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ON 5-REGULAR BIPARTITIONS WITH ODD PARTS DISTINCT

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Abstract. In his work, K. Alladi [1] considered the partition function $pod(n)$, the number of partitions of an integer n with odd parts distinct (the even parts are unrestricted). Later Hirschhorn and Sellers [8] obtained some internal congruences involving infinite families of Ramanujan-type congruences for $pod(n)$. Let $B_o(n)$ denote the number of 5-regular bipartitions of a positive integer n with odd parts distinct. In this paper, we establish many infinite families of congruences modulo powers of 2 for $B_o(n)$. For example,

$$B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0 \pmod{16},$$

for $\alpha,\beta,\gamma\geq 0$ and $t_7\in\{28,92,124,156\}$.

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1. Introduction

A partition of a positive integer n is a sequence of positive integers $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k$ such that $\lambda_1+\lambda_2+\cdots\lambda_k=n$. A partition is an m-regular partition if none of its parts is divisible by m. Let $b_m(n)$ denote the number of m-regular partitions of n with $b_m(0)=1$. The generating function for $b_m(n)$ is

$$\sum_{n=0}^{\infty} b_m\left(n\right)q^n=\frac{f_m}{f_1},$$

where

$$f_m:=(q^m;q^m)_\infty=\prod_{n=1}^{\infty}(1-q^{mn}).$$

Calkin et al. [3] established congruences for 5-regular partitions modulo 2 and for 13-regular partitions modulo 2 and 3 using the theory of modular forms. For more details, one can see [4], [9], [12], [13] and [14].
Recently, the authors [11] obtained infinite families of congruences modulo powers of 2 for 5-regular bipartitions with even parts distinct.
K. Alladi [1] considered the partition function $pod(n)$, the number of partitions of an integer n with odd parts distinct (the even parts are unrestricted). Subsequently Hirschhorn and Sellers [8] obtained some internal congruences involving infinite families of Ramanujan-type congruences for $pod(n)$. For more details, one can see [5], [10], [15], [17] and [18].
Let $B_o(n)$ denote the number of 5-regular bipartitions of n with odd parts distinct and $B_o(0)=1$.

The generating function is given by

$$\sum_{n=0}^{\infty} B_o(n)\, q^n = \left(\frac{(-q;q^2)_\infty}{(-q^5;q^{10})_\infty} \frac{(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty} \right)^2$$
$$= \left(\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty} \frac{(q^5;q^{10})_\infty}{(q^{10};q^{20})_\infty} \frac{(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty} \right)^2$$
$$= \left(\frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}} \right)^2$$

(1)
$$\sum_{n=0}^{\infty} B_o(n)\, q^n = \frac{f_2^2 f_5^2 f_{20}^2}{f_1^2 f_4^2 f_{10}^2}.$$

For example, there are 16 bipartitions with odd parts distinct for $B_o(5)$, namely

(0, 4+1), (4+1, 0), (3+2, 0), (0, 3+2), (0, 2+2+1), (2+2+1, 0), (1, 4), (4, 1), (2+2, 1), (1, 2+2), (3+1, 1), (1, 3+1), (2, 2+1), (2+1, 2), (3, 2), (2, 3).

We prove many congruences of the following form. For $\alpha, \beta \geq 0$,

$$\sum_{n=0}^{\infty} B_o\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta} - 1\right) q^n \equiv 8 f_2^3 f_5^3 \pmod{16}.$$

2. Preliminary results

In this section, we collect many identities which are useful in proving our main results.

Lemma 2.1. *The following* 2*-dissections hold :*

(2)
$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}$$

and

(3)
$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}.$$

The equation (2) is essentially (1.9.4) in [7]. The equation (3) can be obtained from (2) by replacing q by $-q$. Also see [2, p.40].

Lemma 2.2. *The following* 2*-dissections hold :*

(4)
$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}.$$

and

(5)
$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}$$

The equation (4) was proved by Hirschhorn and Sellers [9] ; see also [16]. Replacing q by $-q$ in (4) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (5).

Lemma 2.3. [2, p.345] *We have*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3} \tag{6}$$

$$\equiv f_3 + q f_9^3 \pmod{2}. \tag{7}$$

Lemma 2.4. *We require the 5-dissection due to Ramanujan,*

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{8}$$

where

$$R(q) = \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty}.$$

The equation (8) is the same as (8.1.1) in [7]. Also we can see [6], [19].

