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Application of Quasi-f-Power Increasing Sequence in $|C, \alpha, \gamma; \delta|_k$ Summability

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Abstract. An increasing quasi-power sequence of a wider class has been used to establish a universal theorem on a least set of conditions, which is sufficient for an infinite series to be generalized absolute $|C, \alpha, \gamma; \delta|_k$ summable. Further, a set of new and well-known arbitrary results have been obtained by using the main theorem. Considering suitable conditions a previous result has been obtained, which validates the current findings. In this way, the Bounded Input Bounded Output (BIBO) stability of impulse response has been improved by absolute summability because being absolute summable is the necessary and sufficient condition for BIBO stability. Also, summability plays an important role in signal processing as a digital filter in finite impulse response (FIR).

1 Introduction, definitions and preliminaries

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and is said to be absolute summable, if

$$\lim_{n\to\infty} t_n = s \tag{1}$$

and

$$\sum_{n=1}^{\infty} |\, t_n - t_{n-1}| < \infty, \tag{2}$$

where t_n represent the n^{th} mean (sequence to sequence transformation) of sequence $\{s_n\}$.

Let τ_n represent the n^{th} $(C, 1)$ means of (na_n), then the infinite series $\sum_{n=0}^{\infty} a_n$ is $|C, 1|_k$ summable for $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n} |\tau_n|^k < \infty. \tag{3}$$

If u_n^{α} and τ_n^{α} represent the n^{th} Cesáro means [12] of order $\alpha, \alpha > -1$ of (s_n) and (na_n), respectively, *i.e.*

$$u_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{4}$$

and

$$\tau_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{5}$$

where

$$A_n^\alpha = \begin{cases} O(n^\alpha), & n > 0,\\ 1, & n = 0,\\ 0, & n < 0. \end{cases} \tag{6}$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be $|C,\alpha|_k$ summable for $k \geq 1$ and $\alpha > -1$ [13], if

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty}\frac{1}{n}|\tau_n^\alpha|^k < \infty. \tag{7}$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be $|C,\alpha;\delta|_k$ summable for $\alpha > -1$, $\delta \geq 0$ and $k \geq 1$ [14], if

$$\sum_{n=1}^{\infty} n^{(\delta k+k-1)}|u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\delta k-1}|\tau_n^\alpha|^k < \infty \tag{8}$$

and it is said to be $|C,\alpha,\gamma;\delta|_k$ summable for $k \geq 1$, $\alpha > -1$, $\delta \geq 0$ and γ is real number [30], if

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)}|u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k}|\tau_n^\alpha|^k < \infty. \tag{9}$$

For the sequence $\{\tau_n^\alpha\}$ which is n^{th} Cesáro means of $\{na_n\}$, we take the following sequence ([17])

$$w_n^\alpha = \begin{cases} |\tau_n^\alpha|, & \alpha = 1,\\ \max_{1\leq v\leq n}|\tau_v^\alpha|, & 0 < \alpha < 1. \end{cases}$$

Bor [4, 5, 10, 11] worked out numerous theorems on Cesáro summability by using almost all factors. In 2008, he [6] used almost increasing sequence for establishing a theorem on $|C,\alpha,\gamma;\beta|_k$ summable factor. Yu et. al [31] introduced a new class of matrices and established a general summability factor theorem by using absolute $|A,\alpha_n|_k$ summability. Özarslan [15] modified the result of Bor [7] by adopting generalized Cesáro summability. Sulaiman [27, 28], Sonker & Munjal [18–24], Sonker et al. [25, 26], H. S. Özarslan [29], H. S. Ozarslan & S. Yildiz [16] and S. Yildiz [32, 33] have also worked out some interesting theorems on absolute summability factor. In this paper, an advanced study has been carried out in line with previous studies on absolute summability. The presented theorem has not only generalized the previous result but also increased its applications in the engineering and sciences.

The engineers working with electrical systems often use the concepts of summability theory and have noticed it's value in the stability of the system. For the moving average system, if the impulse response of the system satisfy the condition of absolute summable, then it's frequency response (FR) exists and infinite series will uniformly converge to a continuous function. For a stable sequence, which is absolute summable, it is customary to have Fourier transforms. Hence, in this way the frequency response of a stable system can be made continuous and finite. So the frequency extenuation highlights the smoothness of the input sequence of the system. Also, the system is representing the rough approximation to a low pass filter. This proves the significance of summability in signal processing as a digital filter in finite impulse response (FIR).

2 Known results

Bor [2] proved the following theorem using absolute Cesáro summable.

