Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 2. pp. 249 – 267
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.2.249

On the results of nonlocal Hilfer fractional semilinear differential inclusions

R. Subashini* and C. Ravichandran†

Abstract

In this article, we establish the sufficient condition for the approximate controllability of Hilfer fractional semilinear differential inclusions with nonlocal conditions. The results are obtained by using a fixed point technique, semigroup theory and multivalued investigation. Examples are provided to illustrate the theory.

Keywords: Hilfer derivative; Approximate controllability; Fixed-point technique;

2000 MSC: 26A33, 93B05, 58C30, 45N05.

1 Introduction

Nowadays, significant consideration has been paid to fractional differential equations mainly due to the fact that they need several potential applications in engineering and scientific disciplines because the mathematical modeling of systems and processes within the fields of physics, aeromechanics, chemistry, electrodynamics of advanced medium or chemical compound natural philosophy [1, 2, 3, 7, 8, 18, 33, 34, 35, 36, 37].

Several authors have published in fractional differential equations in recent years, for example, Kilbas et al. [14], Lakshmikantham et al. [15], Miller and Ross [19] and Podlubny [23]. Controllability of deterministic fractional differential equations has been very much created by utilizing various types of strategies, which can be found in [4, 5, 6, 16, 17, 18, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 38].

*Research and Development Center, Bharathiar University, Coimbatore - 641046, Tamil Nadu, India and Department of Mathematics, GTN Arts College, Dindigul - 624004, Tamil Nadu, India. E. Mail-: subavenkat.ks@gmail.com

†Corresponding Author: Post Graduate and Research Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore - 641029, Tamil Nadu, India. E. Mail-: ravibirthday@gmail.com

Today the generalized Riemann-Liouville fractional derivative is is most well known technique utilized by Hilfer [13], which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. Recently, Gu and Trujillo [11] investigated a class of evolution equations via Hilfer fractional derivatives. There are some related research works on Hilfer fractional derivatives, for the so-called Hilfer fractional derivatives, one can see [9, 10, 12, 20, 32]. The focus of this paper is that the study of the approximate controllability of Hilfer fractional semilinear differential inclusion with non local condition in Banach spaces.

$$\mathcal{D}_{0+}^{\gamma,\mu}\chi(\eta) \in \mathcal{A}\chi(\eta) + \mathcal{F}\left(\eta, \chi(\eta), \int_0^\eta k(\eta,\vartheta,\chi(\vartheta))d\vartheta\right) + \mathcal{B}u(\eta), \eta \in \Omega' = (0,b]$$

$$\mathcal{I}_{0+}^{(1-\gamma)(1-\mu)}\chi(\eta)|_{\eta=0} + p(x) = x_0 \quad (1.1)$$

where $\mathcal{D}_{0+}^{\gamma,\mu}$ is the Hilfer fractional derivative, $0 \leq \gamma \leq 1,\ 0 < \mu < 1$, the state $\chi(\cdot)$ takes value in a Banach space X with norm $\|\cdot\|$, and the control function u is given in $\mathcal{L}^2(\Omega, U)$ with U as a Banach space; $\mathcal{A}$ indicates the infinitesimal generator of a strongly continuous semigroup $\{S(\eta)\}_{\eta\geq 0}$; $\mathcal{B}$ means a bounded linear operator from U into X. Let $\mathcal{F} : \Omega \times X \times X \to 2^X \backslash \emptyset$ denotes a multivalued map, $\Omega = [0, b]$, $k : \Omega \times \Omega \times X \to X$ and $h : \mathcal{C}(\Omega, X) \to X$ denotes continuous and compact.

2 Preliminaries

In this section, we have a tendency to mention a notations, definitions, lemmas and preliminary facts required to to establish our main results.

Throughout this paper, by $\mathcal{C}(\Omega, X)$ and $\mathcal{C}(\Omega', X)$ we denote the spaces of all continuous functions from Ω to X and Ω' to X, respectively. Let $\nu = \gamma + \mu - \gamma\mu$, then $1-\nu = (1-\gamma)(1-\mu)$, define $\mathcal{C}_{1-\nu}(\Omega, X) = \{x : \eta^{1-\nu}\chi(\eta) \in \mathcal{C}(\Omega, X)\}$ with norm $\|\cdot\|_\nu$ defined by $\|x\|_\nu = \sup\{\eta^{1-\nu}\|\chi(\eta)\|, \nu = \gamma + \mu - \gamma\mu\}$. Obviously, $\mathcal{C}_{1-\nu}(\Omega, X)$ is a Banach space.

Definition 2.1. *([13]) The left-sided Riemann-Liouville fractional integral of order α with the lower limit a for a function $f : [a, +\infty) \to R$ is defined as*

$$\mathcal{I}_{a+}^{\alpha} f(\eta) = \frac{1}{\Gamma(\alpha)} \int_a^\eta \frac{f(\vartheta)}{(\eta-\vartheta)^{1-\alpha}} d\vartheta, \quad \eta > a, \quad \alpha > 0,$$

provided the right hand-side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *([12]) The left-sided Riemann-Liouville fractional derivative of or-*

der $\alpha \in [n-1, n), n \in Z^+$,*of function* $f : [a, +\infty) \to R$ *is defined as*

$$^{L}\mathcal{D}_{a^+}^{\alpha} f(\eta) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\eta^n} \int_a^{\eta} (\eta - \vartheta)^{n-\alpha-1} f(\vartheta) d\vartheta, \quad \eta > a, \quad n-1 \leq \alpha < n$$

where $\Gamma(\cdot)$ *is the gamma function*

Definition 2.3. *([13]) The left-sided Caputo's derivative of order* $\alpha \in (n-1, n), n \in Z^+$ *of function* $f : [a, +\infty) \to R$ *is defined by*

$$^{C}\mathcal{D}_{a^+}^{\alpha} f(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_a^{\eta} \frac{f^{(n)}(\vartheta) d\vartheta}{(\eta-\vartheta)^{\alpha-n+1}}, \quad \eta > a, \quad n-1 < \alpha < n.$$

Definition 2.4. *([12]) (Hilfer fractional derivative) The generalized Riemann-Liouville fractional derivative of order* $0 \leq \gamma \leq 1$ *and* $0 < \mu < 1$ *of function* $f(\eta)$ *is defined by*

$$\mathcal{D}_{a^+}^{\gamma,\mu} f(\eta) = \left(\mathcal{I}_{a^+}^{\gamma(1-\mu)} \mathcal{D} \left(\mathcal{I}_{a^+}^{(1-\gamma)(1-\mu)} f \right) \right) (\eta)$$

where $\mathcal{D} = \frac{d}{d\eta}$

Remark 2.1. *([13]) (i)When* $\gamma = 0$, $0 < \mu < 1$ *and* $a = 0$, *the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative:* $\mathcal{D}_{0^+}^{0,\mu} f(\eta) = \frac{d}{d\eta} \mathcal{I}_{0^+}^{1-\mu} f(\eta) = \mathcal{D}_{0^+}^{\mu}$
(ii) When $\gamma = 1$, $0 < \mu < 1$ *and* $a = 0$, *the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:* $\mathcal{D}_{0^+}^{1,\mu} = \mathcal{I}_{0^+}^{1-\mu} \frac{d}{d\eta} f(\eta) =^C \mathcal{D}_{0^+}^{\mu} f(\eta)$

Lemma 2.1. *([36]) For* $\sigma \in (0, 1]$ *and* $0 < a \leq b$, *we have* $|a^\sigma - b^\sigma| \leq (b-a)^\sigma$

Lemma 2.2. *([23]) Let* $\mathcal{D}$ *be a nonempty subset of* X, *which is bounded, closed, and convex. Suppose* $\mathcal{G} : \mathcal{D} \to 2^X \backslash \emptyset$ *is upper semi continuous with closed, convex values, and such that* $\mathcal{G}(\mathcal{D}) \subset \mathcal{D}$ *and* $\mathcal{G}(\mathcal{D})$ *is compact. Then* $\mathcal{G}$ *has a fixed point.*