Lemma 2.5. *We require the 7-dissection due to Ramanujan,*

$$f_1 = f_{49} \left(\frac{B(q^7)}{C(q^7)} - q \frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right), \tag{9}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.

Lemma (2.5) is an exercise in [7], see [7, 10.5]. Also we can see [2, p.303, Entry 17(v)].

Lemma 2.6. *For any positive integers* k *and* m,

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}, \tag{10}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \tag{11}$$

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{8}. \tag{12}$$

Proof. We see that

$$(1-q)^2 \equiv 1 - q^2 \pmod{2},$$

from which it follows that (10).
The others (11) and (12) can be proved in similar fashion from

$$(1-q)^4 \equiv (1-q^2)^2 \pmod{4}$$

and

$$(1-q)^8 \equiv (1-q^2)^4 \pmod{8}.$$

□

We prove the following Theorems:

Theorem 2.7. *Let* $t_1 \in \{62, 78\}$, $t_2 \in \{62, 158\}$, $t_3 \in \{166, 214\}$, $t_4 \in \{142, 238\}$, $t_5 \in \{86, 134\}$, $t_6 \in \{10, 26\}$, $t_7 \in \{28, 92, 124, 156\}$, $t_8 \in \{28, 124, 316, 412\}$, $t_9 \in \{92, 188, 284, 476\}$, $t_{10} \in \{124, 156\}$, *then for all* $\alpha, \beta, \gamma \geq 0$, *we have, modulo* 16,

$$\sum_{n=0}^{\infty} B_o\left(16 \cdot 5^{2\beta} n + 6 \cdot 5^{2\beta} - 1\right) q^n \equiv 8 f_1^9 + 8 f_4 f_5, \tag{13}$$

$$\sum_{n=0}^{\infty} B_o\left(16 \cdot 5^{2\beta+1} n + 14 \cdot 5^{2\beta+1} - 1\right) q^n \equiv 8 f_1 f_{20} + 8q f_5^9, \tag{14}$$

$$B_o\left(16\cdot 5^{2\beta+2}n+t_1\cdot 5^{2\beta+1}-1\right)\equiv 0, \tag{15}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)q^n\equiv 8f_2^3f_5^3, \tag{16}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}-1\right)q^n\equiv 8qf_1^3f_{10}^3, \tag{17}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)q^n\equiv 8f_2f_5, \tag{18}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+22\cdot 3^{2\alpha}\cdot 5^{2\beta+1}-1\right)q^n\equiv 8f_1f_{10}, \tag{19}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}-1\right)q^n\equiv 8q^2f_6^3f_{15}^3, \tag{20}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+46\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)q^n\equiv 8qf_2f_{15}^3+8f_5f_6^3, \tag{21}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+38\cdot 3^{2\alpha}\cdot 5^{2\beta+1}-1\right)q^n\equiv 8f_3^3f_{10}+8q^3f_1f_{30}^3, \tag{22}$$

$$B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+t_2\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)\equiv 0, \tag{23}$$

$$B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+t_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}-1\right)\equiv 0, \tag{24}$$

$$B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+t_4\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)\equiv 0, \tag{25}$$

$$B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+t_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}-1\right)\equiv 0, \tag{26}$$

$$B_o\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}n+t_6\cdot 3^{2\alpha+1}\cdot 5^{2\beta}-1\right)\equiv 0, \tag{27}$$

$$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 8f_1^9, \tag{28}$$

$$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}-1\right)q^n\equiv 8q^2f_7^9, \tag{29}$$

$$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}-1\right)q^n\equiv 8qf_5^9, \tag{30}$$

(31) $$B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0,$$

(32) $$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+44\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 8f_2f_3^3,$$

(33) $$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+76\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 8f_1f_6^3,$$

(34) $$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}-1\right)q^n\equiv 8q^2f_{10}f_{15}^3,$$

(35) $$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+92\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 8q^3f_5f_{30}^3,$$

(36) $$B_o\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_8\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0,$$

(37) $$B_o\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_9\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0,$$

(38) $$\sum_{n=0}^{\infty}B_o\left(32\cdot 5^{2\beta}n+28\cdot 5^{2\beta}-1\right)q^n\equiv 8f_1f_{20}+8f_2^3f_5^3,$$

(39) $$\sum_{n=0}^{\infty}B_o\left(32\cdot 5^{2\beta+1}n+12\cdot 5^{2\beta+1}-1\right)q^n\equiv 8f_4f_5+8qf_1^3f_{10}^3,$$

