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The approximate solution of higher-order of fractional differential equations using collocation method based on Chebyshev polynomials of third kind

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Abstract

In this paper, the collocation method for the approximate solution of higher order multi-term fractional differential equations (FDEs) in terms of Chebyshev polynomials. The multi-term FDEs and its initial or boundary conditions are transformed into equations, which correspond to a system of algebraic equations with unknown Chebyshev coefficients. The solution of this system yields the Chebyshev coefficients of the solution formula. Several numerical examples, such as Cauchy and Bagley-Torvik fractional differential equations are provided to confirm the accuracy and the effectiveness of the proposed method. Numerical simulation with the exact solution of FDEs is presented. The graph and approximate solution are calculated in MATLAB.

Key words: Chebyshev polynomials of third kind, Collocation method, Caputo fractional derivative, Convergence analysis

1 Introduction

Fractional calculus, the theory of differentiation and integration to non-integer order, is very useful for the description of various physical phenomena, such as damping laws, diffusion process, etc. Now a days, the area of fractional differential equations (FDEs) can be model successfully in many fields such as engineering, biology, chemistry and physical science. For this reason many researchers are attracted in knowing the properties of fractional differential equations [1-3]. The primary idea of physical interest of fractional differential equations is more important. In fact, it is difficult to find the closed form solution of FDEs. However, numerical approximation scheme [4,5] must be used for reduce the problems. Recently important various scheme have been investigated for the approximate and numerical solution of FDEs. For example, the homotopy-perturbation method (HPM) [6], the variational iteration method (VIM) [7], Adomian decomposition method (ADM) [8,9] and the homotopy analysis method [10] (HAM).

We consider the fractional differential equation in the following form

$$\left[a_0 D^2 + a_1 D^{\mu} + a_2 D\right] u(x) = p(x),\ x \in [0, L], \tag{1}$$

subject to boundary conditions

$$u(0) = \delta_1, \;\; u(L) = \delta_2, \tag{2}$$

where $a_0, a_1, a_2, \delta_1, \delta_2$ are constants, $p(x)$ is continuous on [0,L] and fractional differential operator D^μ in Caputo sense. In similar way we can study for the boundary conditions are

$$\alpha_0 u(0) + \gamma_0 u'(0) = \beta_0, \; \alpha_1 u(L) + \gamma_1 u'(L) = \beta_1, \tag{3}$$

where $\alpha_0, \alpha_1, \gamma_0, \gamma_1, \beta_0$ and β_1 are constants. Many numerical methods for FDEs have developed which are given in [16-21].

In recent decades, the Chebyshev polynomials are most powerful polynomial approximation in numerical analysis we can see from theoretical as well as practical points of view. The Chebyshev polynomials have direct connections with Fourier and Laurent series, due to minimality properties in approximation theory and with orthogonality condition holds for both discrete and continuous in function spaces [11].

This paper is organized as follows. In Section 2, the idea of fractional derivative and some properties of Chebyshev polynomials. In Section 3, main results for fractional derivative and error analysis. In Section 4, collocation method for multi-term FDE. In Section 5, numerical examples are presented for validation of proposed method and finally in Section 6, conclusions are given.

2 Preliminaries

This section is devoted to relevant definition of Chebyshev polynomials and well known results of Caputo fractional derivative is considered for this work.

2.1 Fractional derivative

The fractional derivative operator D^α of order $\alpha > 0$, is defined in the Caputo form [12]

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{(\alpha-n+1)}} dt, \;\; n-1 < \alpha \leqslant n, \; n \in \mathbb{N}, \; x > 0. \tag{4}$$

The Caputo fractional derivative satisfies linearity properties:

$$D^\alpha(\lambda p(x) + \mu q(x)) = \lambda D^\alpha p(x) + \mu D^\alpha q(x), \tag{5}$$

where λ and μ are arbitrary constants.
The Caputo fractional derivative in the power of x is defined as :

$$D^\alpha K = 0, \;\; K \; is \; a \; constant, \tag{6}$$

$$D^\alpha x^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & for \; \beta \in \mathbb{N}_0 \; and \; \beta \geqslant \lceil \alpha \rceil \\ 0, & for \; \beta \in \mathbb{N}_0 \; and \; \beta < \lceil \alpha \rceil, \end{cases} \tag{7}$$

where the function $\lceil \alpha \rceil$ is used to denote the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

2.2 Chebyshev polynomials of the third kind

The polynomials $V_n(x)$ are well known orthogonal third kind of Chebyshev polynomials which is defined in the interval [-1,1] by the relation [13]