Lemma 2.3. *The fractional nonlocal control system (1.1) is equivalent to the integral inclusion*

$$\chi(\eta) \in \frac{x_0 - p(x)}{\Gamma(\gamma + \mu - \gamma\mu)} \eta^{\nu-1} + \frac{1}{\Gamma(\mu)} \int_0^{\eta} (\eta - \vartheta)^{\mu-1} \big[\mathcal{A}\chi(\vartheta) + \mathcal{F}\big(\vartheta, \chi(\vartheta), \int_0^{\vartheta} k(\vartheta, \tau, \chi(\tau)) d\tau\big) + \mathcal{B}u(\vartheta) \big] d\vartheta, \eta \in \Omega' \tag{2.1}$$

The Wright function $\Upsilon_\mu(\varrho)$, which is defined by

$$\Upsilon_\mu(\varrho) = \sum_{n=1}^{\infty} \frac{(-\varrho)^{n-1}}{(n-1)!\Gamma(1-\mu n)}, 0 < \mu < 1, \varrho \in \mathcal{C} \tag{2.2}$$

which satisfies the following equality $\int_0^\infty \theta^\delta \Upsilon_\mu(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)}, for\ \theta \geq 0.$

Lemma 2.4. *If integral inclusion (2.1) holds, then there exists* $f \in \mathcal{L}^1(\Omega, X)$ *such that* $f(\eta) \in \mathcal{F}(\eta, \chi(\eta), \int_0^\eta k(\eta, \vartheta, \chi(\vartheta))d\vartheta)$

$$\chi(\eta) = \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta \mathcal{T}_\mu(\eta - \vartheta)f(\vartheta)d\vartheta + \int_0^\eta \mathcal{T}_\mu(\eta - \vartheta)\mathcal{B}u(\vartheta)d\vartheta, \eta \in \Omega' \tag{2.3}$$

where $\mathcal{T}_\mu(\eta) = \eta^{\mu-1}\mathcal{P}_\mu(\eta), \mathcal{P}_\mu(\eta) = \int_o^\infty \mu\theta\Upsilon_\mu(\theta)\mathcal{Q}(\eta^\mu\theta)d\theta$ *and* $\mathcal{S}_{\gamma,\mu}(\eta) = \mathcal{I}_{0^+}^{\gamma(1-\mu)}\mathcal{T}_\mu(\eta)$

We list out the following hypotheses.

($\mathbf{H_1}$): $\mathcal{Q}(\eta)$ is continuous in the uniform operator topology for $\eta > 0$ and $\{\mathcal{Q}(\eta)\}_{\eta \geq 0}$ is uniformly bounded, i.e., there exists $\Upsilon > 1$ such that $\sup_{\eta \in [0,+\infty)} |\mathcal{Q}(\eta)| < \Upsilon$

Proposition 2.1. *([37]) Under assumption* (H_1), $\mathcal{P}_\mu(\eta)$ *is continuous in the uniform operator topology for* $\eta > 0$.

Proposition 2.2. *([34]) Under assumption* (H_1), *or any fixed* $\eta > 0$, $\{\mathcal{T}_\mu(\eta)\}_{\eta>0}$ *and* $\{\mathcal{S}_{\gamma,\mu}(\eta)\}_{\eta>0}$ *are linear operators, and for any* $x \in X$

$$\|\mathcal{T}_\mu(\eta)x\| \leq \frac{\Upsilon\eta^{\mu-1}}{\Gamma(\mu)}\|x\|, \qquad \|\mathcal{S}_{\gamma,\mu}(\eta)x\| \leq \frac{\Upsilon\eta^{(\gamma-1)(\mu-1)}}{\Gamma(\gamma(1-\mu)+\mu)}\|x\|$$

Proposition 2.3. *([34]) Under assumption (H1),*$\{\mathcal{T}_\mu(\eta)\}_{\eta>0}$ *and* $\{\mathcal{S}_{\gamma,\mu}(\eta)\}_{\eta>0}$ *are strongly continuous, which means that, for any* $x \in X$ *and* $0 < \eta' < \eta'' \leq b$, *we have* $\|\mathcal{T}_\mu(\eta')x - \mathcal{T}_\mu(\eta'')x\| \to 0$ *and* $\|\mathcal{S}_{\gamma,\mu}(\eta')x - \mathcal{S}_{\gamma,\mu}(\eta'')x\| \to 0$ *as* $\eta'' \to \eta'$

Let $(X, \|.\|)$ be a Banach space. We will use the following notations: $\mathcal{P}(X) = \{Y \in 2^Y : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\}$ is closed, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X)\}$ is bounded, $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X)\}$ is convex, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X)\}$ is compact.

Proposition 2.4. *([24])*

(1) *A measurable function* $u : \Omega \to X$ *is Bochner integrable if and only if* $\|u\|$ *is Lebesgue integrable.*

(2) *A multi-valued map* $\mathcal{F} : X \to 2^X$ *is said to be convex-valued (closed-valued) if* $\mathcal{F}(u)$ *is convex (closed) for all* $u \in X$*; is said to be bounded on bounded sets if* $\mathcal{F}(\mathcal{B}) = \bigcup_{u \in \mathcal{B}}$ *is bounded in* X *for all* $\mathcal{B} \in \mathcal{P}_b(X)$.

(3) *A map* $\mathcal{F}$ *is said to be upper semi-continuous (u.s.c.) on* X *if for each* $u_0 \in X$ *the set* $\mathcal{F}(u_0)$ *is a nonempty closed subset of* X, *and if for each open subset* Θ *of* X *containing* $\mathcal{F}(u_0)$, *there exists an open neighborhood* ∇ *of* u_0 *such that* $\mathcal{F}(\nabla) \subseteq \Theta$.

(4) A map $\mathcal{F}$ is said to be completely continuous if $\mathcal{F}(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multi-valued map $\mathcal{F}$ is completely continuous with nonempty compact values, then $\mathcal{F}$ is u.s.c. if and only if $\mathcal{F}$ has a closed graph, i.e., $u_n \to u; y_n \to y; y_n \in \mathcal{F}(u)$. We say that $\mathcal{F}$ has a fixed point if there is $u \in X$ such that $u \in \mathcal{F}(u)$.

(5) A multi-valued map $\mathcal{F} : \Omega \to \mathcal{P}_{cl}(X)$ is said to be measurable if for each $u \in X$ the function $y : \Omega \to R$ defined by $y(\eta) = d(u, \mathcal{F}(\eta)) = inf\{\|u - z\|, z \in \mathcal{F}(\eta)\}$ is measurable.