(40) $$B_o\left(32\cdot 5^{2\beta+1}n+t_{10}\cdot 5^{2\beta}-1\right)\equiv 0.$$

Theorem 2.8. *Let $t_{11}\in\{22,38\}$, $t_{12}\in\{34,66\}$, $t_{13}\in\{26,42,58,74\}$, $t_{14}\in\{44,76\}$, $t_{15}\in\{68,132\}$, $t_{16}\in\{52,84,116,148\}$, then for all $\alpha,\beta,\gamma\geq 0$, we have, modulo 4,*

(41) $$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_1^3,$$

(42) $$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_7^3,$$

(43) $$\begin{aligned}&B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\\ \equiv\ &\begin{cases}2 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z},\\ 0 & \textit{otherwise.}\end{cases}\end{aligned}$$

(44) $$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_3^3,$$

$$B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{45}$$

$$B_o\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+t_{11}\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{46}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_5^3, \tag{47}$$

$$B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+t_{12}\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{48}$$

$$B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{49}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_1^3, \tag{50}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_7^3, \tag{51}$$

$$\begin{aligned}&B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\\ &\equiv \begin{cases}2 & \textit{if } \ n=k(3k+1)/2 \ \textit{ for some } k\in\mathbb{Z},\\ 0 & \textit{otherwise.}\end{cases}\end{aligned} \tag{52}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_3^3, \tag{53}$$

$$B_o\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{54}$$

$$B_o\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+t_{11}\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{55}$$

$$\sum_{n=0}^{\infty}B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_5^3, \tag{56}$$

$$B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_{12}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{57}$$

$$B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+t_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{58}$$

$$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_1^3, \tag{59}$$

$$\sum_{n=0}^{\infty}B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_7^3, \tag{60}$$

$$B_o\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right) \equiv \begin{cases} 2 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z}, \\ 0 & \textit{otherwise.}\end{cases} \tag{61}$$

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_3^3, \tag{62}$$

$$B_o\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{63}$$

$$B_o\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+t_{14}\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{64}$$

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_5^3, \tag{65}$$

$$B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+t_{15}\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{66}$$

$$B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_{16}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{67}$$

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_1^3, \tag{68}$$

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}-1\right)q^n \equiv 2f_7^3, \tag{69}$$

$$B_o\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right) \equiv \begin{cases} 2 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z}, \\ 0 & \textit{otherwise.}\end{cases} \tag{70}$$

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_3^3, \tag{71}$$

$$B_o\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{72}$$

$$B_o\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+t_{14}\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{73}$$

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_5^3, \tag{74}$$

$$B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+t_{15}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{75}$$

$$B_o\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+t_{16}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0. \tag{76}$$

3. Proof of Theorem (2.7)

We have

$$\begin{aligned}\sum_{n=0}^{\infty}B_o(n)\,q^n &= \frac{f_2^2f_5^2f_{20}^2}{f_1^2f_4^2f_{10}^2}\\ &= \frac{f_2^2f_{20}^2}{f_4^2f_{10}^2}\cdot\frac{f_5^2}{f_1^2}\\ &= \frac{f_2^2f_{20}^2}{f_4^2f_{10}^2}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)^2, \end{aligned}\tag{77}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty}B_o(2n+1)\,q^n &= 2\frac{f_1^2f_{10}^2}{f_2^2f_5^2}\cdot\frac{f_4f_{10}^2}{f_1^2f_{20}}\cdot\frac{f_2^3f_5f_{20}}{f_1^3f_4f_{10}}\\ &= 2\frac{f_2f_{10}^3}{f_1^3f_5}\\ &= 2f_2f_{10}^3\cdot\frac{1}{f_1^4}\cdot\frac{f_1}{f_5}\\ &= 2f_2f_{10}^3\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)\left(\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}\right), \end{aligned}\tag{78}$$