Theorem 1. *Let (A_n) be a ξ-quasi-monotone sequence and (X_n) be an increasing quasi-f-power sequence $(for\ some\ \eta\,(0<\eta<1))$ satisfying the conditions:*

$$\sum n\xi_n X_n < \infty, \tag{10}$$

$$\Delta A_n \leq \xi_n, \tag{11}$$

$$|\Delta\lambda_n| \leq |A_n|, \tag{12}$$

$$\sum A_n X_n \ is\ \ convergent\ \ for\ all\ n. \tag{13}$$

If the conditions

$$|\lambda_n| X_n = O(1)\ as\ n \to \infty, \tag{14}$$

and

$$\sum_{n=1}^{m} \frac{(w_n^\alpha)^k}{n} = O(X_m)\ as\ m \to \infty \tag{15}$$

are also satisfied, then the series $\sum a_n\lambda_n$ is $|C,\alpha|_k$ summable for $0<\alpha\leq 1$ and $k \geq 1$.

3 Main results

Cesáro summability $|C,\alpha,\gamma;\delta|_k$ is applied to modernize the previous finding of Bor [2] and to establish the new result.

Theorem 2. *Let (A_n) be a ξ-quasi-monotone sequence and (X_n) be an increasing quasi-f-power sequence $(for\ some\ \eta\,(0<\eta<1))$ satisfying the conditions:*

$$\sum n\xi_n X_n < \infty, \tag{16}$$

$$\Delta A_n \leq \xi_n, \tag{17}$$

$$|\Delta\lambda_n| \leq |A_n|, \tag{18}$$

$$\sum A_n X_n \ is \ \ convergent \ \ for \ all \ n. \tag{19}$$

If the conditions

$$|\lambda_n| X_n = O(1) \ as \ n \to \infty, \tag{20}$$

and

$$\sum_{n=1}^{m} \frac{(w_n^\alpha)^k}{n^{k-\gamma(\delta k+k-1)}} = O(X_m) \ as \ m \to \infty \tag{21}$$

are also satisfied, then the infinite series $\sum a_n \lambda_n$ *is* $|C, \alpha, \gamma; \delta|_k$ *summable for* $0 \leq \delta < \alpha \leq 1$, γ *is a real number,* $k \geq 1$ *and* $k(1+\alpha) - \gamma(\delta k + k - 1) > 1$.

The following Lemmas are required to prove the theorem.

Lemma 1. *[1] If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{22}$$

Lemma 2. *[3] Let* (A_n) *be a* ξ*-quasi-monotone sequence and* (X_n) *be an increasing quasi-*f*-power sequence* $(for \ some \ \eta \, (0 < \eta < 1))$ *with* $\Delta A_n \leq \xi_n$ *and* $\sum n \, \xi_n \, X_n < \infty$, *then*

$$\sum_{n=1}^{\infty} n X_n |\Delta A_n| < \infty, \tag{23}$$

$$n A_n X_n = O(1) \ as \ n \to \infty. \tag{24}$$

4 Proof of Theorem 2

Let T_n^α be the n^{th} (C, α) mean of the sequence $(n a_n \lambda_n)$. The series is $|C, \alpha, \gamma; \delta|_k$ summable, if

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} |T_n^\alpha|^k < \infty. \tag{25}$$

Applying Able's transformation and Lemma 1, we have

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v \\ &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \end{aligned} \tag{26}$$

$$
\begin{aligned}
|T_n^{\alpha}| &\leq \frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\right| + \frac{|\lambda_n|}{A_n^{\alpha}}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\right| \\
&\leq \frac{1}{A_n^{\alpha}}\sum_{v=1}^{n-1}A_v^{\alpha}w_v^{\alpha}|\Delta\lambda_v| + |\lambda_n|w_n^{\alpha} \\
&= T_{n,1}^{\alpha} + T_{n,2}^{\alpha}. \qquad (27)
\end{aligned}
$$

Using Minkowski's inequality,

$$
|T_n^{\alpha}|^k = |T_{n,1}^{\alpha} + T_{n,2}^{\alpha}|^k \leq 2^k\Big(|T_{n,1}^{\alpha}|^k + |T_{n,2}^{\alpha}|^k\Big). \qquad (28)
$$

It is sufficient to show that

$$
\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k}|T_{n,r}^{\alpha}|^k < \infty, \;\; for\ r = 1, 2, 3. \qquad (29)
$$