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}},$$

of degree n in x. The polynomials $V_n(x)$ is defined by the fundamental recurrence relations

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x),\ n = 1, 2, \ldots$$

together with initial conditions

$$V_0(x) = 1,\ V_1(x) = 2x - 1.$$

The polynomials $V_n(x)$ are orthogonal on [-1,1] with the weight function $w(x)$

$$\langle V_n(x), V_m(x)\rangle = \int_{-1}^{1} V_n(x)V_m(x)w(x)dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m, \end{cases} \tag{8}$$

where $w(x) = \sqrt{\frac{1+x}{1-x}}$. By using the properties of Gamma function, the analytical form of $V_n(x)$ is defined as

$$V_n(x) = \sum_{i=0}^{\lceil\frac{2n+1}{2}\rceil} (-1)^i 2^{n-2i}(2n+1)\frac{\Gamma(2n-i+1)}{\Gamma(i+1)\Gamma(2n-2i+2)}(x+1)^{n-i},\ n > 0, \tag{9}$$

where $\lceil\frac{2n+1}{2}\rceil$ denotes the integer part of $(2n+1)/2$.

Since the range of independent variable on [0,1] is more convenient to use instead of [-1,1]. The transformation $t = 2x - 1$, then the polynomials $V_n^*(x)$ is known as shifted Chebyshev polynomials which is defined as

$$V_n^*(x) = V_n(2x-1).$$

These polynomials are orthogonal polynomials with the weight function $w(x)$ on [0,1] with inner product

$$\langle V_n^*(x), V_m^*(x)\rangle = \int_{0}^{1} V_n^*(x)V_m^*(x)w(x) = \begin{cases} 0 & n \neq m, \\ \frac{\pi}{2} & n = m, \end{cases} \tag{10}$$

where $w(x) = \sqrt{\frac{x}{1-x}}$. The polynomials $V_n^*(x)$ is defined by the fundamental recurrence relations

$$V_{n+1}^*(x) = 2(2x-1)V_n^*(x) - V_{n-1}^*(x),\ n = 1, 2, \ldots,$$

together with the initial conditions

$$V_0^*(x) = 1,\ V_1^*(x) = 4x - 3.$$

$$V_n^*(x) = \sum_{i=0}^{n} (-1)^i 2^{2n-2i} \frac{(2n+1)\Gamma(2n-i+1)}{\Gamma(i+1)\Gamma(2n-2i+2)} x^{n-i}, \; n > 0. \tag{11}$$

The square integrable function $y(x)$ in [0,1] can be expanded in terms of $V_m^*(x)$ as follows

$$y(x) = \sum_{i=0}^{\infty} c_i V_i^*(x), \tag{12}$$

where the expansion coefficients $c_i (i = 0, 1, 2, \ldots)$ are unknown which are defined as

$$c_i = \frac{2}{\pi} \int_0^1 y(x) \sqrt{\frac{x}{1-x}} V_i^*(x) dx. \tag{13}$$

For practical purpose we take only first $(m+1)$-terms of $V_n^*(x)$ in approximation which is given

$$y_m(x) = \sum_{i=0}^{m} c_i V_i^*(x), \; i = 0, 1, 2, \ldots, m. \tag{14}$$

3 Main results and error analysis

3.1 Main results for FDE

Theorem 1 The approximation function $y(x)$ can be approximated by the third kind of Chebyshev polynomials which is defined in (14) and assume that $\alpha > 0$ then

$$D^{\nu}(y_m(x)) = \sum_{i=\lceil \nu \rceil}^{m} \sum_{k=0}^{i-\lceil \nu \rceil} c_i w_{i,k}^{(\nu)} x^{i-k-\nu}, \tag{15}$$

where $w_{i,k}^{(\nu)}$ is given by

$$w_{i,k}^{(\nu)} = (-1)^k 2^{(2i-2k)} \frac{(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\nu)}. \tag{16}$$

Proof. Since the Caputo's fractional derivative satisfies linear properties, we have

$$D^{\nu}(y_m(x)) = \sum_{i=0}^{m} c_i D^{\nu}(V_i^*(x)). \tag{17}$$

Applying Eqs. (6) and (7) we get

$$D^{\nu}(V_i^*(x)) = 0, \;\; i = 0, 1, \ldots, \lceil \nu \rceil - 1, \; \nu > 0. \tag{18}$$