from which we extract, for modulo 16,

$$\begin{aligned}\sum_{n=0}^{\infty}B_o(4n+1)\,q^n &= 2f_1f_5^3\left(\frac{f_2^{14}}{f_1^{14}f_4^4}\cdot\frac{f_1f_4f_{10}^3}{f_2f_5^3f_{20}}-4q\frac{f_2^2f_4^4}{f_1^{10}}\cdot\frac{f_2^2f_{20}}{f_4f_5^2}\right)\\ &= 2\frac{f_2^{13}f_{10}^3}{f_1^{12}f_4^3f_{20}}-8q\frac{f_2^4f_4^3f_5f_{20}}{f_1^9}\\ &= 2\left(\frac{f_2^4}{f_1^8}\right)\left(\frac{f_2^8}{f_4^4}\right)\frac{f_2f_4f_{10}^3}{f_1^4f_{20}}-8q\left(\frac{f_2}{f_1^2}\right)^6\left(\frac{f_4}{f_2^2}\right)f_1^3f_4^2f_5f_{20}\\ &\equiv 2\frac{f_2f_4f_{10}^3}{f_1^4f_{20}}+8qf_1^3f_4^2f_5f_{20}\\ &\equiv 2\frac{f_2f_4f_{10}^3}{f_{20}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)+8qf_4^2f_{20}\left(f_2^4+qf_2f_{10}^3\right) \end{aligned}\tag{79}$$

and

$$\sum_{n=0}^{\infty} B_o\,(4n+3)\,q^n = 2f_1f_5^3\left(4\frac{f_2^2f_4^4}{f_1^{10}}\cdot\frac{f_1f_4f_{10}^3}{f_2f_5^3f_{20}} - \frac{f_2^{14}}{f_1^{14}f_4^4}\cdot\frac{f_2^2f_{20}}{f_4f_5^2}\right)$$
$$= 8\frac{f_2f_4^5f_{10}^3}{f_1^8f_{20}} - 2\frac{f_2^{16}f_5f_{20}}{f_1^{13}f_4^5}$$
$$= 8\left(\frac{f_2}{f_1^2}\frac{f_{10}^2}{f_{20}}\frac{1}{f_1^6}\right)f_4^5f_{10} - 2\left(\frac{f_2^8}{f_4^4}\right)\left(\frac{f_2^4}{f_1^8}\right)^2\frac{f_1^3f_5f_{20}}{f_4}$$
$$\equiv 8f_2^7f_{10} - 2f_1^4.\frac{f_5}{f_1}\cdot\frac{f_{20}}{f_4}$$
$$\equiv 8f_2^7f_{10} - 2\frac{f_{20}}{f_4}\left(\frac{f_4^{10}}{f_2^2f_8^4} - 4q\frac{f_2^2f_8^4}{f_4^2}\right)\left(\frac{f_8f_{20}^2}{f_2^2f_{40}} + q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right). \tag{80}$$

From the equation (79), we extract

$$\sum_{n=0}^{\infty} B_o\,(8n+1)\,q^n \equiv 2\frac{f_1f_2f_5^3}{f_{10}}\cdot\frac{f_2^{14}}{f_1^{14}f_4^4} + 8qf_1f_2^2f_5^3f_{10} \tag{81}$$

and

$$\sum_{n=0}^{\infty} B_o\,(8n+5)\,q^n \equiv 8\frac{f_1f_2f_5^3}{f_{10}}\cdot\frac{f_2^2f_4^4}{f_1^{10}} + 8f_1^4f_2^2f_{10}$$
$$\equiv 8\frac{f_2^3f_4^4f_5^3}{f_1^9f_{10}} + 8(f_1^2)^2f_2^2f_5^2$$
$$\equiv 8\left(\frac{f_2}{f_1^2}\right)^3\left(\frac{1}{f_1^2}\right)^3\left(\frac{f_5^2}{f_{10}}\right)f_1^3f_4^4f_5 + 8f_2^4f_5^2$$
$$\equiv 8f_1^3f_2^5f_5 + 8f_8f_{10}$$
$$\equiv 8f_2^5\left(f_2^4 + qf_2f_{10}^3\right) + 8f_8f_{10}. \tag{82}$$

From the equation (82), we extract

$$\sum_{n=0}^{\infty} B_o\,(16n+5)\,q^n \equiv 8f_1^9 + 8f_4f_5 \tag{83}$$

and

$$\sum_{n=0}^{\infty} B_o\,(16n+13)\,q^n \equiv 8f_2^3f_5^3. \tag{84}$$

The congruence (83) is the case $\beta = 0$ of the congruence (13). Suppose that the congruence (13) is true for some integer $\beta \geq 0$. We have

