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MULTIDIMENSIONAL FOURIER TRANSFORM AND FRACTIONAL DERIVATIVE

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Abstract. Some properties of the multidimensional Fourier transform and the suitable fractional derivative are established. The obtained results are used to prove that the inverse problem for a time fractional advection-dispersion equation in a 2–D setting is ill-posed.

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1. Introduction

The Fourier transform plays an important role in engineering and science. In particular, it has some essential applications in image processing, signal analysis and communication theory, interference spectroscopy and spectral line shapes [12, 9, 2]. In inverse problems, Fourier analysis is a useful tool for analyzing ill-posed problems [21].
The inverse heat conduction problems include the determination of boundary conditions, such as temperature and/or heat flux, from temperature measurements taken at selected internal points in a body [21, 30, 24]. The n-dimensional Fourier transform

$$\mathcal{F}(f)(\omega) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\omega} f(x)\, dx, \quad \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n,$$

of a function f of rapid decrease [31] has the property that

$$\mathcal{F}(D^\alpha f)(\omega) = (i)^{|\alpha|}\, \omega^\alpha\, \mathcal{F}(f)(\omega), \quad \omega \in \mathbb{C}^n, \tag{1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index consisting of nonnegative integers α_i $(i = 1, \ldots, n)$, D^α denotes the differentiation operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ of the order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\omega^\alpha = \omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n}$. The property (1) provides an important tool for solving model equations of mathematical physics [31, 34].
Fractional derivatives are integro-differential operators of convolutive type with (weakly singular) kernels of power-law type [29, 25, 6] and they have diverse applications in several branches of science [18, 14]. Fractional differential equations have been applied to many problems in classical mechanics and particle physics: diffusive systems, viscoelastic and disordered media, control theory [27, 11].

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The differential operator of Caputo type [3] is a particular case of the classical Riemann-Liouville operator [29] defined as

$$(2)\qquad {}_a^C D_t^\alpha f(t) := \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-n}}, \quad t\in(a,+\infty),$$

where n is a positive integer satisfying $n-1<\alpha<n$ [25]. Note that, if f is absolutely integrable in $(-\infty,\infty)$ and sufficiently smooth, with $f^{(k)}(-\infty)$ bounded for any $k=0,1,\dots,n-1$, then [25, 15]

$$(3)\qquad \mathcal{F}\left({}_{-\infty}^{\;\;C}D_t^\alpha f\right)(\omega) = (-i\omega)^\alpha\,\mathcal{F}(f)(\omega), \quad \omega\in\mathbb{C}.$$

The Fourier transform of the fractional derivative has been used for analyzing the ill-posedness of inverse problems based on fractional differential equations describing anomalous transport processes in one spatial dimension. These include time fractional inverse diffusion problems [22, 23, 33] and time fractional inverse advection-dispersion problems [37, 38, 36, 35, 20]. In this paper, we present some n-dimensional generalizations of the property (3) which are suitable to prove the ill-posedness of linear inverse problems involving two, three and more spatial dimensions. Besides, we give some applications of the obtained generalized property.

2. Preliminaries and Main Results

From now on, we consider the following α $(0<\alpha<1)$ order fractional derivative of a multivariable function $u(x_1,\dots,x_n)$ [19]:

$$(4)\qquad u_{x_i}^{(\alpha)}(x_1,\dots,x_i,\dots,x_n) = \frac{1}{\Gamma(1-\alpha)}\int_0^{x_i}\frac{u_s(x_1,\dots,x_{i-1},s,x_{i+1},\dots,x_n)}{(x_i-s)^\alpha}ds, \quad x_i>0,$$

where u_s is the derivative of the function u by the variable s and $x_j\in\mathbb{R}$, $j\in\{1,\dots,i-1,i+1,\dots,n\}$. The following expression defines the Fourier transform of the function $u_{x_i}^{(\alpha)}$ by the variable x_i:

$$\mathcal{F}(u_{x_i}^{(\alpha)}(x_1,\dots,x_i,\dots,x_n))(w) = \frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty} e^{-iwt}\left(\int_0^t \frac{u_s(x_1,\dots,x_{i-1},s,x_{i+1},\dots,x_n)ds}{(t-s)^\alpha}\right)dt,$$

where it is assumed that $w\in\mathbb{C}$.

