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Mean-square stability of two classes of theta Milstein methods for nonlinear stochastic differential equations

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Abstract. Recently, there is growing interest in developing new numerical methods for stochastic differential equations (SDEs), in order to improve the stability of approximation solution. There are many numerical methods have been constructed based on a Milstein scheme for SDEs. However, there exists very little results on the stability analysis of Milstein type methods for SDEs. This paper is concerned with mean-square (MS) stability of the semi-implicit theta Milstein methods and drifting split-step theta Milstein methods for nonlinear stochastic differential equations. Under a coupled condition on the drifting and diffusion coefficients, it is proved that, the methods with $\theta > \frac{1}{2}$ are unconditionally preserve the MS-stability of the SDEs. For $\theta \in [0, \frac{1}{2}]$, the methods are MS-stable for some small step-size. This work is different from the previous works such that we could get rid of the restrictions that existed on the step-size of two classes Milstein methods for a symptomatic mean-square stability of the non-linear stochastic differential equations, under Local lipschitz condition. Numerical experiments are given to demonstrate the conclusions.

Keywords. Stochastic differential equations, Split-step methods, Semi-implicit theta methods, Milstein method, Nonlinear mean square stability.

1 Introduction

The Itô stochastic differential equations (SDEs) are considered as the form

$$\mathrm{d}y(t) = f(t, y(t))\mathrm{d}t + g(t, y(t))\mathrm{d}W(t), \qquad y(t_0) = y_0, \qquad t > 0, \tag{1.1}$$

where $f(t, y)$ is the drift coefficient and $g(t, y)$ is the diffusion coefficient and the Wiener process $W(t)$ is defined on a given probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfied the usual conditions.

SDEs have been widely used in many applied fields of industry and science such as biology, financial engineering, neural network and wireless communications with respect to techniques in the study of stochastic stability. Numerical solutions for SDEs have attracted a lot of attention to describe the properties of these stochastic systems. There is an extensive literature concerned with explicit and implicit methods for stiff SDEs (see [2, 3, 13, 18, 19, 21]).

In the literature, the mean-square (MS) stability for scalar linear SDEs was considered in [15, 7]. Buckwar et al. [1] considered the asymptotic MS-stability of the two-step methods. The MS-stability of the stochastic theta methods (STM) was discussed in [14, 16]. Huang [10] introduced the stability results of the two classes of theta methods. The MS-stability of Milstein classes are introduced in [5, 6, 19, 21].

For the nonlinear SDEs, Higham et al. [9] proved the MS exponential stability of the Euler method. Also, he proved the MS exponential stability of the Split-step backward Euler (SSBE) method. Recently, without the global lipschitz condition, the concept of the STM and SSθ methods can reproduce the exponential MS-stability of the exact solution under conditions is presented by Huang [10] and X. Zong et al. [22, 23].

Higham [8] introduced $\theta > 1$ in the semi-implicit theta Milstein schemes of the stability for linear SDEs. In our work [4], we also presented the MS-stability of drifting split-step theta Milstein (DSSθM) methods for linear SDEs with $\theta \geq \frac{3}{2}$. However, to the best knowledge of the authors, there is no work to discuss the MS-stability of the classes of Milstein methods for nonlinear SDEs. In this work, we extend these results for nonlinear SDEs. We prove that, the two classes of theta Milstein schemes are MS-stable for all step-size with $\theta > \frac{1}{2}$ under a coupled condition on the drift and diffusion coefficients.

The rest of this paper is organized as follows. In section 2, we present the semi-implicit theta Milstein schemes and drifting split-step theta Milstein methods. Some necessary notations and preliminaries are introduced. The MS-stability of the two classes of theta Milstein methods is proved in Section 3. In section 4, several examples are considered to illustrate the main theory, which, implies our results generally.