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 5^{2\beta}n + 6\cdot 5^{2\beta} - 1\right)q^n \equiv 8f_1^9 + 8f_4f_5$$

$$\equiv 8f_{25}^9\left(R(q^5)^{-1} - q - q^2R(q^5)\right)^9 + 8f_5f_{100}\left(R(q^{20})^{-1} - q^4 - q^8R(q^{20})\right), \tag{85}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 5^{2\beta+1}n + 14\cdot 5^{2\beta+1} - 1\right)q^n \equiv 8f_1f_{20} + 8qf_5^9$$

$$\equiv 8f_{20}f_{25}\left(R(q^5)^{-1} - q - q^2R(q^5)\right) + 8qf_5^9, \tag{86}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 5^{2\beta+2}n + 6\cdot 5^{2\beta+2} - 1\right)q^n \equiv 8f_1^9 + 8f_4f_5, \tag{87}$$

which implies that the congruence (13) is true for $\beta+1$. By mathematical induction, the congruence (13) is true for all integers β.
Employing (8) in (13) and then collecting the coefficients of q^{5n+4}, we get (14).
Collecting the coefficients of q^{5n+i} for $i = 3, 4$ from (86), we obtain (15).
The equation (84) is the $\alpha = \beta = 0$ case of (16). Suppose the congruence (16) holds for $\alpha \geq 0$ with $\beta = 0$, we have

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}n + 14\cdot 3^{2\alpha} - 1\right)q^n \equiv 8f_2^3f_5^3$$

$$\equiv 8\left(f_6 + q^2f_{18}^3\right)\left(f_{15} + q^5f_{45}^3\right)$$

$$\equiv 8f_6f_{15} + 8q^2f_{15}f_{18}^3 + 8q^5f_6f_{45}^3 + 8q^7f_{18}^3f_{45}^3, \tag{88}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha+1}n + 10\cdot 3^{2\alpha+1} - 1\right)q^n \equiv 8q^2f_6^3f_{15}^3. \tag{89}$$

Collecting the coefficients of q^{3n+2} in (89), we get

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha+2}n + 14\cdot 3^{2\alpha+2} - 1\right)q^n \equiv 8f_2^3f_5^3, \tag{90}$$

which implies that the congruence (16) is true for $\alpha + 1$ with $\beta = 0$. Hence, by induction, the congruence (16) is true for any integer α with $\beta = 0$.
Suppose that the congruence (16) holds for some integers $\alpha, \beta > 0$. We have

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}n + 14\cdot 3^{2\alpha}\cdot 5^{2\beta} - 1\right)q^n \equiv 8f_2^3f_5^3$$

$$\equiv 8f_5^3f_{50}^3\left(R(q^{10})^{-1} - q^2 - q^4R(q^{10})\right)^3, \tag{91}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}-1\right)q^n \equiv 8qf_1^3f_{10}^3$$

$$\equiv 8qf_{10}^3f_{25}^3\left(R(q^5)^{-1}-q-q^2R(q^5)\right)^3. \tag{92}$$

Extracting the coefficients of q^{5n+4} in (92), we get

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1\right)q^n \equiv 8f_2^3f_5^3, \tag{93}$$

which implies that the congruence (16) is true for $\beta+1$. Hence, by induction, the congruence (16) is true for any non-negative integers $\alpha,\beta>0$.
Using (8) in (16) and then collecting the coefficients of q^{5n+1}, we arrive at (17).
Employing (6) in (16) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain respectively (18), (20) and (21).
Employing the equation (8) in the equations (18) and (21), we get respectively (19) and (22).
Using the equations (18) and (19) along with the equation (8), we obtain (23) and (24) respectively.
Using the equations (21) and (22) along with the equation (8), we obtain (25) and (26) respectively.
Collecting the coefficients of q^{3n} and q^{3n+1} in the equation (20), we get (27).
From the equation (80), we extract

$$\sum_{n=0}^{\infty} B_o(8n+3)q^n \equiv 8f_1^7f_5-2\frac{f_2^9f_{10}^3}{f_1^4f_4^3f_{20}}+8q\frac{f_4^3f_5f_{20}}{f_1} \tag{94}$$

and

$$\sum_{n=0}^{\infty} B_o(8n+7)q^n \equiv 8\frac{f_4^5f_{10}^3}{f_2^3f_{20}}-2\frac{f_2^{12}f_5f_{20}}{f_1^5f_4^5}$$

$$\equiv 8\left(\frac{f_4}{f_2^2}\right)\left(\frac{f_{10}^2}{f_{20}}\right)\frac{f_4^4f_{10}}{f_2}-2\left(\frac{f_2^4}{f_1^8}\right)\left(\frac{f_2^8}{f_4^4}\right)\frac{f_1^3f_5f_{20}}{f_4}$$