Also, for $i = \lceil\nu\rceil, \lceil\nu\rceil + 1, \ldots, m$, and by using Eqs. (6) and (7), we get

$$\begin{aligned} D^{\nu}(V_i^*(x)) &= \sum_{k=0}^{i}(-1)^k \frac{(2i+1)\Gamma(2i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)} D^{\nu} x^{i-k} \\ &= \sum_{k=0}^{i-\lceil\nu\rceil}(-1)^k \frac{2^{(2i-2k)}(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\nu)} x^{i-k-\nu}. \end{aligned} \tag{19}$$

A combination of Eqs. (17)-(19) we obtain

$$D^{\nu}(y_m(x)) = \sum_{i=\lceil\nu\rceil}^{m} \sum_{k=0}^{i-\lceil\nu\rceil} c_i(-1)^k \frac{2^{(2i-2k)}(2i+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\nu)} x^{i-k-\nu}, \tag{20}$$

and Eq. (20) can be written as

$$D^{\nu}(y_m(x)) = \sum_{i=\lceil\nu\rceil}^{m} \sum_{k=0}^{i-\lceil\nu\rceil} c_i w_{i,k}^{(\nu)} x^{i-k-\nu},$$

where $w_{i,k}^{(\nu)}$ is given by

$$w_{i,k}^{(\nu)} = (-1)^k 2^{(2i-2k)} \frac{(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\nu)}.$$

3.2 Error analysis

Theorem 2 (Chebyshev truncation theorem) The error in approximation $u(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$u_m(x) = \sum_{i=0}^{m} c_i V_i^*(x) \tag{21}$$

then

$$E_m(x) \equiv \mid u(x) - u_m(x) \mid \leq \sum_{i=m+1}^{\infty} \mid c_i \mid, \tag{22}$$

for all $u(x)$, all m, and all $t \in [0, 1]$.

Proof. The maximum value of $V_i^*(x)$ is one, that is $|V_i^*(x)| \leq 1$ for all $x \in [0, 1]$ and for all i. Therefore the approximating function of the ith term is bounded by coefficients c_i and subtracting the $(m+1)$-terms series from the infinite series, which gives the difference of each terms is bounded by the coefficients and summing the difference of bounding of each terms get the desired result.

4 Collocation method

The Chebyshev collocation method is applied to solve fractional Bagley-Torvik boundary value problem

$$D^{\alpha}v(x) = H\left(x, v(x), D^{\beta_1}v(x), \ldots, D^{\beta_k}v(x)\right), \; x \in (0,1), \tag{23}$$

with initial conditions

$$v^{(i)}(0) = d_i, \; i = 0, 1, 2, \ldots, m-1, \tag{24}$$

where the fractional derivatives are taken to be the Caputo type, $m-1 < \alpha \leqslant m, \; 0 < \beta_1 < \beta_2 < \ldots < \beta_k < \alpha$ and H is nonlinear function. For approximation, we used the shifted Chebyshev polynomials as a basis function and applied collocation method for solving multi-term FDE. Let

$$v_m(x) = \sum_{i=0}^{m} c_i V_m^*(x), \tag{25}$$

where c_i are unknown coefficients and first term of the series is halved. By virtue of Theorem 1, the fractional order derivatives $D^{\alpha}v(x), D^{\beta_1}v(x), \ldots, D^{\beta_k}v(x)$ can be expressed explicitly in terms of the expansion coefficients c_i. For applying the Chebyshev collocation method the criterion for solving Eq.(23) subject to initial conditions (24) is to find $v_m(x) \in S_m(0, L)$ in such a way

$$D^{\alpha}v_m(x) = H\left(x, v_m(x), D^{\beta_1}v_m(x), \ldots, D^{\beta_k}v_m(x)\right) \tag{26}$$

and the fact that $v_m(x)$ must satisfy the fractional differential equation in some suitably chosen collocation points $\theta_i, \; i = 1, 2, \ldots, m$. The convergence of numerical solution and its computational stability gets affected by the particular choice of collocation points. In order to find the unknown coefficients, Chebyshev collocation method with collocation points

$$\theta_i = \frac{1}{2} + \frac{1}{2}\cos(i\frac{\pi}{m}), i = 1, 2, \ldots, m \tag{27}$$

$$\sum_{i=0}^{m} c_i D^{\alpha} V^*(x_{\theta_i}) = \; H \left(x_{\theta_i}, \sum_{i=0}^{m} c_i D^{\alpha} V^*(x_{\theta_i}), \sum_{i=0}^{m} c_i D^{\beta_1} V^*(x_{\theta_i}), \ldots, \sum_{i=0}^{m} c_i D^{\beta_k} V^*(x_{\theta_i} \right), \quad i = 1, 2, \ldots, m, \tag{28}$$

with (21) written in the form

$$\sum_{i=0}^{m} c_i V_i^*(0) = d_i, \;\; i = 0, 1, 2, \ldots, m-1. \tag{29}$$

For finding the unknown coefficients $c_i (i = 0, 1, 2, \ldots, m)$, combine the Eqs. (28)-(29), we get the system of $(m+1)$ nonlinear algebraic equations which can be solved numerically by utilizing any standard iteration method.

