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IRREGULAR LABELING ON TRANSPORTATION NETWORK OF SPLITTING GRAPHS OF STARS

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Abstract. The management of land transportation system in a city is very important to do so that there is no congestion on road segments or at road intersections. To optimize the management of land transportation system we requires an appropriate and efficient design/model. In this study, the land transportation system is modeled as a labeled graph. A graph is a set of vertices and a set of edges, where the vertices of the graph represent road intersections and the edges represent links between two road intersections. A vertex irregular total k-labeling on a graph G is a mapping that maps the set of all vertices and edges from G to the set of integers $\{1, 2, ..., k\}$ such that all vertices have different weight. The minimum k for which the graph G has an vertex irregular total k-labeling is called the total vertex irregularity strength of the graph G. This minimum integer is very useful when displaying the minimum travel time on certain roads. In this paper, we consider the splitting graph of stars as a land transportation system and give the exact value of their total vertex irregularity strength.

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1. Introduction

A graph labeling is a mapping that maps graph elements to the numbers (usually to the positive or non-negative integers). The most common choices of domain are the set of all vertices (vertex labelings), the set of all edges alone (edge labelings), or the set of all vertices and edges (total labelings). Graph labelings have been studied by several authors since the 1960s ([3],[4],[8]). Graph labelings have enormous applications within several areas of computer science and communication networks as well as mathematics. There are several types of graph labeling, for example graceful labeling, harmony labeling, magic labeling, anti-magic labeling, super magic edge labeling, and irregular labeling. An irregular labeling on a graph was introduced by Chartrand, et al. in 1988 [2]. Wallis studied the weight of an element, i.e the sum of all the labels associated with the elements [10]. In 2007, Bača, et al. introduced the total vertex(edge) irregularity strength, based on the total vertex irregular labeling [1].

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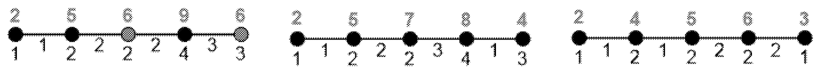
A vertex irregular total k-labeling on a graph G is a mapping that maps the set of all vertices and the set of all edges to the set of integers $\{1, 2, ..., k\}$ such that all vertices have different weight, where the weight of a vertex is the sum of label of the vertex and labels of all edges incident to the vertex. The minimum k for which the graph G has a vertex irregular total k-labeling is called the total vertex irregularity strength of the graph G denoted by $tvs(G)$. For example, in Figure 1, the black numbers in the figure show vertex or edge labels, and the red numbers show the weight of the vertex. Figure 1(a) is not a total vertex irregular labeling on P_5 because there are two vertices having the same weight, that is the red vertices. Figure 1 (b) and(c) are total vertex irregular labelings on P_5. Since the minimum number k such that P_5 have a total vertex irregular k-labeling is 2, the total vertex irregularity strength of P_5 is 2.



(a). It is not a total vertex irregular labeling (b). It is a total vertex irregular 4-labeling (c). It is a total vertex irregular 2-labelling

FIGURE 1. The labeling of the path and the total vertex irregularity strength is 2

As an application of the total vertex irregularity strength, there is the land transportation system which is modeled as a graph labeled. The vertices of the graph represent road junctions and the edges are links between two road junctions. This minimum integer is very useful when displaying the minimum travel time on certain roads.

Only a few graph classes have been studied for their total vertex irregularity strength. Bača et al. studied the total vertex irregularity strengths for some classes of graphs, namely cycles, stars, and prisms [1]. Nurdin et al. studied the total vertex irregularity strengths of a disjoint union of t copies of a path [6], tree graphs [5], and caterpillar graphs [7].

In [5], Nurdin et al. gave a lower bound of $tvs(G)$ for any graph G as follows.

Theorem 1.1. *Let G be a graph with n_i the number vertices of degree i for $i = \delta, \delta+1, \delta+2, \cdots, \Delta$, where δ and Δ are the minimum and maximum degree of G, respectively. Then,*

$$tvs(G) \geq \max\left\{ \left\lceil \frac{\delta + n_\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta+2} \right\rceil, \cdots, \left\lceil \frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta+1} \right\rceil \right\}.$$

2. Total vertex irregularity strength of the splitting graph of $K_{1,n}$

In this section we determine the exact values of the total vertex irregularity strength of the splitting graph of $K_{1,n}$.

Let G be a finite, simple, undirected with vertex set $V(G)$ and edge set $E(G)$. For a graph G, let $V'(G) = \{v' : v \in G\}$ be a copy of $V(G)$. The splitting graph of G, denoted by $Spl(G)$, is defined by a graph with vertex set $V(G) \cup V'(G)$ and edge set

$$\{uv,\ u'v \mid uv \in E(G)\ \}.$$

Figure 2 shows the splitting graph of $K_{1,4}$.

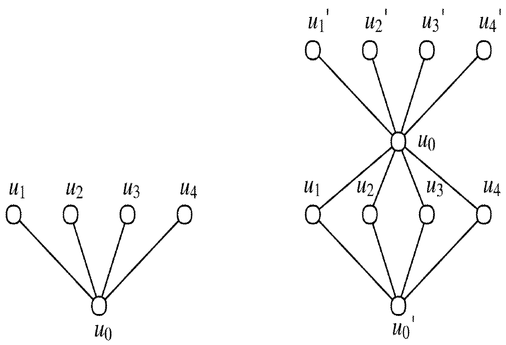


FIGURE 2. $K_{1,4}$ and the slitting graph of $K_{1,4}$

We calculate the total irregularity strength of splitting graphs of stars by obtaining the same lower bound and upper bound for it. The lower bound is analyzed based on the characteristic of the graph and other supporting theorems, while the upper bound is analyzed by the total labeling of the splitting star graph with constructed a total labeling of the graph.

