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S(N)-ALMOST PRIME SUBMODULES

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Abstract. In this article, we focus on some notions of prime submodules over a commutative ring with identity. Suppose R is a commutative ring with identity and M is an R-module. A proper submodule N of M is called prime if rm is element of N implies r element of $(N : M)$ or m element of N. In this article we give a generalized version of prime submodule in its localization which named as $S(N)$-almost prime submodule. This generalization was obtained by creating generalized version of almost prime submodule such that the submodule is not almost prime but its localization is almost prime. Furthermore, some characterizations of $S(N)$-almost prime submodules and its relation to $S(I)$-almost prime ideals are given in this article.

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1. Introduction

In this article, all rings are commutative with identity and all modules are unital. Let R be a ring and M be an R-module. Suppose N is a proper submodule of M. N is called prime submodule if $rm \in N$ implies $r \in (N : M)$ or $m \in N$, where $(N : M) = \{r \in R : rM \subseteq N\}$ is a residual of N by M [2]. Furthermore, N is called weakly prime submodule if $0 \neq rm \in N$ implies $r \in (N : M)$ or $m \in N$ [1]. In 2012, Hani A. Khasan gave a new notion of prime submodule, i.e. a proper submodule N of M is called an almost prime submodule if $rm \in N - (N : M)N$ implies $r \in (N : M)$ or $m \in N$ [9].

A concept which related to prime submodule is multiplication module. An R-module M is called multiplication module if for all submodule N of M, there exists an ideal I of R such that $N = IM$ [2]. Here, I is called representation ideal of N. By this notion, we can define a multiplication between two submodules. Suppose N and K are submodules of M with I and J are representation ideals of N and K respectively. The product of N and K is $NK = IJM$. As for $m, m' \in M$, we have $\langle m\rangle = AM$ and $\langle m'\rangle = BM$ where A and B are representation ideals of $\langle m\rangle$ and $\langle m'\rangle$ respectively. So $mm' = ABM$ [2].

A notion of ring localization was explained by Huishi in [10]. Suppose R be a ring and S be a multiplicatively closed set of R, i.e $rs \in S$ for all $r, s \in S$. We denote R_S as a localization of R in S. It is clear that R_S is a ring with identity. Similarly, if M is an R-module, then M_S denotes a localization of M in S which is an R_S-module. Furthermore, if $S = R - P$ for a maximal ideal P of R, then M_S is called the localization of M at P

and denoted as M_P [10]. The information about these notions are available in [6]. In this article, the localization of module refers to module over its ring localization, i.e. M_S is a module over R_S.

Another related concept of prime submodule is about an element of R which is prime to a submodule of M. Suppose M be an R-module and N be a submodule of M. An element $r \in R$ is called prime to N if $rm \in N$ for $m \in M$ implies $m \in N$. Futhermore, $S(N) = \{r \in R | \exists m \notin N$ such that $rm \in N\}$ is the set of all elements in R which are not prime to N. The submodule N is called a primal submodule if $S(N)$ forms an ideal of R [4]. Combining the notion of prime submodule and localization of module gives us another kind of prime submodule. In this article we introduce $S(N)$-almost prime submodule which is a generalized notion of almost prime submodule.

In this article, we dedicate the second section to introduce $S(N)$-almost prime submodules and give its characterization. In the third section, we give some characterizations of submodules which both $S(N)$-almost prime and primal. Furthermore, we give the condition such that $S(N)$-almost prime submodule is almost prime. In the fourth section we give similar notion of $S(N)$-almost prime submodules in ideals, which is $S(I)$-almost prime ideals. In the last section of this article, we give some relations between $S(I)$-almost prime ideals and $S(N)$-almost prime submodules, with $I = (N : M)$.

2. $S(N)$-Almost Prime Submodules

Definition 2.1. *Let R be a ring and M be an R-module. A proper submodule N of M is called $S(N)$-almost prime submodule if N_P is an almost prime submodule for each maximal ideal P with $S(N) \subseteq P$.*

Corollary 2.1. *Let N be a proper submodule of R-module M. If N is $S(N)$-weakly prime submodule, then N is an $S(N)$-almost prime submodule.*

Proof. Suppose N is an $S(N)$-weakly prime submodule of M and P any maximal ideal of R with $S(N) \subseteq P$. Let $\frac{r}{p} \in R_P, \frac{m}{q} \in M_P$ such that $\frac{rm}{pq} \in N_P - (N_P : M_P)N_P$. Note that $\{0_{M_P}\} \subseteq (N_P : M_P)N_P$, so $\frac{rm}{pq} \in N_P - (N_P : M_P)N_P \subseteq N_P - \{0_{M_P}\}$. Since N_P is weakly prime, then $\frac{r}{p} \in (N_P : M_P)$ or $\frac{m}{p} \in N_P$. Hence N is $S(N)$-almost prime submodule. □