(6) A multi-valued map $\mathcal{F} : X \to 2^X$ is said to be condensing if for any bounded subset $\mathcal{B} \subset X$ with $\beta(\mathcal{B}) \neq 0$ we have $\beta(\mathcal{F}(\mathcal{B})) < \beta(\mathcal{B})$, where $\beta(\cdot)$ denotes the Kuratowski measure of non-compactness defined as follows: $\beta(\mathcal{B}) = inf\{d > 0 : \mathcal{B}$ can be covered by a finite number of balls of radius $d\}$

Definition 2.5. *By a mild solution of system (1.1), we mean a function $x \in \mathcal{C}_{1-\nu}(\Omega, X)$ satisfying:*

(1) $\mathcal{I}_{0+}^{1-\nu}\chi(\eta)|_{\eta=0} + p(x) = x_0 \in X$;

(2) $f \in \mathcal{L}^1(\Omega, X)$ such that $f(\eta) \in \mathcal{F}(\eta, \chi(\eta), \int_0^\eta k(\eta, \vartheta, \chi(\vartheta))d\vartheta)$

$$\chi(\eta) = \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta \mathcal{T}_\mu(\eta - \vartheta)f(\vartheta)d\vartheta + \int_0^\eta \mathcal{T}_\mu(\eta - \vartheta)\mathcal{B}u(\vartheta)d\vartheta, \eta \in \Omega'.$$

Because $\mathcal{T}_\mu(\eta) = \eta^{\mu-1}\mathcal{P}_\mu(\eta)$, then the equation earlier is equivalent to

$$\begin{aligned}\chi(\eta) = \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta (\eta - \vartheta)^{\mu-1}\mathcal{P}_\mu(\eta - \vartheta)f(\vartheta)d\vartheta \\ + \int_0^\eta (\eta - \vartheta)^{\mu-1}\mathcal{P}_\mu(\eta - \vartheta)\mathcal{B}u(\vartheta)d\vartheta, \eta \in \Omega'\end{aligned}$$

In order to review the approximate controllability for the nonlinear system (1.1), we tend to first take into account the approximate controllability of its linear part

$$\begin{aligned}&\mathcal{D}_{0+}^{\gamma,\mu}\chi(\eta) \in \mathcal{A}\chi(\eta) + (\mathcal{B}v)\eta, \quad \eta \in \Omega' = (0, b] \\ &\mathcal{I}_{0+}^{(1-\gamma)(1-\mu)}\chi(\eta)|_{\eta=0} = x_0.\end{aligned} \tag{2.4}$$

Here, $\mathcal{B} : U \to X$ is a linear bounded operator, $v \in \mathcal{L}^2(\Omega, U)$.

Definition 2.6. *The control system (2.3) is said to be approximately controllable on Ω if for all $x_0 \in X$, there is some control $u \in \mathcal{L}^2(\Omega, U)$,the closure of the reachable set $\mathcal{R}(b, x_0)$ is dense in X, that is, $\overline{\mathcal{R}(b, x_0)} = X$, where $\mathcal{R}(b, x_0) = \{\chi(b, u) : u \in$*

$\mathcal{L}^2(\Omega, U), \chi(0,u) = x_0\}$; *which is the reachable set of system (1.1) with the initial value* x_0 *at terminal time* b.

Remark 2.2. *Assume that the linear fractional control system (2.3) is approximate controllable. We recall from [17, 22] that the approximate controllability of (2.3) is such as the convergence of* $a\mathcal{R}(a, \Gamma_0^b) \to 0$ *as* $a \to 0^+$ *in the strong operator topology.*

3 Existence results

We obtain existence and approximate controllability results for the fractional nonlocal control inclusion (1.1). We consider the following hypotheses

(H$_2$) $\mathcal{Q}(\eta)$ is a compact operator for each $\eta > 0$ and $\|a(\mathcal{R}(a, \Gamma_0^b))\| \leq 1$ for $\forall \quad a > 0$

(H$_3$) The multivalued map $\mathcal{F} : \Omega \times X \times X \to \mathcal{P}_{b,cl,cv}(X)$ satisfies the following:

- **(3a)** $\mathcal{F}(\eta, \cdot, \cdot) : X \times X \to X$ is u.s.c and for each $\eta \in \Omega$ and for $x, y \in X$, the function $\mathcal{F}(\cdot, x, y) : \Omega \to X$ is strongly measurable with respect to η for each $x, y \in X$, the set

  $$S_{\mathcal{F},x} = \{f \in \mathcal{L}^1(\Omega, X) : f(\eta) \in \mathcal{F}\left(\eta, \chi(\eta), \int_0^\eta k(\eta, \vartheta, \chi(\vartheta))d\vartheta\right) \text{ for a.e } \eta \in \Omega\}$$

  is nonempty.

- **(3b)** $k : \Omega \times \Omega \times X \to X$ a continuous function, there exist a positive constant k_1 such that $\| \int_0^\eta [k(\eta, \vartheta, x) - k(\eta, \vartheta, y)]d\vartheta\| \leq k_1\|x - y\|$ for each $\eta, \vartheta \in \Omega$ and $x, y \in X$

- **(3c)** There exist a function $n(\eta) \in \mathcal{L}^{\frac{1}{\mu_1}}$, $\mu_1 \in (0, \mu)$ and a continuous nondecreasing function $\phi : [0, \infty) \to (0, \infty)$ such that for any $(\eta, x, y) \in \Omega \times X \times X$, we have $\|\mathcal{F}(\eta, \chi(\eta), \int_0^\eta k(\eta, \vartheta, \chi(\vartheta))d\vartheta)\| = \sup_{\eta \in \Omega}\{\|f(\eta)\| : f(\eta) \in (\mathcal{F}(\eta, \chi(\eta), \int_0^\eta k(\eta, \vartheta, \chi(\vartheta))d\vartheta)\} \leq n(\eta)\phi(\|x\|_\nu)$, $\lim_{r\to\infty} \inf \frac{\phi(r)}{r} = \varpi < \infty$;

(H$_4$) $h : \mathcal{C}(\Omega, X) \to X$ is continuous and compact, and there exists positive constant $\mathcal{L} > 0$ such that $\|p(x)\| \leq \mathcal{L}$ for each $x \in \mathcal{C}(\Omega, X)$

(H$_5$) $\left(\frac{\Upsilon\mathcal{L}}{\Gamma(\gamma(1-\mu)+\mu)} + \frac{b^{1-\nu}\Upsilon}{\Gamma(\mu)} \frac{(1-\mu_1)^{1-\mu_1} b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}} \varpi \|n\|_{\frac{1}{\mu}}\right) \left(1 + \frac{\Upsilon^2 \Upsilon_{\mathcal{B}}^2}{a\Gamma^2(\mu)} \frac{b^{2\mu-1}}{\mu}\right) < 1$

Lemma 3.5. *Let* Ω *be a compact real interval and let* X *be a Banach space. Let* $\mathcal{F}$ *be a multi-valued map satisfying* (H_3)*, and let* Γ *be a linear continuous mapping from* $\mathcal{L}^1(\Omega, X)$ *to* $\mathcal{C}(\Omega, X)$*. Then the operator*

$$\Gamma \circ S_{\mathcal{F}} : \mathcal{C}(\Omega, X) \to \mathcal{P}_{b,cl,cv}(\mathcal{C}(\Omega, X))$$

$x \longmapsto (\Gamma \circ S_{\mathcal{F}})(x) = \Gamma(S_{\mathcal{F},x})$ *is a closed graph operator in* $\mathcal{C}(\Omega, X) \times \mathcal{C}(\Omega, X)$.

To prove our results, we introduce two relevant operators:

$$\Gamma_0^b = \int_0^b (b-\vartheta)^{2\mu-2}\mathcal{P}_\mu(b-\vartheta)\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)d\vartheta, \quad \frac{1}{2} < \mu \leq 1$$

and $R(a, \Gamma_0^b) = (a\mathcal{I} + \Gamma_0^b)^{-1}$, $a > 0$, where $\mathcal{B}^*$ denotes the adjoint of $\mathcal{B}$ and $\mathcal{P}_\mu^*$ is the adjoint of $\mathcal{P}_\mu$. It is straightforward that the operator Γ_0^b is a linear bounded operator. Now, for any $a > 0$, and $x_1 \in X$, we set

$$u(\eta) = (b-\eta)^{\mu-1}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a, \Gamma_0^b)p(\chi(\cdot)),$$

where

$$p(\chi(\cdot)) = x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] + \int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(b-\vartheta)f(\vartheta)d\vartheta.$$

Using this control u, we define the operator $\Psi : \mathcal{C}_{1-\nu}(\Omega, X) \to 2^{\mathcal{C}_{1-\nu}(\Omega,X)}$ as follows

$$\Psi(x) = \Big\{z \in \mathcal{C}_{1-\nu}(\Omega, X) : z(\eta) = \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f(\vartheta)d\vartheta + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)\mathcal{B}u(\vartheta)d\vartheta, f \in S_{\mathcal{F},x}, \eta \in \Omega'\Big\}.$$

