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A CRITERION FOR THE CONTINUITY WITH RESPECT TO THE ORIGINAL GROUP TOPOLOGY OF THE RESTRICTION TO THE COMMUTATOR SUBGROUP FOR A LOCALLY BOUNDED FINITE-DIMENSIONAL REPRESENTATION OF A CONNECTED LIE GROUP

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Abstract. We obtain a criterion for the continuity, with respect to the original topology of the group, of the restriction, of a locally bounded finite-dimensional representation of a connected Lie group, to the commutator subgroup of the group.

§ 1. Introduction

Continuity conditions for locally bounded finite-dimensional representations of connected Lie groups were found in [1, 2]. The specific feature of these conditions is that a locally bounded finite-dimensional representation of a connected Lie group is continuous with respect to the ordinary topology on the radical part of the commutator subgroup of the group, but the restriction to any Levi subgroup need not be continuous with respect to this topology; in general, we can claim only the continuity of this restriction with respect to the intrinsic Lie topology on the group. A natural problem is to find out a

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criterion for the continuity of the restriction to the commutator subgroup of a locally bounded finite-dimensional representation of a connected Lie group with respect to the ordinary group topology. This problem is discussed in the present paper.

§ 2. Preliminaries

Let us recall an important example clarifying the problem. We use the well-known example of a reductive Lie group with nonclosed Levi subgroup (see, e.g., p. 256 in [3]). Let G_0 be the universal covering group of the Lie group $\mathrm{SL}(2,\mathbb{R})$ and let Z be the center of G_0, which is isomorphic to the additive group $\mathbb{Z}$ of integers. There is a discrete subgroup D of $\mathbb{R}\times Z$ such that the quotient group $(\mathbb{R}\times Z)/D$ is isomorphic to the one-dimensional torus $\mathbb{T}$ and the image of the subgroup $\{0\}\times Z$ is dense in $\mathbb{T}$ (for example, for D one can take the kernel of a homomorphism of the form

$$(t,\nu)\mapsto \exp(2\pi i(t+\alpha\overline{\nu})), \qquad t\in\mathbb{R}, \quad \nu\in Z,$$

taking the product $\mathbb{R}\times Z$ onto $\mathbb{T}$, where α is irrational and the mapping $\nu\mapsto\overline{\nu}$ is an isomorphism of Z onto $\mathbb{Z}$). For the group G one can take the quotient group $G=(\mathbb{R}\times G_0)/D$.

Example 1 (see, e.g., [2]). Let us consider the following two-dimensional locally bounded representation of the group G:

$$\pi_2(g)=\pi_1((t,\widetilde{g_0})D)=\theta(g_0)\in\mathrm{SL}(2,\mathbb{R}), \quad g_0\in G_0, \quad t\in\mathbb{R}, \quad g=(t,\widetilde{g_0})D,$$

where θ stands for the canonical covering mapping $\theta\colon G_0\to\mathrm{SL}(2,\mathbb{R})$ (say, the quotient mapping by the even part of the center Z_{G_0} of G_0). The intersection of the radical R with the commutator subgroup S of G (the image of G_0 in G, which is the Levi subgroup $S\subset G$ in this case) is the center of G, and π_2 takes the center to $\pm 1_2$ (1_2 stands for the identity matrix in $\mathrm{SL}(2,\mathbb{R})$) because the center of $\mathrm{SL}(2,\mathbb{R})$ is $\pm 1_2$, and, since the center of S is dense in R and, in particular, the odd powers of the generating element of the center are dense in the radical R, it follows that the restriction of π_2 to R is discontinuous, π_2 is discontinuous on the closure of the center of S (which coincides with R).

Therefore, π_2 is an example of a two-dimensional locally bounded representation of G which is discontinuous on the commutator subgroup of G in the topology induced by the original topology of G.

Thus, the desired continuity conditions need additional assumptions.

§ 3. Main results

Theorem 1. *Let G be a connected Lie group, let R be the radical of G, let L be a Levi subgroup of G, and let π be a finite-dimensional locally bounded representation of G in a space E. The following conditions are equivalent*:

(1) *the restriction of π to L is continuous with respect to the topology on L induced by the original topology of the group G*;

(2) *either the center of the Levi subgroup L is discrete or the center Z of L is nondiscrete and $\pi(z) = 1_E$ for every $z \in Z$ (1_E stands for the identity operator in the space E).*

Proof. It is clear that (1) $\Longrightarrow$ (2). Indeed, let Z be nondiscrete. Note that the π-image $\pi(Z)$ of Z is finite as the center of the linear analytic group $\pi(L)$ (see Exercise 40 to Chap. 3 of [3]). Therefore, some subgroup of Z having finite index in Z is taken by‘π to 1_E. This subgroup is dense in Z in the topology induced by the original topology of the group, and hence $\pi(Z)$=1.

Let us prove now that (2) $\Longrightarrow$ (1). In both the cases of the assumption, the kernel K of the representation $\pi|_L$ is closed in the topology induced by the original topology of the group G. Hence, the representation $\pi|_L$ admits a factorization through the quotient group L/K. Since the Lie group topology on a perfect Lie group is kept by the bounded structure [4], it follows that the mapping of L/K to (onto) $\pi(L)$ defined by π is a topological isomorphism, which is automatically continuous with respect to the topologies of both the groups, and hence the composite mapping $\pi|_L$ is also continuous with respect to the topology induced by the original topology of G, which completes the proof of the theorem.

§ 4. Comments

Theorem 1 has an obvious corollary, which we state in the form of a theorem.

Theorem 2. *Let G be a connected Lie group, let R be the radical of G, let L be a Levi subgroup of G, and let π be a finite-dimensional locally bounded representation of G in a space E. The following conditions are equivalent*:

(1) *the restriction of π to the commutator subgroup G' of G is continuous with respect to the topology on G' induced by the original topology of the group G*;

(2) *either the center of the group G is discrete or the center Z of G is nondiscrete and $\pi(z) = 1_E$ for every $z \in Z$ (1_E stands for the identity*

operator in the space E).

Proof. It is clear that (1) $\Longrightarrow$ (2), see the proof of Theorem 1. Let us prove now that (2) $\Longrightarrow$ (1). By Theorem 1, the restriction of π to L is continuous. On the other hand, as is known, the restriction of π to $G' \cap R$ is continuous with respect to the topology induced by the original topology of G (see, e.g., [6]). Therefore, the restriction of π to the whole G' is continuous (because G' is generated by L and $G' \cap R$). This completes the proof.

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