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SOME IDENTITIES OF CENTRAL FUBINI POLYNOMIALS ARISING FROM NONLINEAR DIFFERENTIAL EQUATION

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Abstract. Recently, several authors have studied central Fubini polynomials associated with central factorial numbers of the second kind. In this paper, we find the nonlinear differential equation arising from the generating function of central Fubini polynomials and we derive some new interesting identities and properties from this differential equation.

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1. Introduction

As is well known that central Fubini polynomials are defined by the generating function to be

$$\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}=\sum_{n=0}^{\infty}F_{n,c}(x)\frac{t^n}{n!},\quad (see[8]). \tag{1.1}$$

From (1.1), we have

$$\begin{aligned}1&=\Big(\sum_{l=0}^{\infty}F_{l,c}(x)\frac{t^l}{l!}\Big)\big(1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})\big)\\&=\sum_{n=0}^{\infty}F_{n,c}(x)\frac{t^n}{n!}-x\Big(\sum_{l=0}^{\infty}F_{l,c}(x)\frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty}\big(\frac{1}{2}\big)^m\frac{t^m}{m!}-\sum_{m=0}^{\infty}\big(-\frac{1}{2}\big)^m\frac{t^m}{m!}\Big)\\&=\sum_{n=0}^{\infty}F_{n,c}(x)\frac{t^n}{n!}-x\Big(\sum_{l=0}^{\infty}F_{l,c}(x)\frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty}\big(\frac{1}{2}\big)^m\big(1-(-1)^m\big)\frac{t^m}{m!}\Big)\\&=\sum_{n=0}^{\infty}F_{n,c}(x)\frac{t^n}{n!}-x\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}F_{l,c}(x)\big(\frac{1}{2}\big)^{n-l}\big(1-(-1)^{n-l}\big)\Big)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\Big(F_{n,c}(x)-x\sum_{l=0}^{n}\binom{n}{l}F_{l,c}(x)\big(\frac{1}{2}\big)^{n-l}\big(1-(-1)^{n-l}\big)\Big)\frac{t^n}{n!}.\end{aligned} \tag{1.2}$$

From (1.2), we get

$$F_{n,c}(x)-x\sum_{l=0}^{n}\binom{n}{l}F_{l,c}(x)\big(\frac{1}{2}\big)^{n-l}\big(1-(-1)^{n-l}\big)=\delta_{n,0} \tag{1.3}$$

and

$$F_{0,c}(x)=1,\quad F_{n,c}(x)=x\sum_{l=0}^{n}\binom{n}{l}F_{l,c}(x)\big(\frac{1}{2}\big)^{n-l}\big(1-(-1)^{n-l}\big),\tag{1.4}$$

where $n\in\mathbb{N}$.

Now, we define the higher-order central Fubini polynomials which are given by the generating function to be

$$\Big(\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\Big)^{r}=\sum_{n=0}^{\infty}F_{n,c}^{(r)}(x)\frac{t^n}{n!},\quad (r\in\mathbb{N}).\tag{1.5}$$

From (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}F_{n,c}^{(r)}(x)\frac{t^n}{n!}&=\Big(\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\Big)^{r}\\&=\underbrace{\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\times\cdots\times\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}}_{r-times}\\&=\Big(\sum_{l_1=0}^{\infty}F_{l_1,c}(x)\frac{t^{l_1}}{l_1!}\Big)\times\cdots\times\Big(\sum_{l_r=0}^{\infty}F_{l_r,c}(x)\frac{t^{l_r}}{l_r!}\Big)\\&=\sum_{n=0}^{\infty}\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\cdots,l_r}F_{l_1,c}(x)\cdots F_{l_r,c}(x)\frac{t^n}{n!}.\end{aligned}\tag{1.6}$$

Thus, we get

$$F_{n,c}^{(r)}(x)=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\cdots,l_r}F_{l_1,c}(x)\cdots F_{l_r,c}(x).\tag{1.7}$$

Recently, many authors have studied central Fubini polynomials. In this paper, we study nonlinear differential equations which are derived from the generating function of central Fubini polynomials. In addition, we give some new identities for the higher-order central Fubini polynomials which are related to central Fubini polynomials.