2 Two classes of theta Milstein methods for SDEs

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which satisfies the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous and each $\{\mathcal{F}_t\}, t \geq 0$, contains all P-null sets in $\mathcal{F}$. Let $(W(t))_{t\geq 0}$ is one-dimensional Brownian motion defined on this probability space, be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. $a \bigvee b$ presents $\max(a, b)$ and $a \bigwedge b$ presents $\min(a, b)$. $\mathbb{N}_+$ represents the positive integer set, namely $\mathbb{N}_+ = \{1, 2, 3, ...\}$. Moreover, we assume y_0 to be $\mathcal{F}_0$-measurable and $E|y_0|^2 < \infty$.

There are many numerical schemes have been constructed to approximate SDEs (1.1). In [15], a class of the semi-implicit theta Milstein methods were proposed. The simplest and most often used for SDEs (1.1) is based on the form

$$y_{n+1} = y_n + (1-\theta)hf(t_n, y_n) + \theta hf(t_{n+1}, y_{n+1}) + g(t_n, y_n)\Delta W_n + \frac{1}{2}\frac{\partial g}{\partial y}(t_n, y_n)g(t_n, y_n)[(\Delta W_n)^2 - h], \tag{2.1}$$

where y_n is an approximation to $y(t_n)$, $h > 0$ is the time step-size, $t_n = nh$, θ is a free parameter, with increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0, h)$-distributed Gaussian random variables and $y(0) = y_0$. Moreover, y_n is $\{\mathcal{F}_{t_n}\}$-measurable at the mesh-point t_n.

Another method is drifting split-step theta Milstein (DSSθM) methods, which are constructed in [4] as follows:

$$y_n^* = y_n + \theta hf(t_n, y_n^*), \tag{2.2}$$

$$y_{n+1} = y_n^* + (1-\theta)hf(t_n, y_n^*) + g(t_n, y_n^*)\Delta W_n + \frac{1}{2}\frac{\partial g}{\partial y}(t_n, y_n^*)g(t_n, y_n^*)[(\Delta W_n)^2 - h], \tag{2.3}$$

Remark 1. *The semi-implicit theta Milstein methods(2.1) with parameter ($\theta = 1$) reduce to the Milstein method in [12]. The DSSθM methods (2.2- 2.3) with parameter ($\theta = 1$) reduce to the DSSBM method [20], while both of the two methods with ($\theta = 0$) are the forward Milstein (SSFM) method [17].*

Now we introduce some stability concepts which will be used later. Let $f, g : \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable functions in (1.1). We Assume that f and g satisfy the following assumptions:

Assumption 1. *(Local Lipshitz Condition) There exists a positive constant k_j for all $j > 0$ such that for all $t \in \mathbb{R}$ and $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^d$ with $|x| \bigvee |y| \leq j$,*

$$|f(t, x) - f(t, y)| \vee |g(t, x) - g(t, y)| \leq k_j|x - y|. \tag{2.4}$$

Assumption 2. *Assume that there exist a constant σ such that for any $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^d$, the function g in (1.1) also satisfies*

$$\left|\frac{\partial g(t, x)}{\partial y}g(t, x) - \frac{\partial g(t, y)}{\partial y}g(t, y)\right|^2 \leq \sigma|x - y|^2. \tag{2.5}$$

Definition 1. *Let $f(0, 0) = g(0, 0) = 0$ in SDEs (1.1), Then the solution is said to be MS-stable if*

$$\lim_{t\to\infty} E|y(t)|^2 = 0.$$

Definition 2. *The numerical method is said to be MS-stable, if there exists a $h^* > 0$, such that any application of the method to SDEs (1.1) generates numerical approximation y_n, which satisfy*

$$\lim_{n\to\infty} E|y_n|^2 = 0, \tag{2.6}$$

for all step-size $h \in (0, h^*)$.