According to assumption(H_3), we can obtain

$$\int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f(\vartheta)d\vartheta \leq \frac{\Upsilon\phi(\|x\|_\nu)}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\|n\|_{\frac{1}{\mu}}$$

Thus

$$\|p(\chi(b))\| \leq \|x_1\| + \frac{\Upsilon b^{\nu-1}}{\Gamma(\gamma(1-\mu))}[\|x_0\| + \mathcal{L}\|x\|_\nu] + \frac{\Upsilon\phi(\|x\|_\nu)}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\|n\|_{\frac{1}{\mu}}$$

Theorem 3.1. *Assume that the hypotheses* $(H_1)-(H_5)$ *are satisfied. Then system* (1.1) *has a mild solution.*

Proof: Now, it will be shown that the operator Ψ has a fixed point. The proof will be divided into six steps.
Step 1: The operator $\Psi(x)$ is convex for each $x \in \mathcal{C}_{1-\nu}(\Omega, X)$.
Let $z_1, z_2 \in \mathcal{C}_{1-\nu}(\Omega, X)$, then there exists $f_1, f_2 \in S_{\mathcal{F},x}$ such that for each $\eta \in \Omega$, we have

$$z_i(\eta) = \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f_i(\vartheta)d\vartheta$$

$$+\int_0^\eta(\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b)$$
$$\times\Big(x_1-\mathcal{S}_{\gamma,\mu}(b)[x_0-p(x)]-\int_0^b(b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau)f_i(\tau)d\tau\Big)d\vartheta,\quad i=1,2.$$

Let $0\le\lambda\le1$, then for each $\eta\in\Omega$ we have

$$\begin{aligned}
&(\lambda z_1+(1-\lambda)z_2)(\eta)\\
&=\mathcal{S}_{\gamma,\mu}(\eta)[x_0-p(x)]+\int_0^\eta(\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)[\lambda f_1(\vartheta)+(1-\lambda)f_2(\vartheta)]d\vartheta\\
&\quad+\int_0^\eta(\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b)\\
&\quad\times\Big(x_1-\mathcal{S}_{\gamma,\mu}(b)[x_0-p(x)]-\int_0^b(b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau)[\lambda f_1(\tau)+(1-\lambda)f_2(\tau)]d\tau\Big)d\vartheta.
\end{aligned}$$

Since $S_{\mathcal{F},x}$ is convex, $\lambda z_1+(1-\lambda)z_2\in S_{\mathcal{F},x}$. Hence $\lambda z_1+(1-\lambda)z_2\in\Psi(x)$.

Step 2: consider a set $\mathcal{B}_r=x\in\mathcal{C}_{1-\nu}(\Omega,X):\|x\|_\nu\le r$, where r is a positive constant. Obviously, $\mathcal{B}_r$ is a bounded, closed and convex set of $\mathcal{C}_{1-\nu}(\Omega,X)$. We claim that there exists a positive number r such that $\Psi(\mathcal{B}_r)\subset\mathcal{B}_r$.

If this is not true, then for each positive number r, there exists function $x^r\in\mathcal{B}_r$, but $\Psi(x^r)\notin\mathcal{B}_r$, i.e.,

$$\begin{aligned}
r&\le\|\Psi(x^r)\|_\nu\\
&\le\sup_{\eta\in\Omega'}\eta^{1-\nu}\|\mathcal{S}_{\gamma,\mu}(\eta)[x_0-p(x)]\|\\
&\quad+\sup_{\eta\in\Omega'}\eta^{1-\nu}\|\int_0^\eta(\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f^r(\vartheta)d\vartheta\|\\
&\quad+\sup_{\eta\in\Omega'}\eta^{1-\nu}\|\int_0^\eta(\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)\mathcal{B}u^r(\vartheta)d\vartheta\|\\
&\le\frac{\Upsilon}{\Gamma(\gamma(1-\mu)+\mu)}[\|x_0\|+\mathcal{L}\|x\|_\nu]+\frac{b^{1-\nu}\Upsilon\phi(\|x^r\|)}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\|n\|_{\frac{1}{\mu}}\\
&\quad+\left[\frac{b^{1-\nu}\Upsilon^2\Upsilon_{\mathcal{B}}^2}{a\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\right]\\
&\quad\times\left(\|x_1\|+\frac{\Upsilon b^{\nu-1}}{\Gamma(\gamma(1-\mu)+\mu)}+[\|x_0\|+\mathcal{L}\|x\|_\nu]+\frac{\Upsilon\phi(\|x^r\|)}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\|n\|_{\frac{1}{\mu}}\right)\\
&\le\frac{\Upsilon}{\Gamma(\gamma(1-\mu)+\mu)}[\|x_0\|+\mathcal{L}r]+\frac{b^{1-\nu}\Upsilon\phi(r)}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\|n\|_{\frac{1}{\mu}}\\
&\quad+\left[\frac{b^{1-\nu}\Upsilon^2\Upsilon_{\mathcal{B}}^2}{a\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\right]\\
&\quad\times\left(\|x_1\|+\frac{\Upsilon b^{\nu-1}}{\Gamma(\gamma(1-\mu)+\mu)}+[\|x_0\|+\mathcal{L}\|x\|_\nu]+\frac{\Upsilon\phi(r)}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\|n\|_{\frac{1}{\mu}}\right).
\end{aligned}$$

Dividing both sides of the above inequality by r and taking the limit as $r \to \infty$ and using (H_5), we get

$$\left(\frac{\Upsilon\mathcal{L}}{\Gamma(\gamma(1-\mu)+\mu)} + \frac{b^{1-\nu}\Upsilon}{\Gamma(\mu)}\frac{(1-\mu_1)^{1-\mu_1}b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}}\varpi\|n\|_{\frac{1}{\mu}}\right)\left(1+\frac{\Upsilon^2\Upsilon_{\mathcal{B}}^2}{a\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\right) \geq 1$$

which is a contradiction to (H_5). Thus, there exists $r > 0$ such that Ψ maps $\mathcal{B}_r$ into itself.

Step 3: Ψ maps bounded sets into equicontinuous sets of $\mathcal{C}_{1-\nu}(\Omega, X)$.

Let $0 < \vartheta < \eta < \eta + h \leq b$ and $\epsilon > 0$. For each $x \in \mathcal{B}_r, z \in \Psi(x)$, there exists a $f \in S_{\mathcal{F},x}$ such that

$$z(\eta) = \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f(\vartheta)d\vartheta + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)\mathcal{B}u(\vartheta)d\vartheta.$$