2. Some identities of central Fubini polynomials arising from nonlinear differential equation

Let

$$Y=Y(t;x)=\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}.\tag{2.1}$$

Then by taking the derivative with respect to t of (2.1), we get

$$\begin{aligned} Y^{(1)} &= \frac{\partial}{\partial t}Y(t;x) = \frac{1}{\big(1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})\big)^2}\big(\frac{x}{2}(e^{\frac{t}{2}}+e^{-\frac{t}{2}})\big) \\ &= -\frac{1}{2}\Big(\frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}\Big)^2\big(1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})-1-2xe^{-\frac{t}{2}}\big) \\ &= -\frac{1}{2}Y+\frac{1}{2}(1+2xe^{-\frac{t}{2}})Y^2. \end{aligned} \tag{2.2}$$

From (2.2), we have

$$2Y^{(1)} = -Y+(1+2xe^{-\frac{t}{2}})Y^2. \tag{2.3}$$

From (2.3), we note that

$$2Y^{(2)} = -Y^{(1)} - xe^{-\frac{t}{2}}Y^2 + 2(1+2xe^{-\frac{t}{2}})YY^{(1)}. \tag{2.4}$$

Multiply by 2 both sides of (2.4) and by (2.3), we have

$$\begin{aligned} 2^2Y^{(2)} &= Y-(1+2xe^{-\frac{t}{2}})Y^2-2xe^{-\frac{t}{2}}Y^2+2(1+2xe^{-\frac{t}{2}})Y \\ &\times(-Y+(1+2xe^{-\frac{t}{2}})Y^2) \\ &= Y-(1+2xe^{-\frac{t}{2}})Y^2-2xe^{-\frac{t}{2}}Y^2-2(1+2xe^{-\frac{t}{2}})Y^2 \\ &+2(1+2xe^{-\frac{t}{2}})^2Y^3 \\ &= Y+(-3-8xe^{-\frac{t}{2}})Y^2+(2+8xe^{-\frac{t}{2}}+8x^2e^{-t})Y^3. \end{aligned} \tag{2.5}$$

Continuing this process, we get

$$2^NY^{(N)} = \sum_{k=1}^{N+1}\sum_{j=0}^{k-1} a_{k,j}(N)x^je^{-\frac{j}{2}t}Y^k. \tag{2.6}$$

By taking the derivative with respect to t of (2.6), we get

$$\begin{aligned} 2^NY^{(N+1)} &= \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}\big(-\frac{j}{2}\big)a_{k,j}(N)x^je^{-\frac{j}{2}t}Y^k \\ &+\sum_{k=1}^{N+1}\sum_{j=0}^{k-1}ka_{k,j}(N)x^je^{-\frac{j}{2}t}Y^{k-1}Y^{(1)}. \end{aligned} \tag{2.7}$$

Multiply by 2 both sides of (2.7) and by (2.3), we have

$$
\begin{aligned}
2^{N+1}Y^{(N+1)} &= \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}(-j)a_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^k \\
&+ \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}ka_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^{k-1}(-Y+(1+2xe^{-\frac{t}{2}})Y^2) \\
&= \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}(-k-j)a_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^k \\
&+ \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}ka_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^{k+1} \\
&+ \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}2ka_{k,j}(N)x^{j+1} e^{-\frac{j+1}{2}t}Y^{k+1} \\
&= \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}(-k-j)a_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^k \\
&+ \sum_{k=2}^{N+2}\sum_{j=0}^{k-2}(k-1)a_{k-1,j}(N)x^j e^{-\frac{j}{2}t}Y^k \\
&+ \sum_{k=2}^{N+2}\sum_{j=1}^{k-1}2(k-1)a_{k-1,j-1}(N)x^j e^{-\frac{j}{2}t}Y^k \\
&= -a_{1,0}(N)Y + (N+1)a_{N+1,0}(N)Y^{N+2} + 2(N+1) \\
&\times a_{N+1,N}(N)x^{N+1}e^{-\frac{N+1}{2}t}Y^{N+2} + \sum_{j=1}^{N}\big((N+1)a_{N+1,j}(N) \\
&+ 2(N+1)a_{N+1,j-1}(N)\big)x^j e^{-\frac{j}{2}t}Y^{N+2} \\
&+ \sum_{k=2}^{N+1}\Big\{\big((-k)a_{k,0}(N) + (k-1)a_{k-1,0}(N)\big) + \big((-2k+1) \\
&\times a_{k,k-1}(N) + 2(k-1)a_{k-1,k-2}(N)\big)x^{k-1}e^{-\frac{k-1}{2}t} \\
&+ \sum_{j=1}^{k-2}\big((-k-j)a_{k,j}(N) + (k-1)a_{k-1,j}(N) \\
&+ 2(k-1)a_{k-1,j-1}(N)\big)x^j e^{-\frac{j}{2}t}\Big\}Y^k.
\end{aligned} \tag{2.8}
$$