Lemma 2.2. *Let M be an R-module and N be a proper submodule of M with $S(N) \subseteq P$ where P is a maximal ideal of R. Then $(N : M)M_P = (N_P : M_P)M_P = (N : M)_P M_P = ((N : M)M)_P$*

Proof. By [7, Theorem 2.21] $(N_P : M_P) = (N : M)_P$. Then $(N_P : M_P)M_P = (N : M)_P M_P$. We will prove $(N : M)M_P = (N : M)_P M_P$. Take arbitrary $m \in (N : M)M_P$. It means $m = \sum_{i=1}^{k} r_i \frac{m_i}{p_i}$ with $r_i \in (N : M)$, $m_i \in M, p_i \notin P \ \forall\ i = 1, 2, ..., k$. Since $1_R \notin P$, we have $m = \sum_{i=1}^{k} \frac{r_i}{1_R} \frac{m_i}{p_i} \in (N : M)_P M_P$. So $(N : M)M_P \subseteq (N : M)_P M_P$.

For another containment, take $\frac{m}{p} \in (N : M)_P M_P$. It means $\frac{m}{p} = \sum_{i=1}^{k} \frac{r_i}{q_i} \frac{m_i}{p_i}$ with $r_i \in (N : M), m_i \in M, p_i, q_i \notin P \ \forall\ i = 1, 2, ..., k$. Since $R - P$ is a multiplicatively closed set, then $q_ip_i \notin P$ $\forall i = 1, 2, ..., k$. So $\sum_{i=1}^{k} r_i \frac{m_i}{q_ip_i} \in (N : M)M_P$. Hence $((N : M)M)_P \subseteq (N : M)M_P$. So, we prove that $(N : M)M_P = ((N : M)M)_P$. We can easily prove $(N : M)_P M_P = ((N : M)M)_P$ by similar argument. Hence $(N : M)M_P = (N_P : M_P)M_P = (N : M)_P M_P = ((N : M)M)_P$. □

Theorem 2.3. *Let M be an R-module. Every almost prime submodule N of M is an $S(N)$-almost prime submodule.*

Proof. Let P be any maximal ideal of R such that $S(N) \subseteq P$. We will prove that N_P is a proper submodule of M_P. Assume the contrary, that $N_P = M_P$. Take arbitrary $m \in M$, it means $\frac{m}{1_R} \in M_P = N_P$, so $\exists p \notin P$ such that $pm \in N$. Since $S(N) \subseteq P$ and $p \notin P$, then p is prime to N. This give us $m \in N$ and $M \subseteq N$, contradiction. Hence N_P is a proper submodule of M_P.

Let $\frac{r}{p} \in R_P, \frac{m}{q} \in M_P$ such that $\frac{rm}{pq} \in N_P - (N_P : M_P)N_P$. So $\exists s \notin P$ such that $srm \in N$. We will prove $srm \in N - (N : M)N$. If $srm \in (N : M)N$, then $\frac{rm}{pq} = \frac{srm}{spq} \in ((N : M)M)_P$. By lemma 2.2, $\frac{rm}{pq} = \frac{srm}{spq} \in ((N : M)M)_P = (N_P : M_P)N_P$, a contradiction. Hence $srm \in N - (N : M)N$.

Since N is an almost prime submodule of M, we have $r \in (N : M)$ or $sm \in N$. If $r \in (N : M)$, then $\frac{r}{p}M_P = (rM)_P \subseteq N_P$ by [7, Corollary 2.9]. It means $\frac{r}{p} \in (N_P : M_P)$. If $sm \in N$, then $\frac{m}{q} = \frac{sm}{sq} \in N_P$. Hence N is an $S(N)$-almost prime submodule of M. □

Theorem 2.4. *Let N be a proper submodule of R-module M. If $S((N : M)N) \subseteq (N : M)$, then N is a prime submodule if and only if N is an almost prime submodule.*

Proof. It is clear that a prime submodule is an almost prime submodule. For the converse part, suppose N is an almost prime submodule. Take an arbitrary $r \in R$ and $m \in M - N$ such that $rm \in N$. Since $m \notin N$ it is sufficient to prove $r \in (N : M)$.