$$\equiv 8f_2^7f_{10}-2\frac{f_1^3f_5f_{20}}{f_4}. \tag{95}$$

From the equations (80) and (95), we obtain

$$B_o(8n+7) \equiv B_o(4n+3).$$

On induction, we get

$$B_o\left(2^{\alpha+2}n+2^{\alpha+2}-1\right) \equiv B_o(4n+3).$$

From the equation (94), we have

$$\sum_{n=0}^{\infty} B_o(8n+3)q^n \equiv 8\frac{f_1(f_1^2)^3f_5^3}{f_{10}}-2\left(\frac{f_2^8}{f_4^4}\right)\frac{f_2f_4f_{10}^3}{f_1^4f_{20}}+8q\frac{f_4^3f_5f_{10}^2}{f_1}$$

$$\equiv 8\frac{f_1f_2^3f_5^3}{f_{10}}-2\frac{f_2f_4f_{10}^3}{f_1^4f_{20}}+8qf_1f_2^5f_5^3f_{10}$$

$$\equiv 8\frac{f_2^3}{f_{10}}\left(f_2^3f_{10}+qf_{10}^4\right)-2\frac{f_2f_4f_{10}^3}{f_{20}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)+8qf_2^5f_{10}\left(f_2^3f_{10}+qf_{10}^4\right), \tag{96}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o(16n+3)\,q^n \equiv 8f_1^6 - 2\frac{f_2^{15}f_5^3}{f_1^{13}f_4^4f_{10}} + 8qf_1^5f_5^5 \tag{97}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} B_o(16n+11)\,q^n &\equiv 8f_1^3f_5^3 + 8\frac{f_2^3f_4^4f_5^3}{f_1^9f_{10}} + 8f_1^8f_5^2\\ &\equiv 8(f_1^2)f_1f_5^3 + 8\left(\frac{f_2}{f_1^2}\right)^3\left(\frac{f_5^2}{f_{10}}\right)f_1^3f_2^5f_5 + 8f_8f_{10}\\ &\equiv 8f_1f_2f_5^3 + 8f_1^3f_2^5f_5 + 8f_8f_{10}\\ &\equiv 8f_2\left(f_2^3f_{10}+qf_{10}^4\right) + 8f_2^5\left(f_2^4+qf_2f_{10}^3\right) + 8f_8f_{10},\end{aligned} \tag{98}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty} B_o(32n+11)\,q^n &\equiv 8f_1^4f_5 + 8f_1^9 + 8f_4f_5\\ &\equiv 8f_4f_5 + 8f_1^9 + 8f_4f_5\end{aligned}$$

$$\sum_{n=0}^{\infty} B_o(32n+11)\,q^n \equiv 8f_1^9 \tag{99}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} B_o(32n+27)\,q^n &\equiv 8f_1f_5^4 + 8f_1^6f_5^3\\ &\equiv 8f_1\left(f_5^2\right)^2 + 8\left(f_1^2\right)^3f_5^3\end{aligned}$$

$$\sum_{n=0}^{\infty} B_o(32n+27)\,q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3. \tag{100}$$

The congruence (99) is $\alpha=\beta=\gamma=0$ case of congruence (28). Suppose that the congruence (28) holds for $\alpha\geq 0$ with $\beta=\gamma=0$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}n + 12\cdot 3^{4\alpha} - 1\right)q^n &\equiv 8f_1^9\\ &\equiv 8\left(f_3+qf_9^3\right)^3,\end{aligned} \tag{101}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha+1}n + 4\cdot 3^{4\alpha+1} - 1\right)q^n &\equiv 8f_1^3 + 8qf_3^9\\ &\equiv 8\left(f_3+qf_9^3\right) + 8qf_3^9.\end{aligned} \tag{102}$$

Collecting the coefficients of q^{3n+1} in (102), we get

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha+2}n+4\cdot 3^{4\alpha+3}-1\right)q^n \equiv 8f_1^9+8f_3^3$$

$$\equiv 8\left(f_3+qf_9^3\right)^3+8f_3^3. \tag{103}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha+3}n+4\cdot 3^{4\alpha+3}-1\right)q^n \equiv 8qf_3^9, \tag{104}$$

which yields

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha+4}n+12\cdot 3^{4\alpha+4}-1\right)q^n \equiv 8f_1^9, \tag{105}$$

which implies that the congruence (28) is true for $\alpha+1$ with $\beta=\gamma=0$. Hence, by mathematical induction, the congruence (28) is true for all $\alpha\ge 0$ with $\beta=\gamma=0$.
Suppose that the congruence (28) holds for $\alpha,\beta\ge 0$ with $\gamma=0$. We have