The main results of this article are related to the application of the one, two and more dimensional Fourier transform to the fractional derivative (4). We start by the one-dimensional case.

Theorem 2.1. *Let $u_{x_i}(x_1,\dots,x_i,\dots,x_n)$ be a function on $L^1(0,+\infty)$ by a variable x_i $(1\le i\le n)$, such that*

$$(5)\qquad \lim_{x_i\to 0} u(x_1,\dots,x_i,\dots,x_n)=0 \text{ and } \lim_{x_i\to\infty} e^{x_i(\operatorname{Im} w)}u(x_1,\dots,x_i,\dots,x_n)=0$$

for any fixed $x_j\in\mathbb{R}$, $j\in\{1,\dots,i-1,i+1,\dots,n\}$ and $w\in G^-=\{z\in\mathbb{C}/\operatorname{Im} z<0\}$. Then

$$\mathcal{F}(u_{x_i}^{(\alpha)}(x_1,\dots,x_i\dots,x_n))(w) = (iw)^\alpha\mathcal{F}(u(x_1,\dots,x_n))(w).$$

Proof. By the inclusion $u_{x_i}(x_1,\dots,x_i,\dots,x_n) \in L^1(0,+\infty)$ it is evident that the function

$$\mathcal{F}(u^{(\alpha)}_{x_i}(x_1,\dots,x_i\dots,x_n))(w) = \frac{1}{\Gamma(1-\alpha)}\int_0^{+\infty} e^{-iwt}\left(\int_0^t \frac{u_s(x_1,\dots,s,\dots,x_n)ds}{(t-s)^\alpha}\right)dt \tag{6}$$

is holomorphic by w in the domain G^- and the integral

$$f(w) = (iw)^{\alpha-1}\int_0^\infty e^{-iws}u_s(x_1,\dots,s,\dots,x_n)ds \tag{7}$$

is absolutely and uniformly convergent in G^-. If $w=-iv\,(v>0)$ in formula (6), then we obtain that

$$\mathcal{F}(u^{(\alpha)}_{x_i}(x_1,\dots,x_i\dots,x_n))(w) = f(-iv), \quad v>0.$$

Thus,

$$\begin{aligned}\mathcal{F}(u^{(\alpha)}_{x_i}(x_1,\dots,x_i\dots,x_n))(w) &= (iw)^{\alpha-1}\int_0^\infty e^{-iws}u_s(x_1,\dots,s,\dots,x_n)ds\\ &= (iw)^{\alpha-1}\mathcal{F}(u_{x_i}(x_1,\dots,x_i,\dots,x_n)(w), \quad w\in G^-.\end{aligned} \tag{8}$$

Besides, by the conditions (5)

$$\lim_{x_i\to 0} u(x_1,\dots,x_i,\dots,x_n)=0 \quad\text{and}\quad \lim_{x_i\to\infty} e^{-x_iv}u(x_1,\dots,x_i,\dots,x_n)=0$$

for $w=-iv\,(v>0)$, and therefore an integration by parts in (7) gives

$$f(-iv) = v^\alpha\int_0^\infty e^{-vs}u(x_1,\dots,s,\dots,x_n)ds, \quad v>0. \tag{9}$$

Hence, by (5), (8), (9) and the uniqueness of the analytic function we get

$$\mathcal{F}(u^{(\alpha)}_{x_i}(x_1,\dots,x_i\dots,x_n))(w) = (iw)^\alpha\mathcal{F}(u(x_1,\dots,x_i,\dots,x_n))(w), \quad w\in G^-. \tag{10}$$

□

Remark 2.2. *Note that Theorem 2.1 can be extended to the boundary of G^-, if, additionally, it is assumed*

$$\lim_{x_i\to\infty} u(x_1,\dots,x_i,\dots,x_n)=0.$$

The similar result for the two-dimensional Fourier transform of $u^{(\alpha)}_{x_i}$ easily follows from the above theorem.