By replacing N by $N+1$ in (2.6), we get

$$\begin{aligned}
2^{N+1}Y^{(N+1)} &= \sum_{k=1}^{N+2}\sum_{j=0}^{k-1} a_{k,j}(N+1)x^j e^{-\frac{j}{2}t}Y^k \\
&= a_{1,0}(N+1)Y + a_{N+2,0}(N+1)Y^{N+2} \\
&+ a_{N+2,N+1}(N+1)x^{N+1}e^{-\frac{N+1}{2}t}Y^{N+2} \\
&+ \sum_{j=1}^{N} a_{N+2,j}(N+1)x^j e^{-\frac{j}{2}t}Y^{N+2} \\
&+ \sum_{k=2}^{N+1}\big(a_{k,0}(N+1) + a_{k,k-1}(N+1)x^{k-1}e^{-\frac{k-1}{2}t} \\
&+ \sum_{j=1}^{k-2} a_{k,j}(N+1)x^j e^{-\frac{j}{2}t}\big)Y^k.
\end{aligned} \tag{2.9}$$

Comparing the coefficients on the both sides of (2.8) and (2.9), we get

$$a_{1,0}(N+1) = -a_{1,0}(N), \tag{2.10}$$

$$a_{N+2,0}(N+1) = (N+1)a_{N+1,0}(N), \tag{2.11}$$

$$a_{N+2,j}(N+1) = (N+1)a_{N+1,j}(N) + 2(N+1)a_{N+1,j-1}(N), \tag{2.12}$$

where $1 \le j \le N$.

$$a_{N+2,N+1}(N+1) = 2(N+1)a_{N+1,N}(N), \tag{2.13}$$

$$a_{k,0}(N+1) = -ka_{k,0}(N) + (k-1)a_{k-1,0}(N), \tag{2.14}$$

where $2 \le k \le N+1$.

$$\begin{aligned}
a_{k,j}(N+1) &= (-k-j)a_{k,j}(N) + (k-1)a_{k-1,j}(N) \\
&+ 2(k-1)a_{k-1,j-1}(N),
\end{aligned} \tag{2.15}$$

where $2 \le k \le N+1$ and $1 \le j \le k-2$.

$$a_{k,k-1}(N+1) = (-2k+1)a_{k,k-1}(N) + 2(k-1)a_{k-1,k-2}(N), \tag{2.16}$$

where $2 \le k \le N+1$.

From (2.3) and (2.6), we get

$$\begin{aligned}
2Y^{(1)} &= \sum_{k=1}^{2}\sum_{j=0}^{k-1} a_{k,j}(1)x^j e^{-\frac{j}{2}t}Y^k \\
&= a_{1,0}(1)Y + (a_{2,0}(1) + a_{2,1}(1)xe^{-\frac{t}{2}})Y^2 \\
&= -Y + (1 + 2xe^{-\frac{t}{2}})Y^2.
\end{aligned} \tag{2.17}$$

By (2.17), we have

$$a_{1,0}(1) = -1, \quad a_{2,0}(1) = 1 \quad \textit{and} \quad a_{2,1}(1) = 2. \tag{2.18}$$

From (2.10),(2.11),(2.13) and (2.18), we get

$$\begin{aligned} a_{1,0}(N+1) &= -a_{1,0}(N) = (-1)^2 a_{1,0}(N-1) = \cdots = (-1)^N a_{1,0}(1) \\ &= (-1)^{N+1}. \end{aligned} \tag{2.19}$$

$$\begin{aligned} a_{N+2,0}(N+1) &= (N+1)a_{N+1,0}(N) = (N+1)_2 a_{N,0}(N-1) = \cdots \\ &= (N+1)_N a_{2,0}(1) = (N+1)!. \end{aligned} \tag{2.20}$$

$$\begin{aligned} a_{N+2,N+1}(N+1) &= 2(N+1)a_{N+1,N}(N) = 2^2(N+1)_2 a_{N,N-1}(N-1) \\ &= \cdots = 2^N (N+1)_N a_{2,1}(1) = 2^{N+1}(N+1)!, \end{aligned} \tag{2.21}$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$.
Therefore, we obtain the following theorem.