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}-1\right)q^n \equiv 8f_1^9$$

$$\equiv 8f_{25}^9\left(R(q^5)^{-1}-q-q^2R(q^5)\right)^9, \tag{106}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+1}n+28\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1\right)q^n \equiv 8qf_5^9, \tag{107}$$

which implies

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta+2}-1\right)q^n \equiv 8f_1^9, \tag{108}$$

which implies that the congruence (28) is true for $\beta+1$ with $\gamma=0$. Hence, by induction, the congruence (28) is true for $\alpha,\beta\ge 0$ with $\gamma=0$.
Suppose that the congruence (28) holds for $\alpha,\beta,\gamma\ge 0$. We have

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n \equiv 8f_1^9$$

$$\equiv 8f_{49}^9\left(\frac{B(q^7)}{C(q^7)}-q\frac{A(q^7)}{B(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right)^9, \tag{109}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}-1\right)q^n \equiv 8q^2f_7^9, \tag{110}$$

which implies

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n \equiv 8f_1^9, \tag{111}$$

which yields that the congruence (28) is true for $\gamma+1$. Hence, by mathematical induction, the congruence (28) is true for all integers $\alpha, \beta, \gamma \geq 0$.
Using (9) in (28) and then collecting the coefficients of q^{7n+4}, we arrive at (29).
Employing (6) in (28) and then collecting the coefficients of q^{3n+1} and q^{3n+2}, we obtain (32) and (33) respectively.
Utilizing (8) in (28) and then collecting the coefficients of q^{5n+4}, we get (30).
Collecting the coefficients of q^{5n+i} for $i=0,2,3,4$ from (30), we arrive at (31).
Using (32) and (33) along with (8), we get (34) and (35) respectively.
From the congruences (34) and (35), we obtain (36) and (37) respectively.
The congruence (100) is $\beta=0$ case of congruence (38). Suppose the congruence (38) is true for $\beta \geq 0$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_o\left(32\cdot 5^{2\beta}n+28\cdot 5^{2\beta}-1\right)q^n &\equiv 8f_1f_{20}+8f_2^3f_5^3\\ &\equiv 8f_{20}f_{25}\left(R(q^5)^{-1}-q-q^2R(q^5)\right)\\ &+8f_5^3f_{50}^3\left(R(q^{10})^{-1}-q^2-q^4R(q^{10})\right)^3,\end{aligned} \tag{112}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty} B_o\left(32\cdot 5^{2\beta+1}n+12\cdot 5^{2\beta+1}-1\right)q^n &\equiv 8f_4f_5+8qf_1^3f_{10}^3\\ &\equiv 8f_5f_{100}\left(R(q^{20})^{-1}-q^4-q^8R(q^{20})\right)\\ &+8qf_{10}^3f_{25}^3\left(R(q^5)^{-1}-q-q^2R(q^5)\right)^3,\end{aligned} \tag{113}$$

which implies

$$\sum_{n=0}^{\infty} B_o\left(32\cdot 5^{2\beta+2}n+28\cdot 5^{2\beta+2}-1\right)q^n \equiv 8f_1f_{20}+8f_2^3f_5^3, \tag{114}$$

which shows that the congruence (38) is true for $\beta+1$. Hence, by induction, the congruence (38) is true for all non-negative integers β.
Employing (8) in (113) and then collecting the coefficients of q^{5n+1} from the resultant equation, we get (39).
Collecting the coefficients of q^{5n+i} for $i=3,4$ from (38) along with (8), we obtain (40).