Corollary 2.3. *Let $u(x_1,\dots,x_n)$ be a function on $L^1(0,+\infty)$ by some of its variables x_i and x_j, $1\le i,j\le n$, $i\ne j$. Besides, let $u(x_1,\dots,x_n)$ satisfy the conditions* (5) *for any $w_1\in G^-$ and $x_j\ge 0$. Then the following equality is true for the two-dimensional Fourier transform of $u^{(\alpha)}_{x_i}(x_1,\dots,x_i,\dots,x_n)$:*

$$\mathcal{F}(u^{(\alpha)}_{x_i}(x_1,\dots,x_i,\dots,x_n))(w_1,w_2) = (iw_1)^\alpha\mathcal{F}(u(x_1,\dots,x_n))(w_1,w_2), \quad w_2\in G^-.$$

It is necessary to note that in Corollary 2.3 the two-dimensional Fourier transform can be calculated by the derivation variable (x_i), while the other variable (x_j) can be arbitrary, by which $u \in L^1(0,+\infty)$. Besides, it is evident that under some additional conditions one can calculate the three, four or more dimensional Fourier transform of $u^{(\alpha)}_{x_i}(x_1,\dots,x_i\dots,x_n)$ in a similar way.

Before proceeding to a property of a more general n-dimensional Fourier transform, we give the definition of the n-dimensional fractional derivative (see e.g. [29, 19, 13]).

Definition 2.4. *For a function $u:\mathbb{R}^n\to\mathbb{R}$ of $C^n(\mathbb{R}^n)$, the n-dimensional fractional derivative is defined as*

(11)

$$u^{(\alpha)}_{x_1\cdots x_n}(x_1,\dots,x_n)=\prod_{k=1}^{n}\frac{1}{\Gamma(1-\alpha_k)}\int_0^{x_n}\!\!\cdots\!\int_0^{x_1}\frac{u_{s_1\cdots s_n}(s_1,\dots,s_n)ds_1\dots ds_n}{(x_1-s_1)^{\alpha_1}\cdots(x_n-s_n)^{\alpha_n}},$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$, $0<\alpha_i<1$ and $x_i\ge 0$, $i=1,\dots,n$.

Theorem 2.5. *If $u:\mathbb{R}^n\to\mathbb{R}$ is a function of $C^n(\mathbb{R}^n)$ and $u_{x_1\dots x_n}(x_1,\dots,x_n)\in L^1(\mathbb{R}^n)$, then*

$$\text{(12)}\quad \mathcal{F}(u^{(\alpha)}_{x_1\dots x_n}(x_1,\dots,x_n))(w_1,\dots,w_n)=(iw_n)^{\alpha_n-1}\dots(iw_1)^{\alpha_1-1}\times \\ \times\mathcal{F}(u_{x_1\dots x_n}(x_1,\dots,x_n))(w_1,\dots,w_n),\ w_i\in G^-,\ 1\le i\le n.$$

Proof. To prove the equality (12), first we calculate the following integral in the same way as formula (8) was proved:

$$\begin{aligned}K_1(w_1,s_2,\dots,s_n)=&\int_0^{+\infty}e^{-iy_1w_1}\left\{\int_0^{y_1}\frac{u_{s_1\cdots s_n}(s_1,\dots,s_n)ds_1}{(y_1-s_1)^{\alpha_1}}\right\}dy_1\\ =&\frac{\Gamma(1-\alpha_1)}{(iw_1)^{1-\alpha_1}}\int_0^{+\infty}e^{-is_1w_1}u_{s_1\cdots s_n}(s_1,\dots,s_n)ds_1\\ =&(iw_1)^{\alpha_1-1}\Gamma(1-\alpha_1)\mathcal{F}(u_{x_1,s_2,\cdots s_n}(x_1,s_2,\dots,s_n))(w_1).\end{aligned}$$