3 MS-stability of two classes of theta Milstein methods

In this section, we discuss the nonlinear stability of the two classes of theta Milstein methods for SDEs (1.1). It is impossible to find a sufficient and necessary condition for analytical stability for nonlinear SDEs. For the purpose of stability, assume that $f(0,0) = g(0,0) = 0$. This shows that SDEs (1.1) admits a trivial solution. Then inequality (2.5) reduces to

$$\left|\frac{\partial g(t,x)}{\partial y} g(t,x)\right|^2 \le \sigma |x|^2. \tag{3.1}$$

To investigate stability of numerical approximations, let us firstly give the stability criterion of SDEs (1.1)

Theorem 1. *(see [11]) Let Assumption 1 hold. If there exists a positive constant γ such that for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^d$,*

$$2xf(t,x) + |g(t,x)|^2 \le -\gamma |x|^2, \tag{3.2}$$

then the solution of (1.1) is MS-stable.

First, we introduce the following stability results for the DSSθM (2.2, 2.3) methods.

Theorem 2. *Let the conditions (3.1, 3.2) hold, Then the DSSθM methods (2.2,2.3) have the following stability results*

1. *If $\theta > \frac{1}{2}$ and for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^d$*

$$(1-2\theta)|f(t,x)|^2 + \frac{1}{2}\left|\frac{\partial g}{\partial x}(t,x) g(t,x)\right|^2 \le 0, \tag{3.3}$$

then the DSSθM methods are MS-stable for all $h > 0$.

2. *If $\theta \in [0, \frac{1}{2}]$ and if there exists a positive constant K such that for any $(t,x) \in [t_0,\infty) \times \mathbb{R}^d$,*

$$|f(t,x)|^2 \le K|x|^2, \tag{3.4}$$

then the DSSθM methods are MS-stable for all $h \in (0, h^*)$,

$$h^* \le \frac{\gamma}{(1-2\theta)K + \frac{1}{2}\sigma}.$$

To prove Theorem 2, we first give the following lemma.

Lemma 1. *Let the conditions (3.1, 3.2) hold. Moreover, the functions f and g satisfy the local Lipschitz condition (2.4) for any $(t,x),(t,y) \in [t_0,\infty) \times \mathbb{R}^d$, with a positive constant k_l for all $l > 0$. Then there exist constants $\bar{K}_l$ for all $l > 0$ and $n > 0$, such that the DSSθM methods (2.2,2.3) have the properties*

$$E|y_n|^2 \le \bar{K}_l \quad \text{and} \quad E|y_n^*|^2 \le \bar{K}_l. \tag{3.5}$$

Proof. For sufficiently large $l > 0$, we define the stopping time

$$\lambda_l := \inf\{i > 0 : |y_i^*| > l \ \text{ or } \ |y_i| > l\}. \tag{3.6}$$

It is observe that, for $n \in [0, \lambda_l]$

$$|y_{n-1}^*| \le l \quad \text{and} \quad |y_{n-1}| \le l. \tag{3.7}$$

From (2.3), we have

$$y_n = y_{n-1}+hf(t_{n-1},y^*_{n-1})+g(t_{n-1},y^*_{n-1})\Delta W_{n-1}+\frac{1}{2}\frac{\partial g}{\partial y}(t_{n-1},y^*_{n-1})g(t_{n-1},y^*_{n-1})[(\Delta W_{n-1})^2-h], \tag{3.8}$$

in view of $|a+b+c+d|^2 \le 4(|a|^2+|b|^2+|c|^2+|d|^2)$, squaring both side in above equation we get

$$\begin{aligned} |y_n|^2 &\le 4(|y_{n-1}|^2 + h^2|f(t_{n-1},y^*_{n-1})|^2 + |g(t_{n-1},y^*_{n-1})|^2|\Delta W_{n-1}|^2 \\ &+ \frac{1}{4}\left|\frac{\partial g}{\partial y}(t_{n-1},y^*_{n-1})g(t_{n-1},y^*_{n-1})\right|^2 ||\Delta W_{n-1}|^2-h|^2). \end{aligned} \tag{3.9}$$