4. Proof of Theorem (2.8)

From the equation (81), we get for modulo 4,

$$\begin{aligned}\sum_{n=0}^{\infty} B_o(8n+1)q^n &\equiv 2\frac{f_2^{15}f_5^3}{f_1^{13}f_4^4f_{10}}\\ &\equiv 2\left(\frac{f_2}{f_1^2}\right)^7\left(\frac{f_2^2}{f_4}\right)^4\frac{f_1f_5^3}{f_{10}}\\ &\equiv 2\frac{f_1f_5^3}{f_{10}}\\ &\equiv 2\frac{1}{f_{10}}\left(f_2^3f_{10}+qf_{10}^4\right), \qquad (115)\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o(16n+1)q^n \equiv 2f_1^3 \qquad (116)$$

and

$$\sum_{n=0}^{\infty} B_o(16n+9)q^n \equiv 2f_5^3. \qquad (117)$$

The equation (116) is the $\alpha=\beta=\gamma=0$ case of (41). Suppose that the congruence (41) is true for $\alpha \geq 0$ with $\beta=\gamma=0$. From (41) with $\beta=\gamma=0$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}n+2\cdot 3^{2\alpha}-1\right)q^n &\equiv 2f_1^3\\ &\equiv 2f_3+2qf_9^3, \qquad (118)\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha+1}n+2\cdot 3^{2\alpha+2}-1\right)q^n \equiv 2f_3^3, \qquad (119)$$

which implies

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha+2}n+2\cdot 3^{2\alpha+2}-1\right)q^n \equiv 2f_1^3, \qquad (120)$$

which implies that the congruence (41) is true for $\alpha+1$ with $\beta=\gamma=0$. Hence, by mathematical induction, the congruence (41) is true for all $\alpha \geq 0$. Suppose that the congruence (41) holds for $\alpha, \beta \geq 0$ with $\gamma=0$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)q^n &\equiv 2f_1^3\\ &\equiv 2f_{25}^3\left(R(q^5)^{-1}-q-q^2R(q^5)\right)^3, \qquad (121)\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1\right)q^n\equiv 2f_5^3, \tag{122}$$

which yields

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1\right)q^n\equiv 2f_1^3, \tag{123}$$

which implies that the congruence (41) is true for $\beta+1$ with $\gamma=0$. By mathematical induction, the congruence (41) is true for all non-negative integers α,β with $\gamma=0$.
Suppose that the congruence (41) holds for $\alpha,\beta,\gamma\geq 0$. We have

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_1^3$$

$$\equiv 2f_{49}^3\left(\frac{B(q^7)}{C(q^7)}-q\frac{A(q^7)}{B(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right)^3, \tag{124}$$

from which we extract

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_7^3. \tag{125}$$

The congruence (125) reduces to

$$\sum_{n=0}^{\infty} B_o\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n\equiv 2f_1^3, \tag{126}$$

which implies that the congruence (41) is true for $\gamma+1$. By mathematical induction, the congruence (41) is true for all integers α,β and γ.
Using (9) in (41) and then collecting the coefficients of q^{7n+6}, we arrive at (42).
Employing (6) in (41) and then collecting the coefficients of q^{3n},q^{3n+1} and q^{3n+2}, we obtain (43), (44) and (45) respectively.
From the congruence (44), we get (46).
Using (8) in (41) and then collecting the coefficients of q^{5n+3}, we get (47).
Collecting the coefficients of q^{5n+2} and q^{5n+4} from (41) along with (8), we obtain (48).
Collecting the coefficients of q^{5n+i} for $i=1,2,3,4$ from (47), we arrive at (49).
From the congruence (117), we get

$$\sum_{n=0}^{\infty} B_o(80n+9)\,q^n\equiv 2f_1^3, \tag{127}$$

which is the $\alpha=\beta=\gamma=0$ case of (50).
The rest of the proofs of the identities (50) - (58) are similar to the proofs of the identities (41) - (49), so we omit the details.

From the congruence (97), we obtain

$$\sum_{n=0}^{\infty} B_o(16n+3)\,q^n \equiv 2\frac{f_2^{15}f_5^3}{f_1^{13}f_4^4f_{10}}$$

$$\equiv 2\left(\frac{f_2}{f_1^2}\right)^7\left(\frac{f_2^2}{f_4}\right)^4\frac{f_1f_5^3}{f_{10}}$$

$$\equiv 2\frac{f_1f_5^3}{f_{10}}$$

(128) $$\equiv 2\frac{1}{f_{10}}\left(f_2^3f_{10}+qf_{10}^4\right);$$

from which we extract

(129) $$\sum_{n=0}^{\infty} B_o(32n+3)\,q^n \equiv 2f_1^3$$

and

(130) $$\sum_{n=0}^{\infty} B_o(32n+19)\,q^n \equiv 2f_5^3:$$

The rest of the proofs of the identities (59) - (76) are similar to the proofs of the identities (41) - (49), so we omit the details.

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