for $w_1\in G^-$ and $s_i\ge 0$, $2\le i\le n$. Hence, by Definition 2.4

$$\int_0^{+\infty}e^{-iy_1w_1}u^{(\alpha)}_{y_1\dots y_n}(y_1,\dots,y_n)dy_1=$$
$$\prod_{k=1}^{n}\frac{1}{\Gamma(1-\alpha_k)}\int_0^{y_n}\cdots\int_0^{y_2}\big(K_1(w_1,s_2,\dots,s_n)\big)\frac{ds_2\dots ds_n}{(y_2-s_2)^{\alpha_2}\cdots(y_n-s_n)^{\alpha_n}}=$$
$$\prod_{k=2}^{n}\frac{(iw_1)^{\alpha_1-1}}{\Gamma(1-\alpha_k)}\int_0^{y_n}\cdots\int_0^{y_2}\frac{\mathcal{F}(u_{x_1s_2\dots s_n}(x_1,s_2,\dots,s_n))(w_1)ds_2\cdots ds_n}{(y_2-s_2)^{\alpha_2}\cdots(y_n-s_n)^{\alpha_n}}.$$

Similarly, for $w_2 \in G^-$ we get

$$\int_0^{+\infty} e^{-iy_2w_2}\left(\int_0^{+\infty} e^{-iw_1y_1} u^{(\alpha)}_{y_1\dots y_n}(y_1,\dots,y_n)dy_1\right)dy_2$$
$$= \prod_{k=3}^{n} \frac{(iw_1)^{\alpha_1-1}(iw_2)^{\alpha_2-1}}{\Gamma(1-\alpha_k)}\times$$
$$\times \int_0^{y_n}\cdots\int_0^{y_3} \frac{\mathcal{F}(u_{x_1x_2s_3\dots s_n}(x_1,x_2,s_3,\dots,s_n))(w_1,w_2)ds_3\cdots ds_n}{(y_3-s_3)^{\alpha_3}\cdots(y_n-s_n)^{\alpha_n}}.$$

Thus, the same procedure leads to the proof of our theorem:

$$\mathcal{F}(u_{x_1\dots x_n}{}^{(\alpha)}(x_1,\dots,x_n))(w_1,\dots,w_n)$$
$$= \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-i(y_1w_1+\dots+y_nw_n)} u^{(\alpha)}_{y_1\dots y_n}(y_1,\dots,y_n)dy_1\dots dy_n$$
$$= (iw_1)^{\alpha_1-1}\cdots(iw_n)^{\alpha_n-1}\mathcal{F}(u_{x_1\dots x_n}(x_1,\dots,x_n))(w_1,\dots,w_n).$$

□

The following corollary is proved just in the same way as the previous theorem.

Corollary 2.6. *If $u:\mathbb{R}^{m+n+k}\to\mathbb{R}$ is a function in $C^n(\mathbb{R}^n)$, such that*

$$u_{x_1\dots x_n}(t_1,\dots,t_m,x_1,\dots,x_n,s_1,\dots,s_k)\in L^1\left((\mathbb{R}^+)^n\right)$$

by the variables $x_1,\dots,x_n$, and some fixed $t_1,\dots,t_m,s_1,\dots,s_k\in\mathbb{R}$, then the following formula is true for the Fourier transform of $u^{(\alpha)}_{x_1\dots x_n}(t_1,\dots,t_m,x_1,\dots,x_n,s_1,\dots,s_k)$ by the variables $x_1,\dots,x_n$:

$$\mathcal{F}(u^{(\alpha)}_{x_1\dots x_n}(t_1,\dots,t_m,x_1,\dots,x_n,s_1,\dots,s_k))(w_1,\dots,w_n)$$
$$= \prod_{r=1}^{n}(iw_r)^{\alpha_r-1}\mathcal{F}(u_{x_1\dots x_n}(t_1,\dots,t_m,x_1,\dots,x_n,s_1,\dots,s_k))(w_1,\dots,w_n),$$

where $(x_1,\dots,x_n)\in(\mathbb{R}^+)^n$ and $(w_1,\dots,w_n)\in(G^-)^n$.