Clearly,

$$\begin{aligned}
\|z(\eta+h)-z(\eta)\| \leq\ & \sup_{\eta\in\Omega'} \eta^{1-\nu}\|\mathcal{S}_{\gamma,\mu}(\eta+h)[x_0-p(x)] - \mathcal{S}_{\gamma,\mu}(\eta)[x_0-p(x)]\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_\eta^{\eta+h}(\eta+h-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta+h-\vartheta)f(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_{\eta-\epsilon}^{\eta}(\eta+h-\vartheta)^{\mu-1}(\mathcal{P}_\mu(\eta+h-\vartheta)-\mathcal{P}_\mu(\eta-\vartheta))f(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_{\eta-\epsilon}^{\eta}((\eta+h-\vartheta)^{\mu-1}-(\eta-\vartheta)^{\mu-1})\mathcal{P}_\mu(\eta-\vartheta)f(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_0^{\eta-\epsilon}(\eta+h-\vartheta)^{\mu-1}(\mathcal{P}_\mu(\eta+h-\vartheta)-\mathcal{P}_\mu(\eta-\vartheta))f(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_0^{\eta-\epsilon}((\eta+h-\vartheta)^{\mu-1}-(\eta-\vartheta)^{\mu-1})\mathcal{P}_\mu(\eta-\vartheta)f(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_\eta^{\eta+h}(\eta+h-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta+h-\vartheta)\mathcal{B}u(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_{\eta-\epsilon}^{\eta}(\eta+h-\vartheta)^{\mu-1}(\mathcal{P}_\mu(\eta+h-\vartheta)-\mathcal{P}_\mu(\eta-\vartheta))\mathcal{B}u(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_{\eta-\epsilon}^{\eta}((\eta+h-\vartheta)^{\mu-1}-(\eta-\vartheta)^{\mu-1})\mathcal{P}_\mu(\eta-\vartheta)\mathcal{B}u(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_0^{\eta-\epsilon}(\eta+h-\vartheta)^{\mu-1}(\mathcal{P}_\mu(\eta+h-\vartheta)-\mathcal{P}_\mu(\eta-\vartheta))\mathcal{B}u(\vartheta)d\vartheta\right\| \\
&+ \sup_{\eta\in\Omega'} \eta^{1-\nu}\left\|\int_0^{\eta-\epsilon}((\eta+h-\vartheta)^{\mu-1}-(\eta-\vartheta)^{\mu-1})\mathcal{P}_\mu(\eta-\vartheta)\mathcal{B}u(\vartheta)d\vartheta\right\| \\
&= \sum_{i=1}^{11}\mathcal{I}_i
\end{aligned}$$

Now, we only need to check $\mathcal{I}_i \to 0$ as $h \to 0, i = 1, 2, ..., 11$.
For $\mathcal{I}_1$, by Proposition 1.3, $\mathcal{I}_1 \to 0$ as $h \to 0$.
Let $q = \frac{\mu-1}{1-\mu_1} \in (-1, 0)$. In view of Proposition 1.2, for $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5$ and $\mathcal{I}_6$ we have

$$\mathcal{I}_2 \leq \frac{b^{1-\nu}\Upsilon\phi(r)}{\Gamma(\mu)} \frac{h^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|n\|_{\frac{1}{\mu_1}}$$

$$\mathcal{I}_3 \leq \frac{b^{1-\nu}\Upsilon\phi(r)}{\Gamma(\mu)} \left(\int_{\eta-\epsilon}^{\eta} (\eta+h-\vartheta)^{\mu-1} d\vartheta\right) \|n\|_{\frac{1}{\mu_1}} \leq \frac{b^{1-\nu}\Upsilon\phi(r)}{\Gamma(\mu)} \frac{(2h)^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|n\|_{\frac{1}{\mu_1}}$$

$$\mathcal{I}_4 \leq \frac{b^{1-\nu}\Upsilon\phi(r)}{\Gamma(\mu)} \left(\int_{\eta-\epsilon}^{\eta} [(\eta+h-\vartheta)^{\mu-1} - (\eta-\vartheta)^{\mu-1}]^{\frac{1}{1-\mu_1}} d\vartheta\right)^{1-\mu_1} \|n\|_{\frac{1}{\mu_1}}$$
$$\leq \frac{b^{1-\nu}\Upsilon\phi(r)}{\Gamma(\mu)} \frac{(2h)^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|n\|_{\frac{1}{\mu_1}}$$

$$\mathcal{I}_5 \leq b^{1-\nu} \sup_{\vartheta\in[0,\eta-\epsilon]} \|\mathcal{P}_\mu(\eta+h-\vartheta) - \mathcal{P}_\mu(\eta-\vartheta)\| \left(\int_0^{\eta-\epsilon} (\eta+h-\vartheta)^{\mu-1} d\vartheta\right) \phi(r)\|n\|_{\frac{1}{\mu_1}}$$
$$\leq b^{1-\nu} \sup_{\vartheta\in[0,\eta-\epsilon]} \|\mathcal{P}_\mu(\eta+h-\vartheta) - \mathcal{P}_\mu(\eta-\vartheta)\| \frac{[(\eta+h)^{q+1} - (h+\epsilon)^{q+1}]^{1-\mu_1}}{(q+1)^{1-\mu_1}} \phi(r)\|n\|_{\frac{1}{\mu_1}}$$

In a similar way, for $\mathcal{I}_7, \mathcal{I}_8, \mathcal{I}_9, \mathcal{I}_{10}$ and $\mathcal{I}_{11}$ we obtain

$$\mathcal{I}_7 \leq \frac{b^{1-\nu}\Upsilon\Upsilon_{\mathcal{B}}}{\Gamma(\mu)} \frac{h^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|u\|$$

$$\mathcal{I}_8 \leq \frac{b^{1-\nu}\Upsilon\Upsilon_{\mathcal{B}}}{\Gamma(\mu)} \frac{(2h)^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|u\|$$

$$\mathcal{I}_9 \leq \frac{b^{1-\nu}\Upsilon\Upsilon_{\mathcal{B}}}{\Gamma(\mu)} \frac{(2h)^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|u\|$$

$$\mathcal{I}_{10} \leq b^{1-\nu}\Upsilon_{\mathcal{B}} \sup_{\vartheta\in[0,\eta-\epsilon]} \|\mathcal{P}_\mu(\eta+h-\vartheta) - \mathcal{P}_\mu(\eta-\vartheta)\| \frac{[(\eta+h)^{q+1} - (h+\epsilon)^{q+1}]^{1-\mu_1}}{(q+1)^{1-\mu_1}} \|u\|$$

$$\mathcal{I}_{11} \leq \frac{b^{1-\nu}\Upsilon\Upsilon_{\mathcal{B}}}{\Gamma(\mu)} \frac{(2h)^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}} \|u\|$$

It can be easily seen that $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_6, \mathcal{I}_7, \mathcal{I}_8, \mathcal{I}_9, \mathcal{I}_{11}$ tends to zero as $h \to 0$. By Proposition 2.3, $\mathcal{I}_5$ and $\mathcal{I}_{10}$ tends to zero. Thus $\|z(\eta+h) - z(\eta)\| \to 0$ as $h \to 0$ for all $x \in \mathcal{B}_r$. This implies that $\Psi(\mathcal{B}_r) \subset \mathcal{C}_{1-\nu}(\Omega, X)$ is equicontinuous.
Step 4: Next we show that the set $V(\eta) = \{z(\eta) : z \in \Psi(\mathcal{B}_r)\}$ is relatively compact in X. The case $\eta = 0$ is trivial. Let $\eta \in (0, b]$ be fixed and for each $\lambda \in (0, \eta)$ and $\forall\delta > 0$, $z \in \Psi(\mathcal{B}_r)$, define an operator

$$z^{\lambda,\delta}(\eta) = \frac{1}{\Gamma(\gamma(1-\mu))} \int_0^{\eta-\lambda} \int_\delta^\infty \mu\theta\Upsilon_\mu(\theta)(\eta-\vartheta)^{\gamma(1-\mu)-1}\vartheta^{\mu-1}\mathcal{Q}(\vartheta^\mu\theta)[x_0 - p(x)]d\theta d\vartheta$$

$$+\mathcal{Q}(\lambda^{\mu}\theta)\int_{0}^{\eta-\lambda}\int_{\delta}^{\infty}\mu\theta(\eta-\vartheta)^{\mu-1}\Upsilon_{\mu}(\theta)\mathcal{Q}((\eta-\vartheta)^{\mu}\theta-\lambda^{\mu}\theta)[f(\vartheta)+\mathcal{B}u(\vartheta)]d\theta d\vartheta$$