Boundary value problem

For the boundary value problem the boundary conditions (for m is even) is defined as

$$v^{(i)}(0) = a_i,\ v^{(i)}(L) = b_i,\ i = 0, 1, \ldots, \frac{m}{2} - 1. \quad (30)$$

We can use the same technique as described in section (4), for the nonlinear multi-term FDE (23) but Eq. (24) shall be changed to be (30). In order to finding the unknown coefficients $c_i (i = 0, 1, 2, \ldots, m)$, combine the Eqs. (28) and (30), we get the system of $(m + 1)$ linear or nonlinear algebraic equations which can be solved numerically by iteration method.

5 Numerical examples

In this section, the proposed scheme is implemented using numerical examples for multi-term initial and boundary conditions.

Example 1 Consider the fractional order Bagley-Torvik equation [14]

$$D^2 u(x) + D^{\frac{3}{2}} u(x) + u(x) = 1 + x, \quad x \in (0, 1), \quad (31)$$

together with the boundary conditions

$$u(0) = 1, \quad u(1) = 2, \quad (32)$$

where the exact solution is $u(x) = 1 + x$. The results obtained by the proposed method which is described in Section 4, for $m = 2$ and $\alpha = 1.5$ of Eq. (31), get the exact solution where as in [14], solved this problem with $N = 9$ and $\alpha = 0.5$ using Bessel collocation method and get maximum absolute error $4.2834e - 015$, it seems that the proposed method needs only few terms of third kind of Chebyshev polynomials and get exact solution. The comparison of exact and approximate solutions are given in Fig. 1.

Example 2 Consider the fractional order IVP [15]

$$D^{(2.2)} u(x) + 1.3 D^{(1.5)} u(x) + 2.6 u(x) = \sin(2x),\ x \in (0, 1), \quad (33)$$

subject to the initial conditions

$$u(0) = u'(0) = u''(0) = 0. \quad (34)$$

Eq.(33) have series solution (see,[15]). The numerical solution obtained by proposed method are given in Tables 1-2. In Table 1, the comparison between series solution [15] and approximate solution. In Table 2, the absolute error are given for different choices of m.

6 Conclusion

An efficient, Chebyshev collocation method is applied for solving multi-term FDE. The Caputo fractional derivatives are considered in this problem. The properties of third kind of Chebyshev polynomials are utilized to reduce multi-term FDE into system of algebraic equations

Table 1: The comparison for different values of m with series solution [15] for Example 2

x	[15]	$m = 10$	$m = 15$	$m = 20$
0	0	0	0	0
0.1	0.0001478	0.0001543	0.0001493	0.0001483
0.2	0.0012750	0.0012920	0.0012790	0.0012770
0.3	0.0043990	0.0044310	0.0044110	0.0044070
0.4	0.0104100	0.0104800	0.0104500	0.0104400
0.5	0.0199600	0.0201400	0.0201000	0.0201000
0.6	0.0334500	0.0338900	0.0338500	0.0338400
0.7	0.0509200	0.0519500	0.0519000	0.0518900
0.8	0.0720400	0.0742300	0.0741800	0.0741700
0.9	0.0960400	0.1004000	0.1003000	0.0100300

Table 2: The absolute error for different values of m with series solution [15] for Example 2

x	Absolute error $m = 10$	Absolute error $m = 15$	Absolute error $m = 20$
0	0	0	0
0.1	6.5020e-6	1.5590e-6	5.2560e-7
0.2	1.7050e-5	4.0650e-6	1.7150e-6
0.3	3.2310e-5	1.1830e-5	8.2260e-6
0.4	7.0380e-5	4.2620e-5	3.7820e-5
0.5	0.0001754	0.0001412	0.0001353
0.6	0.0004387	0.0003988	0.0003920
0.7	0.0010230	0.0009780	0.0009704
0.8	0.0021930	0.0021440	0.0021350
0.9	0.0043510	0.0042990	0.0042900