Remark 2.7. *To get a function without partial derivatives in the right-hand side of formula (12), i.e. inside the Fourier transform, some strong conditions are necessary to be put, like those in Corollary 2.3.*

3. Applications

The classical one-dimensional advection-dispersion equation

$$u_t(x,t)+a\,u_x(x,t)=d\,u_{xx}(x,t) \tag{13}$$

models the transport of a sorbing solute through a one-dimensional homogeneous porous medium under steady flow conditions [4], where $u(x,t)$ is the solute concentration, x is the spatial coordinate in the direction of flow, t is time, $d>0$ is the dispersion coefficient and $a>0$ is the the average fluid velocity. Experimental data indicate that solutes moving through highly heterogeneous media exhibit anomalous dispersion processes which can be better described by fractional differential equations [1, 10, 8].
The time fractional advection-dispersion equation

$$u_t^{(\alpha)}(x,t)+a\,u_x(x,t)=d\,u_{xx}(x,t)$$

arises by replacing the first-order time partial derivative in the advection-dispersion equation (13) with the time fractional Caputo derivative $u_t^{(\alpha)}$ of order α $(0 < \alpha < 1)$ given by (4). In recent years, existence and uniqueness of solutions to direct problems for time fractional differential equations (initial/boundary value problems) have been established [7, 5, 28, 17], as well as numerical methods for obtaining rigorous solutions [16, 39].
On the other hand, a wide range of inverse problems for time fractional diffusion (and advection-dispersion) equations are ill-posed problems whose solutions, if they exist, does not depend continuously on the data. These problems include inverse source problems and backward problems, see e.g. [32, 26] and the references therein.
Next, we analyze a backward problem in the frequency domain. The ill-posedness of the problem will be proved as consequence of the results obtained in Section 2.

3.1. **Statement of the problem.** We introduce a two-dimensional time fractional inverse advection-dispersion problem in a semi-infinite slab, consisting in recovering the solute concentration u and dispersion flux u_x on the right-hand side, from measured data on the left-hand side surface. This problem is a generalization of the classical two-dimensional inverse heat conduction problem considered by Murio in [21], by allowing the time fractional order of differentiation α, to vary between 0 and 1. The formulation of the problem is as follows:

$$
\begin{aligned}
& u_t^{(\alpha)}(x,y,t) + a\left(u_x(x,y,t) + u_y(x,y,t)\right) = \\
&\qquad\qquad d\,\Delta u(x,y,t),\ 0 < x < x_0,\ y > 0,\ t > 0, \\
&u(0,y,t) = \Phi(y,t), \quad y > 0,\ t > 0, \quad \text{datum}, \\
&u_x(0,y,t) = \Psi(y,t), \quad y > 0,\ t > 0, \quad \text{datum}, \\
&u(x,y,0) = 0, \quad 0 \le x \le x_0,\ y > 0, \\
&u(x,y,t) = 0, \quad 0 \le x \le x_0,\ y > 0,\ t \to \infty, \\
&u(x_0,y,t) = \varphi(y,t), \quad y > 0,\ t > 0, \quad \text{unknown}, \\
&u_x(x_0,y,t) = \psi(y,t), \quad y > 0,\ t > 0, \quad \text{unknown}, \\
&u(x,0,t) = 0, \quad 0 \le x \le x_0,\ t > 0,
\end{aligned}
\tag{14}
$$