From the compactness of $\mathcal{Q}(\lambda^{\mu}\theta), \lambda^{\mu}\theta > 0$, we obtain that for $\forall\,\lambda \in (0,\eta)$ and $\forall\,\delta > 0$, the set $V^{\epsilon,\delta}(\eta) = \{z^{\lambda,\delta}(\eta), z^{\lambda,\delta} \in \Psi^{\lambda,\delta}(x), x \in \mathcal{B}_r\}$ is relatively compact set in X.
Moreover, for each $x \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\|z(\eta)-z^{\lambda,\delta}(\eta)\|\\
&=\sup_{\eta\in\Omega'}\eta^{1-\nu}\Big\|\frac{1}{\Gamma(\gamma(1-\mu))}\int_{0}^{\eta}(\eta-\vartheta)^{\gamma(1-\mu)-1}\vartheta^{\mu-1}\int_{0}^{\delta}\mu\theta\Upsilon_{\mu}(\theta)\mathcal{Q}(\vartheta^{\mu}\theta)[x_0-p(x)]d\theta d\vartheta\\
&\quad+\frac{1}{\Gamma(\gamma(1-\mu))}\int_{0}^{\eta}(\eta-\vartheta)^{\gamma(1-\mu)-1}\vartheta^{\mu-1}\int_{\delta}^{\infty}\mu\theta\Upsilon_{\mu}(\theta)\mathcal{Q}(\vartheta^{\mu}\theta)[x_0-p(x)]d\theta d\vartheta\\
&\quad-\frac{1}{\Gamma(\gamma(1-\mu))}\int_{0}^{\eta-\lambda}(\eta-\vartheta)^{\gamma(1-\mu)-1}\vartheta^{\mu-1}\int_{\delta}^{\infty}\mu\theta\Upsilon_{\mu}(\theta)\mathcal{Q}(\vartheta^{\mu}\theta)[x_0-p(x)]d\theta d\vartheta\\
&\quad+\int_{0}^{\eta}\int_{0}^{\delta}\mu\theta(\eta-\vartheta)^{\mu-1}\Upsilon_{\mu}(\theta)\mathcal{Q}((\eta-\vartheta)^{\mu}\theta)[f(\vartheta)+\mathcal{B}u(\vartheta)]d\theta d\vartheta\\
&\quad+\int_{0}^{\eta}\int_{\delta}^{\infty}\mu\theta(\eta-\vartheta)^{\mu-1}\Upsilon_{\mu}(\theta)\mathcal{Q}((\eta-\vartheta)^{\mu}\theta)[f(\vartheta)+\mathcal{B}u(\vartheta)]d\theta d\vartheta\\
&\quad-\int_{0}^{\eta-\lambda}\int_{\delta}^{\infty}\mu\theta(\eta-\vartheta)^{\mu-1}\Upsilon_{\mu}(\theta)\mathcal{Q}((\eta-\vartheta)^{\mu}\theta)[f(\vartheta)+\mathcal{B}u(\vartheta)]d\theta d\vartheta\Big\|\\
&\le\sup_{\eta\in\Omega'}\eta^{1-\nu}\Big\|\frac{1}{\Gamma(\gamma(1-\mu))}\int_{0}^{\eta}(\eta-\vartheta)^{\gamma(1-\mu)-1}\vartheta^{\mu-1}\int_{0}^{\delta}\mu\theta\Upsilon_{\mu}(\theta)\mathcal{Q}(\vartheta^{\mu}\theta)[x_0-p(x)]d\theta d\vartheta\Big\|\\
&\quad+\sup_{\eta\in\Omega'}\eta^{1-\nu}\Big\|\frac{1}{\Gamma(\gamma(1-\mu))}\int_{\eta-\lambda}^{\eta}(\eta-\vartheta)^{\gamma(1-\mu)-1}\vartheta^{\mu-1}\int_{\delta}^{\infty}\mu\theta\Upsilon_{\mu}(\theta)\mathcal{Q}(\vartheta^{\mu}\theta)[x_0-p(x)]d\theta d\vartheta\Big\|\\
&\quad+\sup_{\eta\in\Omega'}\eta^{1-\nu}\Big\|\int_{0}^{\eta}\int_{0}^{\delta}\mu\theta(\eta-\vartheta)^{\mu-1}\Upsilon_{\mu}(\theta)\mathcal{Q}((\eta-\vartheta)^{\mu}\theta)[f(\vartheta)+\mathcal{B}u(\vartheta)]d\theta d\vartheta\Big\|\\
&\quad+\sup_{\eta\in\Omega'}\eta^{1-\nu}\Big\|\int_{\eta-\lambda}^{\eta}\int_{\delta}^{\infty}\mu\theta(\eta-\vartheta)^{\mu-1}\Upsilon_{\mu}(\theta)\mathcal{Q}((\eta-\vartheta)^{\mu}\theta)[f(\vartheta)+\mathcal{B}u(\vartheta)]d\theta d\vartheta\Big\|\\
&\le\frac{\mu\Upsilon}{\Gamma(\gamma(1-\mu))}[\|x_0+\mathcal{L}r\|]\mathcal{B}(\gamma(\mu-1),\mu-1)\int_{0}^{\delta}\theta\Upsilon_{\mu}(\theta)d\theta\\
&\quad+\frac{\mu\Upsilon b^{\gamma(1-\mu)-1}}{\Gamma(\gamma(1-\mu))\Gamma(1+\mu)}\frac{\lambda^{\mu}}{\mu}[\|x_0+\mathcal{L}r\|]\int_{\delta}^{\infty}\theta\Upsilon_{\mu}(\theta)d\theta\\
&\quad+\frac{b^{1-\nu}\Upsilon b^{(q+1)(1-\mu_1)}}{(q+1)^{(1-\mu_1)}}(\phi(r)\|n\|_{\frac{1}{\mu_1}}+\Upsilon_{\mathcal{B}}\|u\|)\int_{0}^{\delta}\theta\Upsilon_{\mu}(\theta)d\theta\\
&\quad+\frac{b^{1-\nu}\Upsilon\lambda^{(q+1)(1-\mu_1)}}{\Gamma(1+\mu)(q+1)^{(1-\mu_1)}}(\phi(r)\|n\|_{\frac{1}{\mu_1}}+\Upsilon_{\mathcal{B}}\|u\|)
\end{aligned}$$

This implies that these are relatively compact sets arbitrarily close to the set $V(\eta)$ for each $\eta \in (0,b]$. Thus $V(\eta)$ is relatively compact in X for all $\eta \in (0,b]$. Since it is compact at $\eta = 0$, hence $V(\eta)$ is relatively compact in X for all $\eta \in \Omega$.

Step 5: Ψ has a closed graph.

Let $x_n \to x_*$ as $n \to \infty$, $z_n \in \Psi(x_n)$ and $z_n \to z_*$ as $n \to \infty$. We shall show that $z_* \in \Psi(x_*)$. Since $z_n \in \Psi(x_n)$, there exists a $f_n \in S_{\mathcal{F},x_n}$ such that

$$\begin{aligned} z_n(\eta) &= \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta) f_n(\vartheta) d\vartheta \\ &+ \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \\ &\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_n(\tau) d\tau\Big) d\vartheta \end{aligned}$$

We must prove that there exist $f_* \in S_{\mathcal{F},x_*}$ such that for each $\eta \in \Omega$,

$$\begin{aligned} z_*(\eta) &= \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta) f_*(\vartheta) d\vartheta \\ &+ \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \\ &\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_*(\tau) d\tau\Big) d\vartheta \end{aligned}$$

Clearly,

$$\begin{aligned} &\Big\|\Big[z_n(\eta) - \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] \\ &- \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \\ &\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_n(\tau) d\tau\Big) d\vartheta\Big] \\ &- \Big[z_*(\eta) - \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] \\ &- \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \\ &\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_*(\tau) d\tau\Big) d\vartheta\Big]\Big\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consider the linear continuous operator $\Gamma : \mathcal{L}^{\frac{1}{\mu}}(\Omega, X) \to \mathcal{C}_{1-\nu}(\Omega, X)$

$$\begin{aligned} (\Gamma f)(\eta) &= \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta) f_n(\vartheta) d\vartheta \\ &+ \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \end{aligned}$$

$$\times \Big(\int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_*(\tau)d\tau\Big) d\vartheta$$