Theorem 2.1. ***(Main Theorem)***
For $n \geq 3$, then $tvs(Spl(K_{1,n})) = \lceil \frac{2n+1}{3} \rceil$.

Proof. Proof. We aim to prove the theorem by obtaining the same lower and upper bound of irregularity strength. By Theorem 1, we have that

$$tvs(Spl(K_{1,n})) \geq \max\left\{ \left\lceil \frac{1+n}{2} \right\rceil, \left\lceil \frac{1+2n}{3} \right\rceil, \left\lceil \frac{2+2n}{n+1} \right\rceil, \left\lceil \frac{3+2n}{2n+1} \right\rceil \right\} \quad (1)$$

$$= \left\lceil \frac{1+2n}{3} \right\rceil \quad (2)$$

Next, to find the upper bound, we have to construction a total vertex irregular labeling on $Spl(K_{1,n})$. Define the total labeling on $Spl(K_{1,n})$ as follow:

$$
\begin{aligned}
f(u_i') &= \left\lceil \frac{i+1}{2} \right\rceil, \text{ for } i \;=\; 1,2,\cdots,n;\\
f(u_i) &= \left\lceil \frac{n+i}{3} \right\rceil, \text{ for } i \;=\; 1,2,\cdots,n;\\
f(u_0) &= \left\lceil \frac{i+1}{2} \right\rceil;\\
f(u_0') &= \left\lceil \frac{n+i}{3} \right\rceil;\\
f(u_0u_i') &= \left\lceil \frac{i}{2} \right\rceil, \text{ for } i \;=\; 1,2,\cdots,n;\\
f(u_0u_i) &= \left\lceil \frac{n+i-1}{3} \right\rceil, \text{ for } i \;=\; 1,2,\cdots,n;\\
f(u_0'u_i) &= \left\lceil \frac{n+i+1}{3} \right\rceil, \text{ for } i \;=\; 1,2,\cdots,n;
\end{aligned}
$$

Base on this labeling, we find the weight of all vertices as follows:

$$
\begin{aligned}
wt(u_0) &= 1 + \sum_{i=1}^{n} \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{n+i-1}{3} \right\rceil;\\
wt(u_i') &= \left\lceil \frac{i+1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil;\\
wt(u_0') &= 1 + \sum_{i=1}^{n} \left\lceil \frac{n+i+1}{3} \right\rceil;\\
wt(u_i) &= \left\lceil \frac{i+1}{3} \right\rceil + \left\lceil \frac{n+i-1}{3} \right\rceil + \left\lceil \frac{n+i+1}{3} \right\rceil.
\end{aligned}
$$

Base on the formula for the weight of all vertices, we can find that

$$
\begin{aligned}
& wt(u_1') \leq wt(u_2') < wt(u_3') < \cdots < wt(u_{n-1}') < wt(u_n') < wt(u_1)\\
& < wt(u_2) < wt(u_3) < \cdots < wt(u_{n-1}) < wt(u_n) < wt(u_0') < wt(u_0).
\end{aligned}
$$

Therefore, the $Spl(K_{1,n})$ have a total vertex irregular labeling. Next we have to find that the largest positive integer number used in the labeling. Note that:

$$f(u_i') = \left\lceil \frac{i+1}{2} \right\rceil \le \left\lceil \frac{n+1}{2} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil, \text{ for } i = 1, 2, \cdots, n;$$
$$f(u_i) = \left\lceil \frac{n+i}{3} \right\rceil \le \left\lceil \frac{2n}{3} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil, \text{ for } i = 1, 2, \cdots, n;$$
$$f(u_0) = \left\lceil \frac{i+1}{2} \right\rceil \le \left\lceil \frac{n+1}{2} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil;$$
$$f(u_0') = \left\lceil \frac{n+i}{3} \right\rceil \le \left\lceil \frac{2n}{3} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil;$$
$$f(u_0u_i') = \left\lceil \frac{i}{2} \right\rceil \le \left\lceil \frac{n}{2} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil, \text{ for } i = 1, 2, \cdots, n;$$
$$f(u_0u_i) = \left\lceil \frac{n+i-1}{3} \right\rceil \le \left\lceil \frac{2n-1}{3} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil, \text{ for } i = 1, 2, \cdots, n;$$
$$f(u_0'u_i) = \left\lceil \frac{n+i+1}{3} \right\rceil \le \left\lceil \frac{2n+1}{3} \right\rceil, \text{ for } i = 1, 2, \cdots, n;$$

Therefore, we found that the labeling f is a total vertex irregular k-labeling on $Spl(K_{1,n})$, where $k = \lceil \frac{2n+1}{3} \rceil$. That is

$$tvs(Spl(K_{1,n})) \le \left\lceil \frac{2n+1}{3} \right\rceil. \tag{3}$$

By Equations (2) and (3), $tvs(Spl(K_{1,n})) = \lceil \frac{2n+1}{3} \rceil$.

□

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