where $u(x,y,t)$ is the solute concentration in (x,y) at time t, $u_t^{(\alpha)}(x,y,t)$ denotes the time fractional Caputo derivative of order α $(0 < \alpha < 1)$ defined in formula (4) and $\Delta u = u_{xx} + u_{yy}$. Here, it is assumed that $u_t(x,y,t) \in L^1(0,\infty)$ over the variable t. The constants d $(d > 0)$ and a $(a \ge 0)$ represent the dispersion coefficient and the average fluid velocity, respectively.
The exact data functions Ψ and Φ are not available, but perturbed data Ψ^ε and Φ^ε satisfying the $L^2(\mathbb{R}^2)$ error bounds for some $\varepsilon > 0$

$$\|\Psi - \Psi^\varepsilon\| < \varepsilon \qquad \text{and} \qquad \|\Phi - \Phi^\varepsilon\| < \varepsilon.$$

The inverse source problem (14) is to determine the unknown solute concentration φ and the dispersion flux ψ on the right-hand side $x = x_0$, from solute concentration data Ψ^{ε} and dispersion flux data Φ^{ε} measured on the left-hand side $x = 0$.

3.2. **Ill-posed problem.** In order to perform a Fourier analysis of problem (14), we assume $u(\cdot, y, t)$ and all the function involved to be functions in $L^2(\mathbb{R}^2)$, by defining them to be zero everywhere in $\{(y,t) \mid t \leq 0, y \leq 0\}$. For each fixed $x \in (0, x_0)$, the two dimensional Fourier transform of the function $u(x,y,t)$ is given by

$$\mathcal{F}(u)(x,\omega_1,\omega_2) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(x,z)e^{-i\omega\cdot z}\,dz, \quad \omega = (\omega_1,\omega_2). \tag{15}$$

Differentiating under the integral sign with respect to x we obtain

$$\begin{aligned} \mathcal{F}(u_x)(x,\omega_1,\omega_2) &= \mathcal{F}(u)_x(x,\omega_1,\omega_2), \\ \mathcal{F}(u_{xx})(x,\omega_1,\omega_2) &= \mathcal{F}(u)_{xx}(x,\omega_1,\omega_2), \end{aligned} \tag{16}$$

and applying property (1) we arrive at

$$\begin{aligned} \mathcal{F}(u_y)(x,\omega_1,\omega_2) &= i\omega_1\,\mathcal{F}(u)(x,\omega_1,\omega_2), \\ \mathcal{F}(u_{yy})(x,\omega_1,\omega_2) &= -\omega_1^2\,\mathcal{F}(u)(x,\omega_1,\omega_2). \end{aligned} \tag{17}$$

By conditions stated in problem (14), it follows by Remark 2.2 and Corollary 2.3 that

$$\mathcal{F}\big(u_t^{(\alpha)}\big)(x,\omega_1,\omega_2) = (i\omega_2)^{\alpha}\mathcal{F}(u)(x,\omega_1,\omega_2). \tag{18}$$

By taking Fourier transform of the fractional differential equation in (14) we get

$$\begin{aligned} &d\,\mathcal{F}(u)_{xx}(x,\omega_1,\omega_2) - \\ &a\,\mathcal{F}(u)_x(x,\omega_1,\omega_2) - \left(\omega_1^2 + a\,i\omega_1 + (i\omega_2)^{\alpha}\right)\mathcal{F}(u)(x,\omega_1,\omega_2) = 0. \end{aligned} \tag{19}$$

For each fixed $\omega \in \mathbb{R}^2$, (19) is a second-order differential equation whose general solution is of the form

$$\mathcal{F}(u)(x,\omega_1,\omega_2) = A(\omega_1,\omega_2)e^{k_1 x} + B(\omega_1,\omega_2)e^{k_2 x}, \tag{20}$$

where k_1 y k_2 are the square roots of the complex number $\sigma := a^2 + 4d\left(\omega_1^2 + ai\omega_1 + (i\omega_2)^{\alpha}\right)$ given by

$$k_1 := |\sigma|^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad \text{and} \qquad k_2 := -k_1,$$