Note that y_n is $\mathcal{F}_{t_n}$-measurable at the mesh point t_n, we easily know from (2.2, 2.3) that y^*_n is also $\mathcal{F}_{t_n}$-measurable at related meash-point, ΔW_n is independent of $\mathcal{F}_{t_n}$. Making use of property $E[|\Delta W_n|^{2i}] = (2i-1)!!h^i$, we have $E[|\Delta W_{n-1}|^4] = 3h^2$ and $E[|(\Delta W_{n-1})^2 - h|^4] \le 2^3(E[|\Delta W_{n-1}|^8]+h^4) = 2^3(7!!+1)h^4$, where $(2i-1)!! = (2i-1)(2i-3)...3\cdot 1$ for $i = 1,2,\dots$. By Assumption 1, Equations (3.1) and (3.7), it is easy to deduce that

$$\begin{aligned} |f(t_{n-1},y^*_{n-1})|^2 1_{[0,\lambda_l]}(n) &\le k_l^2|y^*_{n-1}|^2 \le k_l^2 l^2 \\ |g(t_{n-1},y^*_{n-1})|^4 1_{[0,\lambda_l]}(n) &\le k_l^4|y^*_{n-1}|^4 \le k_l^4 l^4 \\ \left|\frac{\partial g}{\partial y}(t_{n-1},y^*_{n-1})g(t_{n-1},y^*_{n-1})\right|^4 1_{[0,\lambda_l]}(n) &\le \sigma^2|y^*_{n-1}|^4 \le \sigma^2 l^4. \end{aligned}$$

Taking expectation on (3.9) and using Hölder inequality, we obtained

$$\begin{aligned} E[|y_n|^2 1_{[0,\lambda_l]}(n)] &\le 4l^2 + 4h^2E[|f(t_{n-1},y^*_{n-1})|^2 1_{[0,\lambda_l]}(n)] + 4E[|g(t_{n-1},y^*_{n-1})|^2|\Delta W_{n-1}|^2 1_{[0,\lambda_l]}(n)] \\ &+ E\left[\left|\frac{\partial g}{\partial y}(t_{n-1},y^*_{n-1})g(t_{n-1},y^*_{n-1})\right|^2 ||\Delta W_{n-1}|^2-h|^2 1_{[0,\lambda_l]}(n)\right] \\ &\le 4l^2 + 4h^2E[|f(t_{n-1},y^*_{n-1})|^2 1_{[0,\lambda_l]}(n)] \\ &+ 4(E[|g(t_{n-1},y^*_{n-1})|^4 1_{[0,\lambda_l]}(n)]E[|\Delta W_{n-1}|^4])^{\frac{1}{2}} \\ &+ \left(E\left[\left|\frac{\partial g}{\partial y}(t_{n-1},y^*_{n-1})g(t_{n-1},y^*_{n-1})\right|^4 1_{[0,\lambda_l]}(n)\right]E[||\Delta W_{n-1}|^2-h|^4]\right)^{\frac{1}{2}} \\ &\le 4l^2 + 4h^2k_l^2l^2 + 4\sqrt{3}k_l^2l^2h + \sqrt{2^3(7!!+1)}\sigma l^2h^2 \\ &\le \bar{K}_l. \end{aligned} \tag{3.10}$$

From (2.2), we get

$$y_n = y^*_n - \theta h f(t_n, y^*_n).$$

Squaring both side of above equation and using Equation (3.2), yield

$$\begin{aligned} |y_n|^2 &= |y^*_n|^2 - 2\theta h y^*_n f(t_n,y^*_n) + \theta^2h^2|f(t_n,y^*_n)|^2 \\ &\ge |y^*_n|^2 + \theta h\gamma|y^*_n|^2 \\ &\ge (1+\theta h\gamma)|y^*_n|^2. \end{aligned} \tag{3.11}$$

By (3.10), we obtained

$$E[|y^*_n|^2 1_{[0,\lambda_l]}(n)] \le \bar{K}_l. \tag{3.12}$$

The proof of Lemma is completed.

