(a^{4k}\right)^{i+1} f\left(\frac{x}{(a^k)^{i+1}}\right), t\right) \\
&\quad \leq \prod_{i=0}^{n-1} \nu\left(\left(a^{4k}\right)^i f\left(\frac{x}{(a^k)^i}\right) - \left(a^{4k}\right)^{i+1} f\left(\frac{x}{(a^k)^{i+1}}\right), t\right) \\
&\quad \leq N(x, \alpha t),
\end{aligned} \tag{18}$$

where $\prod_{j=1}^{n} = a_1 \star a_2 \star \cdots \star a_n$ and $\coprod_{j=1}^{n} = a_1 \diamond a_2 \diamond \cdots \diamond a_n$. From (17) and (18), we obtain

$$
\begin{aligned}
&\tau(f(x)-Q(x),t)\\
&\quad \geq \tau\left(f(x)-\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right),t\right)\\
&\qquad \star\, \tau\left(\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right)-Q(x),t\right)\\
&\quad \geq M(x,\alpha t),\\
&\nu(f(x)-Q(x),t)\\
&\quad \leq \nu\left(f(x)-\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right),t\right)\\
&\qquad \diamond\, \nu\left(\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right)-Q(x),t\right)\\
&\quad \leq N(x,\alpha t)
\end{aligned}
\tag{19}
$$

for each $x \in X$, $t > 0$ and for sufficiently large n; that is, (7) holds. Also, replacing x, y by $a^{-kn}x, a^{-kn}y$ in equations (6) and using (5), we get

$$
\begin{aligned}
&\tau\begin{pmatrix}
(a^{4k})^n f(a^{-kn}(ax+y)) + (a^{4k})^n f(a^{-kn}(ax-y))\\
+(a^{4k})^n f(a^{-kn}(x+ay)) + (a^{4k})^n f(a^{-kn}(x-ay))\\
-2a^2[(a^{4k})^n f(a^{-kn}(x+y)) + (a^{4k})^n f(a^{-kn}(x-y))]\\
-2(a^2-1)^2[(a^{4k})^n f(a^{-kn}x) + (a^{4k})^n f(a^{-kn}y)], \left|a^{4k}\right|^n t
\end{pmatrix}\\
&\quad \geq \tau'\left(\psi\left(a^{-kn}x, a^{-kn}y\right), t\right), \forall x \in X,\ t>0,\\
&\tau\begin{pmatrix}
Q(ax+y)+Q(ax-y)+Q(x+ay)+Q(x-ay)\\
-2a^2[Q(x+y)+Q(x-y)] - 2(a^2-1)^2[Q(x)+Q(y)], t
\end{pmatrix}\\
&\quad \geq \tau'\left(\psi(x,y), \frac{t}{\left|a^{4k}\right|^n}\right), \forall x \in X,\ t>0\\
&\quad \geq \tau'\left(\psi(x,y), \frac{\alpha^n t}{\left|a^k\right|^n}\right),
\end{aligned}
\tag{20}
$$

similarly

$$
\begin{aligned}
&\nu\begin{pmatrix}
(a^{4k})^n f(a^{-kn}(ax+y)) + (a^{4k})^n f(a^{-kn}(ax-y))\\
+(a^{4k})^n f(a^{-kn}(x+ay)) + (a^{4k})^n f(a^{-kn}(x-ay))\\
-2a^2[(a^{4k})^n f(a^{-kn}(x+y)) + (a^{4k})^n f(a^{-kn}(x-y))]\\
-2(a^2-1)^2[(a^{4k})^n f(a^{-kn}x) + (a^{4k})^n f(a^{-kn}y)], \left|a^{4k}\right|^n t
\end{pmatrix}\\
&\quad \geq \nu'\left(\psi\left(a^{-kn}x, a^{-kn}y\right), t\right), \forall x \in X,\ t>0,\\
&\nu\begin{pmatrix}
Q(ax+y)+Q(ax-y)+Q(x+ay)+Q(x-ay)\\
-2a^2[Q(x+y)+Q(x-y)] - 2(a^2-1)^2[Q(x)+Q(y)], t
\end{pmatrix}\\
&\quad \leq \nu'\left(\psi(x,y), \frac{t}{\left|a^{4k}\right|^n}\right), \forall x \in X,\ t>0\\
&\quad \leq \nu'\left(\psi(x,y), \frac{\alpha^n t}{\left|a^k\right|^n}\right),
\end{aligned}
\tag{21}
$$

for all $x \in X$, $t > 0$ and for large n. Since

$$
\lim_{n\to\infty} \tau'\left(\psi(x,y), \frac{\alpha^n t}{|a^k|^n} t\right) = 1,\ \lim_{n\to\infty} \nu'\left(\psi(x,y), \frac{\alpha^n t}{|a^k|^n} t\right) = 0,
\tag{22}
$$

which shows that Q is quartic. For the uniqueness of Q, let $Q' : X \to Y$ be another quartic mapping such that

$$\tau(Q'(x) - f(x), t) \geq M(x,t), \ \nu(Q'(x) - f(x), t) \leq N(x,t), \tag{23}$$

for all $x \in X, t > 0$. Then we have, for all $x \in X, t > 0$,

$$\begin{aligned}
&\tau(Q(x) - Q'(x), t) \\
&\geq \tau\left(Q(x) - \left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right), t\right) \\
&\qquad \star \tau\left(\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right) - Q'(x), t\right) \\
&\geq \tau\left((a^{4k})^n Q(a^{-kn}x) - \left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right), t\right) \\
&\qquad \star \tau\left(\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right) - (a^{4k})^n Q'(a^{-kn}x), t\right) \\
&\geq \tau\left(Q(a^{-kn}x) - f\left(\frac{x}{(a^k)^n}\right), \frac{t}{|a^{4k}|^n}\right) \\
&\qquad \star \tau\left(f\left(\frac{x}{(a^k)^n}\right) - Q'(a^{-kn}x), \frac{t}{|a^{4k}|^n}\right) \\
&\geq M\left(a^{-kn}x, \frac{\alpha t}{|a^{4k}|^n}\right) \geq M\left(x, \frac{\alpha^{n+1}t}{|a^k|^n}\right),
\end{aligned}$$