with $\theta = \arg(\sigma)$. Since $(i\omega_2)^{\alpha} = |\omega_2|^{\alpha}e^{i\varsigma\pi\alpha/2}$ with $\varsigma := \operatorname{sgn}(\omega_2)$, it follows that

$$\begin{aligned} \sigma &= a^2 + 4d\omega_1^2 + 4adi\omega_1 + 4d(i\omega_2)^{\alpha} \\ &= a^2 + 4d\omega_1^2 + 4adi\omega_1 + 4d|\omega_2|^{\alpha}e^{i\varsigma\pi\alpha/2} \\ &= \left(a^2 + 4d\omega_1^2 + 4d|\omega_2|^{\alpha}\cos\frac{\varsigma\pi\alpha}{2}\right) + i\left(4ad\omega_1 + 4d|\omega_2|^{\alpha}\sin\frac{\varsigma\pi\alpha}{2}\right), \end{aligned} \tag{21}$$

which implies that

$$\theta = \arg(\sigma) = \tan^{-1}\left(\frac{4ad\omega_1 + 4d|\omega_2|^{\alpha}\sin\frac{\varsigma\pi\alpha}{2}}{a^2 + 4d\omega_1^2 + 4d|\omega_2|^{\alpha}\cos\frac{\varsigma\pi\alpha}{2}}\right). \tag{22}$$

Setting $\mu := |\sigma|^{1/2}$ and $\nu := \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$, the general solution (20) and its derivative can be rewritten in the form

$$\mathcal{F}(u)(x,\omega_1,\omega_2) = A(\omega_1,\omega_2)e^{\mu\nu x} + B(\omega_1,\omega_2)e^{-\mu\nu x} \tag{23}$$

and

$$\mathcal{F}(u)_x(x,\omega_1,\omega_2) = \mu\nu A(\omega_1,\omega_2)e^{\mu\nu x} - \mu\nu B(\omega_1,\omega_2)e^{-\mu\nu x}. \tag{24}$$

Using data from inverse problem (14) in (23) and (24) when $x = 0$, we obtain the linear system

$$\mathcal{F}(\Phi)(\omega_1,\omega_2) = A(\omega_1,\omega_2) + B(\omega_1,\omega_2),$$
$$\mathcal{F}(\Psi)(\omega_1,\omega_2) = \mu\nu A(\omega_1,\omega_2) - \mu\nu B(\omega_1,\omega_2),$$

whose solution is given by

$$\begin{bmatrix} A(\omega_1,\omega_2) \\ B(\omega_1,\omega_2) \end{bmatrix} = \frac{1}{2\mu\nu}\begin{bmatrix} \mu\nu & 1 \\ \mu\nu & -1 \end{bmatrix}\begin{bmatrix} \mathcal{F}(\Phi)(\omega_1,\omega_2) \\ \mathcal{F}(\Psi)(\omega_1,\omega_2) \end{bmatrix}. \tag{25}$$

On the other hand, using the unknowns of the inverse problem (14) in (23) and (24) when $x = x_0$, we get

$$\begin{bmatrix} \mathcal{F}(\varphi)(\omega_1,\omega_2) \\ \mathcal{F}(\psi)(\omega_1,\omega_2) \end{bmatrix} = \begin{bmatrix} e^{\mu\nu} & e^{-\mu\nu} \\ \mu\nu e^{\mu\nu} & -\mu\nu e^{-\mu\nu} \end{bmatrix}\begin{bmatrix} A(\omega_1,\omega_2) \\ B(\omega_1,\omega_2) \end{bmatrix}. \tag{26}$$