By using Hölder's inequality, Abel's transformation and conditions of Lemmas, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{\;\gamma(\delta k+k-1)-k}|T_{n,1}^{\alpha}|^k &\leq \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k}\frac{1}{(A_n^{\alpha})^k}\left(\sum_{v=1}^{n-1}A_v^{\alpha}w_v^{\alpha}|\Delta\lambda_v|\right)^k \\
&\leq \sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k(1+\alpha)}\left(\sum_{v=1}^{n-1}v^{\alpha k}(w_v^{\alpha})^k|A_v|\right) \times \left(\sum_{v=1}^{n-1}|A_v|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(w_v^{\alpha})^k|A_v|\sum_{n=v+1}^{m+1}\frac{1}{n^{k(1+\alpha)-\gamma(\delta k+k-1)}} \\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(w_v^{\alpha})^k|A_v|\int_{v}^{\infty}\frac{dx}{x^{k(1+\alpha)-\gamma(\delta k+k-1)}} \\
&= O(1)\sum_{v=1}^{m}v|A_v|(w_v^{\alpha})^k v^{\gamma(\delta k+k-1)-k} \\
&= O(1)m|A_m|\sum_{v=1}^{m}(w_v^{\alpha})^k v^{\gamma(\delta k+k-1)-k} \\
&\quad +O(1)\sum_{v=1}^{m-1}\Delta(v|A_v|)\sum_{r=1}^{v}(w_r^{\alpha})^k r^{\gamma(\delta k+k-1)-k} \\
&= O(1)m|A_m|X_m + O(1)\sum_{v=1}^{m-1}\Big|(v+1)\Delta|A_v| + |A_v|\Big|X_v \\
&= O(1)m|A_m|X_m + O(1)\sum_{v=1}^{m-1}v|\Delta A_v|X_v + O(1)\sum_{v=1}^{m-1}|A_v|X_v \\
&= O(1)\ as\ \ m \to \infty, \qquad (30)
\end{aligned}
$$

$$\sum_{n=2}^{m} n^{\ \gamma(\delta k+k-1)-k}|T_{n,2}^{\alpha}|^k = O(1)\sum_{n=1}^{m}|\lambda_n|(w_n^{\alpha})^k n^{\gamma(\delta k+k-1)-k}$$
$$= O(1)|\lambda_m|\sum_{n=1}^{m}(w_n^{\alpha})^k n^{\gamma(\delta k+k-1)-k} + O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}(w_v^{\alpha})^k v^{\gamma(\delta k+k-1)-k}$$
$$= O(1)|\lambda_m|X_m + O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n$$
$$= O(1)|\lambda_m|X_m + O(1)\sum_{n=1}^{m-1}|A_n|X_n$$
$$= O(1) \text{ as } m\to\infty. \tag{31}$$

Collecting (25) - (31), we have

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k}|T_n^{\alpha}|^k < \infty. \tag{32}$$

Hence proof of the theorem is completed.

5 Special cases

Corollary 1. *Let (A_n) be a ξ-quasi-monotone sequence and (X_n) be an increasing quasi-f-power sequence $(for\ some\ \eta\,(0<\eta<1))$ satisfying (16)-(20) and the following condition:*

$$\sum_{n=1}^{m}\frac{(w_n^{\alpha})^k}{n^{1-\delta k}} = O(X_m) \text{ as } m\to\infty \tag{33}$$

then the series $\sum a_n\lambda_n$ is $|C,\alpha;\delta|_k$ summable for $0\le\delta<\alpha\le 1,\ k\ge 1$.

Remark: On putting $\gamma = 1$ in Theorem 2, we will get (33) instead of (21).

Corollary 2. **[2]** *Let (A_n) be a ξ-quasi-monotone sequence and (X_n) be an increasing quasi-f-power sequence $(for\ some\ \eta\,(0<\eta<1))$ satisfying (16)-(20) and the following condition:*

$$\sum_{n=1}^{m}\frac{(w_n^{\alpha})^k}{n} = O(X_m) \text{ as } m\to\infty \tag{34}$$

then the series $\sum a_n\lambda_n$ is $|C,\alpha|_k$ summable for $0<\alpha\le 1,\quad k\ge 1$.

Remark: On putting $\gamma = 1$ and $\delta = 0$ in Theorem 2, we will get (34) instead of (21).

6 Conclusion

This study is based on a generalized absolute Cesáro $|C, \alpha, \gamma; \delta|_k$ summability factor. The absolute Cesáro $|C, \alpha, \gamma; \delta|_k$ summability is applicable in signal processing as a digital filter in infinite impulse and finite impulse response. The derived theorem is a generalized version which can be reduced to well known summabilities. If we take (X_n) as an almost increasing sequence such that $|\Delta X_n| = O\left(\frac{X_n}{n}\right)$ in Theorem 2, then we get a result for $|C, \alpha, \gamma; \delta|_k$ summability (see [8]). Also, if we take (X_n) as an almost increasing sequence such that $|\Delta X_n| = O\left(\frac{X_n}{n}\right)$ in Corollary 1, then we get a result for $|C, \alpha; \delta|_k$ summability (see [9]). In this case, condition "$\Delta A_n \leq \xi_n$" is not needed. For the validation of the presented theorem, we have assumed special values $\gamma = 1$ and $\delta = 0$ for getting Corollary 2 (Bor, [2]).

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