$$\begin{aligned}
&\nu(Q(x) - Q'(x), t) \\
&\leq \nu\left(Q(x) - \left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right), t\right) \\
&\qquad \diamond \nu\left(\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right) - Q'(x), t\right) \\
&\leq \nu\left((a^{4k})^n Q(a^{-kn}x) - \left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right), t\right) \\
&\qquad \diamond \nu\left(\left(a^{4k}\right)^n f\left(\frac{x}{(a^k)^n}\right) - (a^{4k})^n Q'(a^{-kn}x), t\right) \\
&\leq \nu\left(Q(a^{-kn}x) - f\left(\frac{x}{(a^k)^n}\right), \frac{t}{|a^{4k}|^n}\right) \\
&\qquad \diamond \nu\left(f\left(\frac{x}{(a^k)^n}\right) - Q'(a^{-kn}x), \frac{t}{|a^{4k}|^n}\right) \\
&\leq N\left(a^{-kn}x, \frac{\alpha t}{|a^{4k}|^n}\right) \leq N\left(x, \frac{\alpha^{n+1}t}{|a^k|^n}\right).
\end{aligned} \tag{24}$$

Therefore, from (16), we conclude that $Q = Q'$. This completes the proof. □

Corollary 3.2. *Let K be a non-Archimedean field, X a linear space over K and $(Y, \|\cdot\|)$ a non-Archimedean normed space. Let $\psi : X \times X \to R^+$ satisfies*

$$\psi(a^{-k}x, a^{-k}y) \leq \alpha^{-1}\psi(x,y), \tag{25}$$

for all $x, y \in X$, where $\alpha > 0$ and k is an integer with $|a^k| < \alpha$. If a map $f : X \to Y$ satisfies

$$(26)\quad \begin{aligned} &\big\| f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \\ &- 2a^2\{f(x+y) + f(x-y)\} - 2(a^2-1)^2\{f(x) + f(y)\}\big\| \\ &\le \psi(x, y), \end{aligned}$$

for all $x, y \in X$, then there exists a unique quartic mapping $Q : X \to Y$ satisfies

$$(27)\quad \|f(x) - Q(x)\| \le \frac{1}{\alpha}\max\{\psi(x,0) \star \psi(ax,0) \star \cdots \star \psi(a^{k-1}x, 0)\}$$

for all $x \in X$.

Proof. Consider the non-Archimedean intuitionistic fuzzy norm

$$\tau(y,t) = \begin{cases} \frac{t}{t+\|y\|} & , if\, t > 0; \\ 0 & , if\, t \le 0, \end{cases} \quad , \qquad \nu(y,t) = \begin{cases} \frac{\|y\|}{t+\|y\|} & , if\, t > 0; \\ 1 & , if\, t \le 0, \end{cases} .$$

on Y. Let $Z = R$ and let the function $\tau', \nu' : R \times R \to [0, 1]$ be defined by

$$\tau'(z,t) = \begin{cases} \frac{t}{t+\|z\|} & , if\, t > 0; \\ 0 & , if\, t \le 0, \end{cases} \quad , \qquad \nu'(z,t) = \begin{cases} \frac{|z|}{t+\|z\|} & , if\, t > 0; \\ 1 & , if\, t \le 0, \end{cases} .$$

Then (τ', ν') is a non-Archimedean intuitionistic fuzzy norm on R. The result follows from the fact that (25), (26) and (27) are equivalent to (5), (6) and (7), respectively. □

Now, we give an example to validate the main result as follows:

Example 3.3. *Let K be a non-Archimedean field, X a linear space over K and $(Y, \|\cdot\|)$ a non-Archimedean normed space. If a map $f : X \to Y$ satisfies*

$$(28)\quad \begin{aligned} &\big\| f(ax+y) + f(ax-y) + f(x+ay) + f(x-ay) \\ &- 2a^2\{f(x+y) + f(x-y)\} - 2(a^2-1)^2\{f(x) + f(y)\}\big\| \\ &\le \|x\|^p + \|y\|^p, \end{aligned}$$

for all $x, y \in X$ and $p \in [0, 1)$. Suppose that there exists an integer k with $|a^k| < 1$. Since $p < 1$, by applying Corollary 3.2 for $\psi(x, y) = \|x\|^p + \|y\|^p$, we observe that (25) holds for $\alpha = |a^k|^p$. Inequality (27) assures the existence of a unique quartic mapping $Q : X \to Y$ such that

$$(29)\quad \begin{aligned} \|f(x) - Q(x)\| &\le \frac{1}{\alpha}\max\{\|x\|^p \star \|ax\|^p \star \cdots \star \|a^{k-1}x\|^p\} \\ &\le \frac{a^{(k-1)p}}{\alpha}\|x\|^p \le \frac{a^{(k-1)p}}{|a^k|^p}\|x\|^p, \end{aligned}$$

for all $x \in X$.

4. Conclusion

In this paper, we introduced a new generalized A-quartic functional equation in order to obtain its general solution, and also investigated its generalized Hyers-Ulam-Rassias and Hyers-Ulam-Rassias stabilities in non-Archimedean intuitionistic fuzzy normed spaces with a suitable example. Compared to some other investigations already done on the Hyers-Ulam-Rassias stability of functional equations, the results of this paper seem to provide better results on fuzzy normed spaces.

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