Combining the former expressions, we obtain

$$\begin{bmatrix} \mathcal{F}(\varphi)(\omega_1,\omega_2) \\ \mathcal{F}(\psi)(\omega_1,\omega_2) \end{bmatrix} = \frac{1}{2\mu\nu}\begin{bmatrix} e^{\mu\nu} & e^{-\mu\nu} \\ \mu\nu e^{\mu\nu} & -\mu\nu e^{-\mu\nu} \end{bmatrix}\begin{bmatrix} \mu\nu & 1 \\ \mu\nu & -1 \end{bmatrix}\begin{bmatrix} \mathcal{F}(\Phi)(\omega_1,\omega_2) \\ \mathcal{F}(\Psi)(\omega_1,\omega_2) \end{bmatrix},$$

so that

$$\begin{bmatrix} \mathcal{F}(\varphi)(\omega_1,\omega_2) \\ \mathcal{F}(\psi)(\omega_1,\omega_2) \end{bmatrix} = \begin{bmatrix} \cosh\mu\nu & \frac{1}{\mu\nu}\sinh\mu\nu \\ \mu\nu\sinh\mu\nu & \cosh\mu\nu \end{bmatrix}\begin{bmatrix} \mathcal{F}(\Phi)(\omega_1,\omega_2) \\ \mathcal{F}(\Psi)(\omega_1,\omega_2) \end{bmatrix}.$$

By (21), we arrive at

$$\begin{aligned}\mu &= \left|\left(a^2 + 4d\omega_1^2 + 4d|\omega_2|^\alpha \cos\frac{\varsigma\pi\alpha}{2}\right) + i\left(4ad\omega_1 + 4d|\omega_2|^\alpha \sin\frac{\varsigma\pi\alpha}{2}\right)\right|^{1/2} \\ &= \left[\left(a^2 + 4d\omega_1^2 + 4d|\omega_2|^\alpha \cos\frac{\pi\alpha}{2}\right)^2 + \left(4ad\omega_1 + 4d|\omega_2|^\alpha \sin\frac{\varsigma\pi\alpha}{2}\right)^2\right]^{1/4} \\ &\geq \left(a^2 + 4d\omega_1^2 + 4d|\omega_2|^\alpha \cos\frac{\pi\alpha}{2}\right)^{1/2} \to \infty \quad \text{as} \quad |\omega_1| + |\omega_2| \to \infty.\end{aligned}$$

Furthermore, by (22) we have

$$\theta = \arg(\sigma) = \tan^{-1}\left(\frac{4ad\omega_1 + 4d|\omega_2|^\alpha \sin\frac{\varsigma\pi\alpha}{2}}{a^2 + 4d\omega_1^2 + 4d|\omega_2|^\alpha \cos\frac{\varsigma\pi\alpha}{2}}\right) \to 0 \text{ as } |\omega_1| + |\omega_2| \to \infty,$$

hence

$$\nu = \cos\frac{\theta}{2} \pm i\sin\frac{\theta}{2} \to 1 \quad \text{as} \quad |\omega_1| + |\omega_2| \to \infty.$$

Finally, it follows

$$|\cosh\mu\nu| \to \infty \quad \text{as} \quad |\omega_1| + |\omega_2| \to \infty,$$
$$|\sinh\mu\nu| \to \infty \quad \text{as} \quad |\omega_1| + |\omega_2| \to \infty,$$

and therefore, obtaining $\mathcal{F}(\varphi)$ and $\mathcal{F}(\psi)$ from data $\mathcal{F}(\Phi)$ and $\mathcal{F}(\Psi)$, amplifies the errors in the high-frequency component by the factor

$\exp\left(\left|a^2 + 4d\omega_1^2 + 4d|\omega_2|^\alpha \cos\frac{\pi\alpha}{2}\right|^{1/2}\right)$, showing that the inverse problem (14) is ill posed in the high-frequency components.

4. Final remarks

The generalizations of some well-known properties of the Fourier transform of several fractional derivatives of functions are used in many research papers for getting the solutions of many fractional differential equations. This paper establishes some extensions of the Fourier transform applied to a multidimensional fractional derivative of a function. The obtained results are used to prove that an inverse problem is ill posed. The obtained generalizations can be used to get solutions for the 3D settings problem in the future.

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