Theorem 2.1. *For $N \in \mathbb{N}$, the following differential equation*

$$2^N Y^{(N)} = \sum_{k=1}^{N+1} \sum_{j=0}^{k-1} a_{k,j}(N) x^j e^{-\frac{j}{2}t} Y^k \tag{2.22}$$

have a solution $Y = Y(t;x) = \frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}$, where

$$a_{1,0}(N) = (-1)^N$$

$$a_{N+1,0}(N) = N!$$

$$a_{N+1,j}(N) = Na_{N,j}(N-1) + 2Na_{N,j-1}(N-1)$$

where $1 \leq j \leq N-1$

$$a_{N+1,N}(N) = 2^N N!$$

$$a_{k,0}(N) = -ka_{k,0}(N-1) + (k-1)a_{k-1,0}(N-1)$$

where $2 \leq k \leq N$

$$\begin{aligned} a_{k,j}(N) &= (-k-j)a_{k,j}(N-1) + (k-1)a_{k-1,j}(N-1) \\ &\quad + 2(k-1)a_{k-1,j-1}(N-1) \end{aligned}$$

where $2 \leq k \leq N$ and $1 \leq j \leq k-2$

$$a_{k,k-1}(N) = (-2k+1)a_{k,k-1}(N-1) + 2(k-1)a_{k-1,k-2}(N-1)$$

where $2 \leq k \leq N$.

By (1.1), we have

$$\begin{aligned} 2^N Y^{(N)} &= 2^N \frac{\partial^N}{\partial t^N} Y(t;x) = 2^N \frac{\partial^N}{\partial t^N} \sum_{n=0}^{\infty} F_{n,c}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} 2^N F_{n+N,c}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.23}$$

Also, from (1.5) and (2.6), we get

$$\begin{aligned}\sum_{k=1}^{N+1}\sum_{j=0}^{k-1} a_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^k &= \sum_{k=1}^{N+1}\sum_{j=0}^{k-1} a_{k,j}(N)x^j\Big(\sum_{m=0}^{\infty}\big(-\frac{j}{2}\big)^m\frac{t^m}{m!}\Big)\\ &\times\Big(\sum_{l=0}^{\infty}F_{l,c}^{(k)}(x)\frac{t^l}{l!}\Big)\\ &= \sum_{k=1}^{N+1}\sum_{j=0}^{k-1} a_{k,j}(N)x^j\sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}F_{l,c}^{(k)}(x)\\ &\times\big(-\frac{j}{2}\big)^{n-l}\Big)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{N+1}\sum_{j=0}^{k-1}\sum_{l=0}^{n}\binom{n}{l}a_{k,j}(N)x^jF_{l,c}^{(k)}(x)\\ &\times\big(-\frac{j}{2}\big)^{n-l}\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.24}$$

Therefore, by (2.23) and (2.24), we get the following theorem.

Theorem 2.2. *For $N\in\mathbb{N}$, $n\in\mathbb{N}\cup\{0\}$, we have*

$$2^N F_{n+N,c}(x) = \sum_{k=1}^{N+1}\sum_{j=0}^{k-1}\sum_{l=0}^{n}\binom{n}{l}a_{k,j}(N)x^jF_{l,c}^{(k)}(x)\big(-\frac{j}{2}\big)^{n-l},$$

Now, we consider the inversion formula of Theorem 1. From (2.3), we get

$$(1+2xe^{-\frac{t}{2}})Y^2 = Y + 2Y^{(1)}. \tag{2.25}$$

Then by taking the derivative with respect to t of (2.25), we get

$$-xe^{-\frac{t}{2}}Y^2 + 2(1+2xe^{-\frac{t}{2}})YY^{(1)} = Y^{(1)} + 2Y^{(2)}. \tag{2.26}$$

Multiply by $2(1+2xe^{-\frac{t}{2}})$ both sides of (2.26) and by (2.25), we have

$$\begin{aligned}2!(1+2xe^{-\frac{t}{2}})^3 &= (2+6xe^{-\frac{t}{2}})Y + (6+16xe^{-\frac{t}{2}})Y^{(1)}\\ &+ (4+8xe^{-\frac{t}{2}})Y^{(2)}.\end{aligned} \tag{2.27}$$

Also, by (2.25) and (2.27), we get

$$\begin{aligned}3!(1+2xe^{-\frac{t}{2}})^5Y^4 &= (6+36xe^{-\frac{t}{2}}+60x^2e^{-t})Y\\ &+ (22+124xe^{-\frac{t}{2}}+184x^2e^{-t})Y^{(1)}\\ &+ (24+120xe^{-\frac{t}{2}}+144x^2e^{-t})Y^{(2)}\\ &+ (8+56xe^{-\frac{t}{2}}+32x^2e^{-t})Y^{(3)}.\end{aligned} \tag{2.28}$$

Continuing this process, we get

$$N!(1+2xe^{-\frac{t}{2}})^{2N-1}Y^{N+1} = \sum_{k=0}^{N}\sum_{j=0}^{N-1} b_{k,j}(N)x^je^{-\frac{j}{2}t}Y^{(k)}. \tag{2.29}$$

Also, if we repeat the above process, we can get the following theorem.