Proof of Theorem 2

By (2.2), we get

$$|y_n^*|^2 \quad = \quad |y_n|^2 + 2\theta h y_n^* f(t_n, y_n^*) - \theta^2 h^2 |f(t_n, y_n^*)|^2. \tag{3.13}$$

Squaring both side of (2.3), and substituting (3.13), we have

$$\begin{aligned} |y_{n+1}|^2 &= |y_n|^2 + 2hy_n^* f(t_n, y_n^*) + h|g(t_n, y_n^*|^2 \\ &+ (1-2\theta)h^2|f(t_n, y_n^*)|^2 + \frac{1}{2}h^2 \left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 + M_n, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} M_n &= |g(t_n, y_n^*)|^2[|\Delta W_n|^2 - h] + \frac{1}{4}\left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 [|\Delta W_n|^4 - 3h^2] \\ &+ 2y_n^* g(t_n, y_n^*)\Delta W_n + y_n^* \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*)[|\Delta W_n|^2 - h] \\ &+ 2(1-\theta)hf(t_n, y_n^*)g(t_n, y_n^*)\Delta W_n + (1-\theta)hf(t_n, y_n^*)\frac{\partial g}{\partial y}(t_n, y_n^*)g(t_n, y_n^*)[|\Delta W_n|^2 - h] \\ &+ g(t_n, y_n^*)\Delta W_n \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*)[|\Delta W_n|^2 - h] \end{aligned}$$

Note that y_n is $\mathcal{F}_{t_n}$-measurable at the mesh point t_n, we easily know from (2.2, 2.3) that y_n^* is also $\mathcal{F}_{t_n}$-measurable at related meash-point, ΔW_n is independent of $\mathcal{F}_{t_n}$. So, $E[\triangle W_n] = 0$, $E[|\triangle W_n|^2] = h$, $E[|\triangle W_n|^4] = 3h^2$ and $E[|M_n|] = 0$.

Case I: If $\theta > \frac{1}{2}$, then $(1-2\theta)|f(t_n, y_n^*)|^2 < 0$. Hence, we need to discuss the sign of the following term

$$(1-2\theta)|f(t_n, y_n^*)|^2 + \frac{1}{2}\left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2. \tag{3.15}$$

Firstly: If $(1-2\theta)|f(t_n, y_n^*)|^2 + \frac{1}{2}\left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 \leq 0$, we obtain from (3.14, with respect to conditions (3.2) and Lemma 1, the DSSθM methods are MS-stable for all $h > 0$.

Secondly: If $(1-2\theta)|f(t_n, y_n^*)|^2 + \frac{1}{2}\left| \frac{\partial g}{\partial y}(t_n, y_n^*) g(t_n, y_n^*) \right|^2 > 0$, using condition (3.1), we obtain from (3.14)

$$|y_{n+1}|^2 \leq |y_n|^2 + h\left[\frac{1}{2}\sigma h - \gamma\right]|y_n^*|^2 + M_n. \tag{3.16}$$

Taking expectation and summation on both side of (3.16) over n from $n = 0$ with respect to Lemma 1, gives

$$E[|y_{n+1}|^2] + h\left[\gamma - \frac{1}{2}\sigma h\right]\sum_{i=0}^{n} E[|y_n^*|^2] \leq E[|y_0|^2]. \tag{3.17}$$

Let

$$h_1 \leq \frac{2\gamma}{\sigma}, \tag{3.18}$$

Then for any $h \in (0, h_1)$, the DSSθM methods are MS-Stable.

Case II: If $\theta \in [0, \frac{1}{2}]$, using conditions (3.1-3.23), we get from (3.14)

$$|y_{n+1}|^2 \leq |y_n|^2 + h[((1-2\theta)K + \frac{1}{2}\sigma)h - \gamma]|y_n^*|^2 + M_n, \tag{3.19}$$

Let

$$h^* \leq \frac{\gamma}{(1-2\theta)K + \frac{1}{2}\sigma}, \tag{3.20}$$

with respect to Lemma 1. We get, The DSSθM methods are MS-Stable for any $h \in (0, h^*)$.