Clearly, it follows from Lemma 2.4 that $\Gamma \circ S_{\mathcal{F},x}$ is a closed graph operator. Also from the definition of Γ, we have that

$$\Big[z_n(\eta) - \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] - \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b)$$
$$\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_n(\tau)d\tau\Big) d\vartheta\Big] \in \Gamma(S_{\mathcal{F},x_n})$$

Since $f_n \to f_*$, it follows from Lemma 3.5 that

$$\Big[z_*(\eta) - \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - p(x)] - \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b)$$
$$\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - p(x)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f_*(\tau)d\tau\Big) d\vartheta\Big] \in \Gamma(S_{\mathcal{F},x_*})$$

Therefore Ψ has a closed graph. As a consequence of step 1 to step 5 with the ArzelaAscoli theorem, Ψ is a completely continuous multivalued map with compact value and hence Ψ is u.s.c.. Hence by Lemma 2.2, Ψ has a fixed point $\chi(\cdot)$ on $\mathcal{B}_r$, which is the mild solution of the system (1.1)

Theorem 3.2. *Assume that $(H_1) - (H_5)$ hold and multivalued function $\mathcal{F}(\eta, \chi(\eta), \int_0^\eta k(\eta,\vartheta,\chi(\vartheta))d\vartheta)$ is uniformly bounded. Moreover, assume that the corresponding linear system (2.3) is approximately controllable on Ω, then system (1.1) is approximately controllable on Ω.*

Proof: Under the above hypotheses, we know the operator Ψ has a fixed point in $\mathcal{B}_r$. Let x^a be a fixed point of Ψ in $\mathcal{B}_r$, this means there exists $f^a \in S_{\mathcal{F},x}$ such that

$$\begin{aligned} x^a(\eta) =& \mathcal{S}_{\gamma,\mu}(\eta)[x_0 - h(x^a)] + \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta) f^a(\vartheta)d\vartheta \\ &+ \int_0^\eta (\eta-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)(b-\vartheta)^{\mu-1}\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \\ &\times \Big(x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - h(x^a)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f^a(\tau)d\tau\Big) d\vartheta, \quad \eta \in \Omega' \end{aligned}$$

Define

$$p(x^a) = x_1 - \mathcal{S}_{\gamma,\mu}(b)[x_0 - h(x^a)] - \int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau) f^a(\tau)d\tau$$

Noting that $\mathcal{I}-\Gamma_0^b R(a,\Gamma_0^b)=aR(a,\Gamma_0^b)$ we get

$$\begin{aligned}
x^a(b) &= \mathcal{S}_{\gamma,\mu}(b)[x_0-h(x^a)]+\int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f^a(\vartheta)d\vartheta \\
&\quad+\int_0^b (b-\vartheta)^{2(\mu-1)}\mathcal{P}_\mu(b-\vartheta)\mathcal{B}\mathcal{B}^*\mathcal{P}_\mu^*(b-\vartheta)R(a,\Gamma_0^b) \\
&\quad\times\Big(x_1-\mathcal{S}_{\gamma,\mu}(b)[x_0-h(x^a)]-\int_0^b (b-\tau)^{\mu-1}\mathcal{P}_\mu(b-\tau)f^a(\tau)d\tau\Big)d\vartheta \\
&= \mathcal{S}_{\gamma,\mu}(b)[x_0-h(x^a)]+\int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f^a(\vartheta)d\vartheta+\Gamma_0^b R(a,\Gamma_0^b)p(x^b) \\
&= \mathcal{S}_{\gamma,\mu}(b)[x_0-h(x^a)]+\int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(\eta-\vartheta)f^a(\vartheta)d\vartheta+p(x^a)-aR(a,\Gamma_0^b)p(x^a) \\
&= x_1-aR(a,\Gamma_0^b)p(x^a)
\end{aligned}$$

In addition, by our hypotheses, there exists constant $\widetilde{\mathcal{L}}<\infty$ such that $\|f^a(\vartheta)\|\le\widetilde{\mathcal{L}}$. Consequently, the sequence $\{f^a(\vartheta)\}$ has subsequence still denoted by $\{f^a(\vartheta)\}$, weakly converges to say $\{f(\vartheta)\}$.
Denote

$$w=x_1-\mathcal{S}_{\gamma,\mu}(b)[x_0-p(x)]-\int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(b-\vartheta)f(\vartheta)d\vartheta$$

We derive that

$$\begin{aligned}
\|p(x^a)-w\| &= \|h(x^a-p(x)\|+\Big\|\int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(b-\vartheta)[f^a(\vartheta)-f(\vartheta)]d\vartheta\Big\| \\
&\le \mathcal{L}\|x^a-x\|_\nu+\Big\|\int_0^b (b-\vartheta)^{\mu-1}\mathcal{P}_\mu(b-\vartheta)[f^a(\vartheta)-f(\vartheta)]d\vartheta\Big\|
\end{aligned}$$

From the compactness of the operator $\{\mathcal{P}_\mu(\eta),\eta>0\}$ and the uniform boundedness of $\{f^a(\vartheta)\}$ that there exists some $f(\vartheta)\in\mathcal{L}^1(\Omega,X)$ such that as $a\to0^+$

$$\mathcal{P}_\mu(b-\vartheta)f^a(\vartheta)d\vartheta\to\mathcal{P}_\mu(b-\vartheta)f(\vartheta)$$

Moveover, by the approximate controllability of system (2.3) and Remark 2.2, we all know that $a(a\mathcal{I}+\Gamma_0^b)^{-1}$ tends to zero as $a\to0^+$ in the strong operator topology. Thus we can obtain that as $a\to0^+$,

$$\begin{aligned}
\|x^a(b)-x_1\| &\le \|aR(a,\Gamma_0^b)(w)\|+\|aR(a,\Gamma_0^b)\|\|p(x^a)-w\| \\
&\le \|aR(a,\Gamma_0^b)(w)\|+\|p(x^a)-w\| \\
&\to 0
\end{aligned}$$

Therefore, system (1.1) is approximately controllable on Ω. The proof is complete.

4 Application

As a use of our outcomes, we consider the fractional differential inclusion

$$\begin{aligned} &\mathcal{D}_{0^+}^{\gamma,\frac{3}{4}}(\chi(\eta,\theta) \in \chi_{\theta\theta}(\eta,\theta) + \overline{\mathcal{F}}\left(\eta, \chi(\eta,\theta), \int_0^\eta g(\eta,\vartheta,\chi(\vartheta,\theta))d\vartheta\right) + \mathcal{B}u(\eta,\theta), \\ &\chi(\eta,0) = \chi(\eta,\pi) = 0, \quad \eta \in \Omega = [0,b] \\ &\mathcal{I}_{0^+}^{(\frac{1}{4})(1-\mu)}(\chi(0,\theta)) + \sum_{i=0}^{m}\int_0^\pi w(\theta,z)\chi(\eta_i,z)dz = x_0(\theta), \theta \in [0,\pi] \end{aligned} \tag{4.1}$$

where $\mathcal{D}_{0^+}^{\gamma,\frac{3}{4}}$ is the Hilfer fractional derivative of order $\frac{3}{4}$ and type γ, $\mathcal{I}_{0^+}^{(\frac{1}{4})(1-\mu)}$ is the Riemann-Liouville integral of order $\frac{1}{4}(1-\mu)$, $\overline{\mathcal{F}}\left(\eta, \chi(\eta,\theta), \int_0^\eta g(\eta,\vartheta,\chi(\vartheta,\theta))d\vartheta\right)$ and $\mathcal{B}u(\eta,\theta)$ are given functions, $w(\theta,z) \in \mathcal{L}^2([0,\pi]\times[0,\pi], R^+)$, m is a positive integer and $0 < \eta_0 < \eta_1 < ... < \eta_m \leq b, x_0(\theta) \in X = \mathcal{L}^2([0,\pi], R)$.