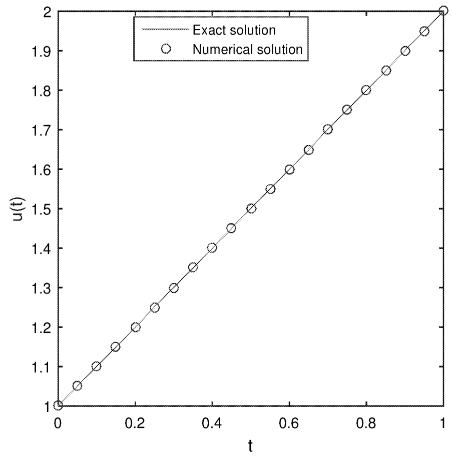

Figure 1: The comparison of the numerical solution with analytical solution with $m = 2$ for Example 1

which is solved numerically. Also, we discussed error analysis and convergence of the derived formula. The present approach is high accuracy and numerical simulation is fastest. The proposed method is characterise by its simplicity, efficiency and high accuracy. For validation of present scheme is verified through number of examples and compared with exiting methods. The present scheme is extended for linear or nonlinear fractional partial differential equations.

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References

[1] Magin RL (2006) Fractional Calculus in Bioengineering, Begell House Publishers.

[2] Miller KS, Ross B (1993) An Introduction to the fractional calculus and fractional differential equations, New York, John Wiley.

[3] Ortigueira M (2000) Introduction to fraction linear systems. Part 2: discrete-time case, IEE Proc, Vis, Image Signal Process 147: 71-78.

[4] Deng J, Ma L (2010) Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Appl Math Lett 23: 676-680

[5] Kilbas AA, Srivastava HM, Trujillo JJ (2006) Theory and applications of fractional differential equations, San Diego, Elsevier.

[6] Abdulaziz O, Hashim I, Momani S (2008) Application of homotopy-perturbation method to fractional IVPs, J Comput Appl Math 216: 574-584.

[7] Yang S, Xiao A, Su H (2010) Convergence of the variational iteration method for solving multi-order fractional differential equations, Comput Math Appli 60: pp. 2871-2879.

[8] Ray SS, Bera RK (2004) Solution of an extraordinary differential equation by Adomian decomposition method, J Appl Math 4: 331-338.

[9] El-Sayed AM, El-Kalla IL, Ziada EA (2010) Analytical and numerical solutions of multi-term nonlinear fractional orders differential equations, Appli Numer Math 60: 788-797.

[10] Odibat Z, Momani S, Xu H (2010) A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations, Appl Math Model 34:593-600.

[11] J.P. Boyd, Chebyshev and Fourier spectral methods, Courier Corporation, 2001.

[12] Podlubny I (1999) Fractional differential equations. New York, Academic Press.

[13] Mason JC, Handscomb DC (2003) Chebyshev polynomials. New York, NY, CRC, Boca Raton, Chapman and Hall.

[14] Yzba , Numerical solution of the Bagley-Torvik equation by the Bessel collocation method, Mathematical Methods Appl Sci 36: (2013) 300-312.

[15] Mdallal QM, Syam MI, Anwar MN (2010) A collocation-shooting method for solving fractional boundary value problems, Commun Nonlinear Sci Numer Simul 15: 3814-3822.

[16] Doha EH, Bhrawy AH (2009) Jacobi spectral-Galerkin method for the integrated forms of fourth-order elliptic differential equations, Numer Methods Partial Differ Equ 25: 712-739.

[17] Doha EH, Bhrawy AH, Hafez RM (2011) A Jacobi-Jacobi dual-Petrov-Galerkin method for third- and fifth-order differential equations, Math Comput Modell 53: 1820-1832.

[18] Doha EH, Bhrawy AH, Ezzeldeen SS (2011) Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations, Appl Math Model 35: 5662-5672.

[19] Bhrawy AH, Alofi AS, Ezzeldeen SS (2011) A quadrature tau method for variable coefficients fractional differential equations, Appl Math Lett 24: 2146-2152.

[20] F. Mohammadi, Numerical solution of Bagley-Torvik equation using Chebyshev wavelet operational matrix of fractional derivative, Int. J. Adv. Appl. Math. Mech. 2(1) (2014) 83-91.

[21] Q.M. Al-Mdallal, M.I. Syam, M.N. Anwar, A collocation-shooting method for solving fractional boundary value problems, Commun Nonlinear Sci Numer Simulat 15(12) (2010) 3814-3822.