Theorem 2.3. *For $N \in \mathbb{N}$, the following differential equation*

$$N!(1+2xe^{-\frac{t}{2}})^{2N-1}Y^{N+1} = \sum_{k=0}^{N}\sum_{j=0}^{N-1} b_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^{(k)}$$

have a solution $Y = Y(t;x) = \frac{1}{1-x(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}$, where

$$b_{0,0}(N) = N!$$

$$b_{0,j}(N) = (N-j)b_{0,j}(N-1) + (6N-2j-4)b_{0,j-1}(N-1)$$

where $1 \le j \le N-2$

$$b_{0,N-1}(N) = \frac{(2N-1)!}{(N-1)!}$$

$$b_{N,0}(N) = 2^N$$

$$b_{N,j}(N) = 2b_{N-1,j}(N-1) + 4b_{N-1,j-1}(N-1)$$

where $1 \le j \le N-2$

$$b_{N,N-1}(N) = 2^{2N-1}$$

$$b_{k,0}(N) = Nb_{k,0}(N-1) + 2b_{k-1,0}(N-1)$$

where $1 \le k \le N-1$

$$\begin{aligned} b_{k,j}(N) &= (N-j)b_{k,j}(N-1) + (6N-2j-4)b_{k,j-1}(N-1) \\ &+ 2b_{k-1,j}(N-1) + 4b_{k-1,j-1}(N-1) \end{aligned}$$

where $1 \le k \le N-1$ and $1 \le j \le N-2$

$$b_{k,N-1}(N) = (4N-2)b_{k,N-2}(N-1) + 4b_{k-1,N-2}(N-1)$$

where $1 \le k \le N-1$.

From (1.5) and (2.29), we have

$$\begin{aligned} &N!(1+2xe^{-\frac{t}{2}})^{2N-1}Y^{N+1} \\ &= N!\Big(\sum_{l=0}^{\infty}\binom{2N-1}{l}2^l x^l e^{-\frac{l}{2}t}\Big)\Big(\sum_{m=0}^{\infty}F_{m,c}^{(N+1)}(x)\frac{t^m}{m!}\Big) \\ &= N!\Big(\sum_{l=0}^{\infty}\binom{2N-1}{l}2^l x^l \sum_{i=0}^{\infty}(-1)^i l^i 2^{-i}\frac{t^i}{i!}\Big)\Big(\sum_{m=0}^{\infty}F_{m,c}^{(N+1)}(x)\frac{t^m}{m!}\Big) \\ &= N!\Big(\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\binom{2N-1}{l}2^{l-i}x^l(-1)^i l^i\frac{t^i}{i!}\Big)\Big(\sum_{m=0}^{\infty}F_{m,c}^{(N+1)}(x)\frac{t^m}{m!}\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\sum_{l=0}^{\infty}\binom{n}{m}\binom{2N-1}{l}N!2^{l-n+m}(-1)^{n-m}x^l l^{n-m} \\ &\times F_{m,c}^{(N+1)}(x)\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.30}$$

Also, (2.23) and (2.29), we get

$$\begin{aligned}
&\sum_{k=0}^{N}\sum_{j=0}^{N-1} b_{k,j}(N)x^j e^{-\frac{j}{2}t}Y^{(k)} \\
&= \sum_{k=0}^{N}\sum_{j=0}^{N-1} b_{k,j}(N)x^j \Big(\sum_{m=0}^{\infty}\big(-\frac{j}{2}\big)^m \frac{t^m}{m!}\Big)\Big(\sum_{i=0}^{\infty} F_{i+k,c}(x)\frac{t^i}{i!}\Big) \\
&= \sum_{k=0}^{N}\sum_{j=0}^{N-1} b_{k,j}(N)x^j \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}F_{i+k,c}(x)\big(-\frac{j}{2}\big)^{n-i}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{N}\sum_{j=0}^{N-1}\sum_{i=0}^{n}\binom{n}{i} b_{k,j}(N)x^j F_{i+k,c}(x)\big(-\frac{j}{2}\big)^{n-i}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{2.31}$$

From (2.30) and (2.31), we get the following theorem.

Theorem 2.4. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned}
&\sum_{m=0}^{n}\sum_{l=0}^{\infty}\binom{n}{m}\binom{2N-1}{l} N! 2^{l-n+m}(-1)^{n-m}x^l l^{n-m} F_{m,c}^{(N+1)}(x) \\
&\qquad = \sum_{k=0}^{N}\sum_{j=0}^{N-1}\sum_{i=0}^{n}\binom{n}{i} b_{k,j}(N)x^j F_{i+k,c}(x)\big(-\frac{j}{2}\big)^{n-i}.
\end{aligned}$$

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