□

By using the same idea and similar process, we can get the stability results to the semi-implicit theta Milstein methods (2.1) as follows

Lemma 2. *Let the conditions (3.1, 3.2) hold. Moreover, the functions f and g satisfy the local Lipschitz condition (2.4) for any $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^d$, with a positive constant k_l for all $l > 0$. Then there exist constants $\tilde{K}_l$ for all $l > 0$ and $n > 0$, such that the DSSθM methods (2.2,2.3) have the properties*

$$E|y_n|^2 \leq \tilde{K}_l \qquad \text{and} \qquad E|\tilde{u}_n|^2 \leq \tilde{K}_l, \tag{3.21}$$

where $\tilde{u}_n = y_n - \theta h f(t_n, y_n)$.

Theorem 3. *Let the conditions (3.1,3.2) hold, Then the semi-implicit theta Milstein methods (2.1) have the following stability results*

1. *If $\theta > \frac{1}{2}$ and for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^d$*

$$(1 - 2\theta)|f(t, x)|^2 + \frac{1}{2}\left|\frac{\partial g}{\partial x}(t, x)g(t, x)\right|^2 \leq 0, \tag{3.22}$$

then the semi-implicit theta Milstein methods are MS-stable for all $h > 0$.

2. *If $\theta \in [0, \frac{1}{2}]$ and if there exists a positive constant K such that for any $(t, x) \in [t_0, \infty) \times \mathbb{R}^d$,*

$$|f(t, x)|^2 \leq K|x|^2, \tag{3.23}$$

then the semi-implicit theta Milstein methods are MS-stable for all $h \in (0, h^*)$,

$$h^* \leq \frac{\gamma}{(1 - 2\theta)K + \frac{1}{2}\sigma}.$$

As, extension of Higham [8] and our work [4] of MS-stability to the semi-implicit theta Milstein methods (2.1) and the DSSθM methods (2.2-2.3) for linear SDEs, respectively. We know that, the stability for the methods with $\theta \in [0, 1]$ are strictly contained in that for the problem. Theorem 2, 3 show that these behavior extends to the numerical methods with $\theta > \frac{1}{2}$ when the diffusion term dominates. Also, if the drift term dominates, then the unconditional stability holds. The following remark shows that for $\theta \geq \frac{3}{2}$, the numerical methods are MS-stable.

Remark 2. *In Theorem 2, 3, if $\theta \geq \frac{3}{2}$, then the semi-implicit theta Milstein methods (2.1) and the DSSθM methods (2.2-2.3) are MS-stable for all $h > 0$, under a coupled condition on the drift and diffusion coefficients (3.3). In case of $\theta = \frac{3}{2}$ this condition reduces to*

$$-2|f(t_n, y_n^*)|^2 + \frac{1}{2}\left|\frac{\partial g}{\partial y}(t_n, y_n^*)g(t_n, y_n^*)\right|^2 \leq 0 \tag{3.24}$$

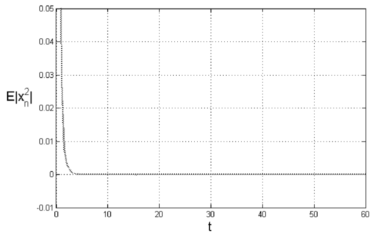
From Remark 2, we get that, The approximation solutions can reproduce the MS-stability of trivial solution, when $\theta \geq \frac{3}{2}$. To illustrate these results, we consider the following nonlinear SDEs

$$\mathrm{d}y(t) = (-y - y^3)\mathrm{d}t + bsin(y)\mathrm{d}W(t), \qquad t \geq 0. \tag{3.25}$$

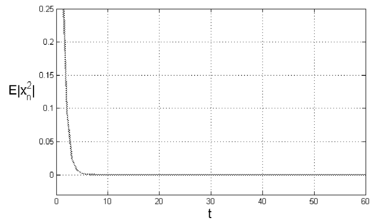
Clearly, the coefficients of 3.25 satisfy conditions (2.4) and (2.5). Now, we explain that, the stability condition (3.24) of the numerical methods satisfy the exact solution stability condition (3.2).