Let $U = X = \mathcal{L}^2([0,\pi], R)$ and Define an operator by $Az = z''$ with the domain, $D(A)$ is given by $\{z \in X : z, z'$ are absolutely continuous, $z'' \in X, z(0) = z(\pi) = 0\}$. Then $Az = \sum_{n=1}^{\infty} -n^2 \langle z, e_n\rangle e_n, z \in D(A)$, where $e_n(\theta) = \sqrt{\frac{2}{\pi}}\sin n\theta,\ n = 1,2,....$ It is known that A generates a compact semigroup $\mathcal{Q}(\eta), \eta > 0$, in A and is given by

$$\mathcal{Q}(\eta)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n.$$

Let $\chi(\eta)(\theta) = \chi(\eta,\theta), \eta \in \Omega = [0,b], \theta \in [0,\pi]$. Now for any $x \in X = \mathcal{L}^2([0,\pi], R), \theta \in [0,\pi]$, we define function $\mathcal{F} : \Omega\times X\times X \to Z$, the bounded linear operator $\mathcal{B} : U \to Z$ respectively by $\mathcal{F}\left(\eta, \chi(\eta), \int_0^\eta k(\eta,\vartheta,\chi(\vartheta))d\vartheta\right)(\theta) = \overline{\mathcal{F}}\left(\eta, \chi(\eta,\theta), \int_0^\eta g(\eta,\vartheta,\chi(\vartheta,\theta))d\vartheta\right)$ and $p(x)(\theta) = \sum_{i=0}^m \int_0^\pi w(\theta,z)\chi(\eta_i,z)dz$. Therefore, system (4.1) can be reformulated as the nonlocal Cauchy problem (1.1). Hence(4.1) is approximately controllable on Ω.

References

[1] A. Ansari, Fractional exponential operators and time-fractional telegraph equation, *Bound. Value Probl.*, 125(1) (2012), 1-6.

[2] D. Baleanu, Z.B. Gunvenc and J.A.T. Machado, *New Trends in Nanotechnology and Fractional Calculus Applications* , Springer(2010).

[3] D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional Calculus*

Models and Numerical Methods, Series on Complexity, Nonlinearity and Chaos , World Scientific, (2012).

[4] A.E. Bashirov and N.I. Mahmudov, On concepts of controllability for deterministic and stochastic systems, *SIAM J. Control Optim*, 37 (1999), 1808-1821.

[5] A. Debbouche and D. Baleanu, Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems, *Comput. Math. Appl*, 62 (2011), 1442-1450.

[6] A. Debbouche and D.F.M. Torres, Approximate controllability of fractional delay dynamic inclusions with nonlocal control conditions, *Appl. Math. Comput.*, 243 (2014), 161-175.

[7] D. Baleanu, J. A.T. Machado and A.C.J. Luo, *Fractional Dynamics and Control*, Springer, (2012).

[8] M. Feckan, J.R. Wang and Y. Zhou, Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators, *J. Optim. Theory Appl*, 156 (2013), 79-95.

[9] K.M. Furati, M.D. Kassim and N.E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput.Math.Appl*, 64 (2012), 1616-1626.

[10] K.M. Furati, M.D. Kassim and N.E. Tatar, Non-existence of global solutions for a differential equation involving Hilfer fractional derivative, *Electron.J.Diff.Eq*, 235 (2013), 1-10.

[11] H.B. Gu and J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, 257 (2015), 344-354.

[12] R. Hilfer, Applications of Fractional Calculus in Physics, *World Scientific*, Singapore(2000).

[13] R. Hilfer, Y. Luchko and Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, *Fract.Calc.Appl.Anal*, 12 (2009), 289-318.

[14] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204 (2006), Elsevier Science Amsterdam.

[15] V. Lakshmikantham, S. Leela and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers(2009).

[16] X.H. Liu, Z.H. Liu and M.J. Bin, Approximate controllability of impulsive fractional neutral evolution equations with Riemann-Liouville fractional derivatives, *J. Comput. Anal. Appl.*, 17 (2014), 468-485.

[17] NI. Mahmudov and MA. McKibben, the approximate controllability of fractional evolution equations with generalized Riemann-Liouville fractional derivative, *J. Funct. Spaces*, (2015) Art. ID 263823, 1-9.

[18] F. Mainardi and A. Carpinteri, Fractional Calculus: Integral and Differential Equations of Fractional Order, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, (1997), 291-348.

[19] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York (1993).

[20] M. Yang and Q.R. Wang, Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions, *Math. Meth. Appl. Sci*, 40(4)(2016), 1126-1138.

[21] G. Mophou and G.N. Gurkata, Controllability of semilinear neutral fractional functional evolution equations with infinite delay, *Nonlinear Stud*, 18 (2011), 195-209.

[22] G. Mophou, Controllability of a backward fractional semilinear differential equation, *Appl. Math. Comput*, 242 (2014), 168-178.

[23] I. Podlubny, *Fractional Differential Equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, San Diego (1999).

[24] C. Ravichandran and D. Baleanu, On the controllability of fractional functional integro-differential systems with an infinite delay in Banach spaces, *Adv. Difference Equ.*, 291 (2013), 1-13.

[25] C. Ravichandran, K. Jothimani, H.M. Baskonus and N. Valliammal, New results on nondensely characterized integro-differential equations with fractional order, *Eur. Phys. J. Plus*, 133(3)109 (2018), 1-9.

[26] C. Ravichandran, N. Valliammal and Juan J. Nieto, New results on exact controllability of a class of fractional neutral integro-differential

systems with state-dependent delay in Banach spaces, *J Franklin Inst*, 356(3)(2019), 1535-1565

[27] K. Rykaczewski, Approximate controllability of differential inclusions in Hilbert spaces, *Nonlinear Anal.*, 75 (2012), 2701-2712.

[28] R. Subashini, S. Vimal Kumar, S. Saranya and C. Ravichandran, On the controllability of non-densely defined fractional neutral functional differential equations in Banach spaces, *Int. J. Pure Appl. Math*, 118(11) (2018), 257-276.

[29] N. Valliammal, C. Ravichandran and Ju.H. Park, On the controllability of fractional neutral integro-differential delay equations with nonlocal conditions, *Math. Methods Appl. Sci.*, 40(4)(2017), 5044-5055.

[30] V. Vijayakumar, C. Ravichandran and R. Murugesu, Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay, *Nonlinear Stud.* 20 (2013), 511-530.

[31] V. Vijayakumar, C. Ravichandran, R. Murugesu and J.J. Trujillo, Controllability results for a class of fractional semilinear integro-differential inclusions via resolvent operators, *Appl. Math. Comput.*, 247 (2014), 152-161.

[32] J.R. Wang and Y.R. Zhang, Nonlocal initial value problems for differential equations with Hilfer fractional derivative, *Appl. Math. Comput.*, 266 (2015), 850-859.

[33] Y. Zhou, F. Jiao and J. Li, Existence and uniqueness for fractional neutral differential equations with infinite delay, *Nonlinear Anal*, TMA 71(7)(2009), 3249-3256.

[34] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal,* 11(5)(2010), 4465-4475.

[35] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput.Math.Appl*, 59 (2010), 1063-1077.

[36] Y. Zhou, L. Zhang and X.H. Shen, Existence of mild solutions for fractional evolution equations, *J.Int.Equ.Appl*, 25 (2013), 557-585.

[37] Y. Zhou, Basic Theory of Fractional Differential Equations, *World Scientific*, Singapore, 2014.

[38] Y. Zhou, V. Vijayakumar, C. Ravichandran and R. Murugesu, Controllability results for fractional order neutral functional differential inclusions with infinite delay, *Fixed Point Theory*, 18(2)(2017), 773-798.