The exact solution stability condition:

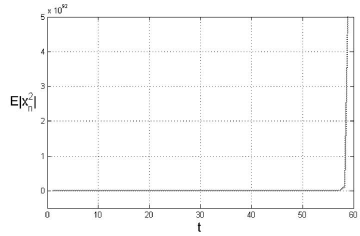
$$\begin{aligned} 2x(-y - y^3) + |bsin(y)|^2 &\leq 0 \\ (-2 + b^2)y^2 - 2y^4 &\leq 0 \\ 4 &\geq b^4 \end{aligned} \tag{3.26}$$



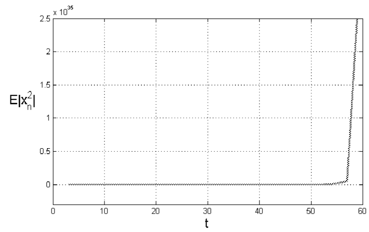
(a) $h = 0.5 < h^* = 1.0282$.



(b) $h = 1 < h^* = 1.0282$.



(c) $h = 1.1 > h^* = 1.0282$.



(d) $h = 3 > h^* = 1.0282$.

Figure 1: Simulating of DSSθM with fixed parameter $\theta = 0.1$, for (4.1)

The stability condition for two classes of theta Milstein methods:

$$-2(-y-y^3)^2 + \frac{1}{2}|b^2 sin(y)cos(y)|^2 \quad \leq \quad 0,$$

we get

$$\begin{aligned} (-2+\frac{1}{2}b^4)y^2 - 4y^4 - 2y^6 &\leq 0 \\ 4 &\geq b^4 \qquad (3.27) \end{aligned}$$

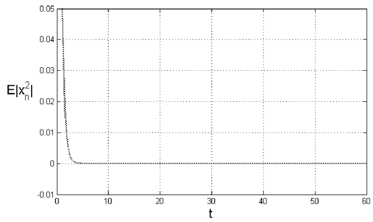
4 Numerical experiments

Since, the semi-implicit theta Milstein methods (2.1) and DSSθM methods (2.2, 2.3) have the same stability results. In this section, some simulations to illustrate the stability properties of DSSθM methods are presented. The data used in the following figures is obtained by the MS of data from 4000 trajectories; that is: $\frac{1}{4000}\sum_{i=1}^{4000} E|y_n(w_i)|^2$.

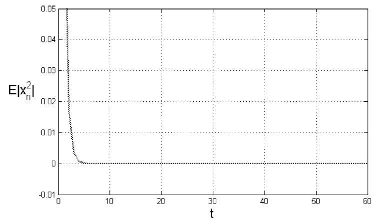
Example 1: We consider the nonlinear SDEs

$$\begin{aligned} \mathrm{d}y(t) &= (-y-y^3)\mathrm{d}t + \frac{1}{2}sin(y)\mathrm{d}W(t), \qquad t \in [0,T], \\ y(0) &= 1. \qquad (4.1) \end{aligned}$$

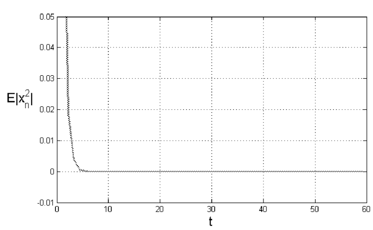
We test the MS-stability for DSSθM methods for $\theta = 0.1$ and 0.3, we obtain $h^* = 1.0282$ and $h^* = 2.8275$, respectively. We first fix the parameter $\theta = 0.1$ and 0.3 and change the step size h (see Figures 1, 2, respectively). For $\theta = 0.5$, we obtain $h^* = 56$. We fix the parameter $\theta = 0.5$ and change the step size h (see Figure 3. These are shown that the DSSθM methods are MS-stable for any $\theta \in [0, \frac{1}{2}]$ if $h \in (0, h^*)$. Figure 4 explains that for any $\theta > \frac{1}{2}$ or $\theta \geq \frac{3}{2}$ with respect to condition (3.15), the DSSθM methods is MS-stable for any $h > 0$.



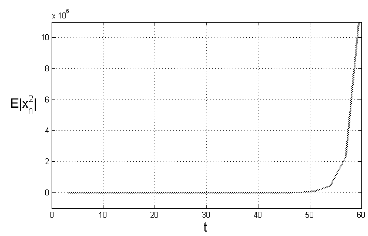
(a) $h = 0.5 < h^* = 2.8275$.



(b) $h = 1 < h^* = 2.8275$.

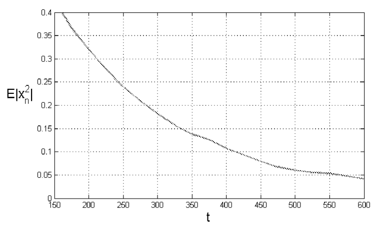


(c) $h = 1.1 < h^* = 2.8275$.

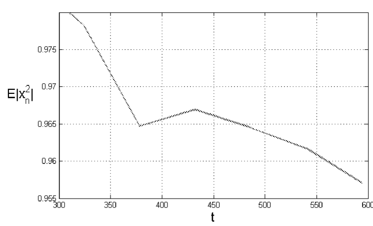


(d) $h = 3 > h^* = 2.8275$.

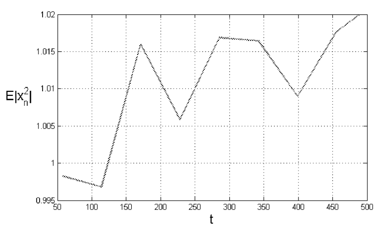
Figure 2: Simulating of DSSθM with fixed parameter $\theta = 0.3$, for (4.1)



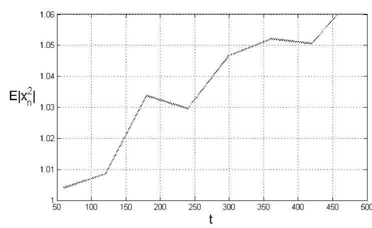
(a) $h = 25 < h_1^* = 56$.



(b) $h = 55 < h_1^* = 56$.

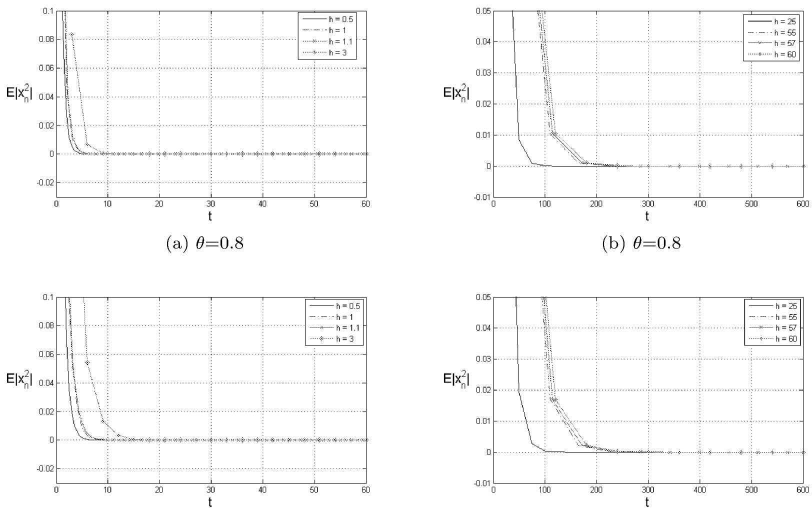


(c) $h = 57 > h_1^* = 56$.



(d) $h = 60 > h_1^* = 56$.

Figure 3: Simulating of DSSθM with fixed parameter $\theta = 0.5$, for (4.1)



(a) θ=0.8 (b) θ=0.8

(c) θ=1.5 (d) θ=1.5

Figure 4: Simulating of DSSθM with fixed parameter θ for (4.1)

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