(1) Case 1: $rm \in N - (N : M)$. Since N is an almost prime submodule, then $r \in (N : M)$.
(2) Case 2: $rm \in (N : M)N$. Since we have $m \notin N$ and $rm \in (N : M)N$, then $r \in S((N : M)N) \subseteq (N : M)$

Both cases give us $r \in (N : M)$. It means N is a prime submodule of M. □

Theorem 2.5. *Let M be an R-module and P is a maximal ideal of R. Suppose $\bar{N}$ is a submodule of M_P. Consider $N = \{x \in M | \frac{x}{1_R} \in \bar{N}\}$. If $S((N : M)N) \subseteq P$ and $S(N) \subseteq P$, then N is a submodule of M and $\bar{N} = N_P$. Furthermore, if $\bar{N}$ is an almost prime submodule of M_P, then N is an almost prime submodule of M.*

Proof. By [7, Proposition 2.16] $N_P = \bar{N}$. Furthermore, suppose N_P is an almost prime submodule of M_P. We will prove $N \neq M$. Assume the contrary that $N = M$. By [7, Corollary 2.2] $N_P = M_P$, contradiction. So $N_P = \bar{N}$ is a proper submodule of M. Let $r \in R, m \in M$ such that $rm \in N - (N : M)N$.

We will prove $\frac{rm}{1_R} \in N_P - (N_P : M_P)N_P$. It is clear that $\frac{rm}{1_R} \in N_P$. If $\frac{rm}{1_R} \in (N_P : M_P)N_P$, then by Lemma 2.2 $\frac{rm}{1_R} \in (N_P : M_P)N_P = ((N : M)N)_P$. It means $\exists\ q \notin P$ such that $qrm \in (N : M)N$. Since $q \notin P$, it means $q \notin S((N : M)N)$. So $rm \in (N : M)N$, a contradiction. Hence $\frac{rm}{1_R} \in N_P - (N_P : M_P)N_P$.

Since $\bar{N} = N_P$ is an almost pime submodule, then $\frac{m}{1_R} \in N_P$ or $\frac{r}{1_R} \in (N_P : M_P)$.

(1) If $\frac{m}{1_R} \in N_P$, then $\exists s \notin P$ such that $sm \in N$. Note that $s \notin P$ implies $s \notin S(N)$. So $m \in N$.
(2) If $\frac{r}{1_R} \in (N_P : M_P)$, then $\frac{r}{1_R}M_P \subseteq N_P = \bar{N}$. Furthermore, $\forall x \in M$, $\frac{rx}{1_R} \in (rM)_P \subseteq N_P$. So $rx \in N\ \forall x \in M$ which means $r \in (N : M)$.

Hence N is an almost prime submodule of M. □

Lemma 2.6. *Let M be an R-module and N be a submodule of M. Then $(N : M)^2 \subseteq ((N : M)N : M)$.*

Proof. Take arbitrary $m \in M$ and $r \in (N : M)^2$, it means $r = \sum_{i=1}^{n} s_i t_i$ with $s_i, t_i \in (N : M), n \in \mathbb{N}$. Since $t_i \in (N : M)\ \forall i \in \{1, 2, ..., n\}$, then $t_i m \in N\ \forall i \in \{1, 2, ..., n\}$, so $rm = \sum_{i=1}^{n}(s_i t_i)m = \sum_{i=1}^{n} s_i(t_i m) \in (N : M)N$. Hence $(N : M)^2 \subseteq ((N : M)N : M)$. □

Theorem 2.7. *Let N be a submodule of R-module M and P be a maximal ideal of R with $S(N) \subseteq P$.*

(1) *If $(N : M)$ be an almost prime ideal of R,then $(N : M)_P$ is an almost prime ideal of R_P.*
(2) *If $(N : M)_P$ is an almost prime ideal of R_P, $S((N : M)N) \subseteq P$ and M is a multiplication module, then $(N : M)$ is an almost prime ideal of R.*

Proof. Suppose N be a submodule of M.

(1) Let $(N : M)$ be an almost prime ideal of R. First, we need to verify that $(N : M)_P \neq R_P$. If $(N : M)_P = R_P$, then $1_{R_P} = \frac{1_R}{1_R} \in (N : M)_P$. It means $\exists p \notin P$ such that $p \cdot 1_R \in (N : M)$. By [7, Lemma 2.7] $p \in (N : M) \subseteq S(N) \subseteq P$, contradiction. Hence $(N : M)_P \neq R_P$. Let $\frac{r}{p}, \frac{s}{q} \in (N : M)_P$ such that $\frac{rs}{pq} \in (N : M)_P - (N : M)_P^2$ with $r, s \in R$ and $p, q \notin P$. It means $\exists u \notin P$ such that $urs \in (N : M)$. Now, we need to prove that $urs \notin (N : M)^2$. Assume the contrary that $urs \in (N : M)^2$. It means $\frac{rs}{pq} = \frac{urs}{upq} \in (N : M)_P^2$, contradiction. So $urs \in (N : M) - (N : M)^2$. Note that $(N : M)$ is an almost prime ideal in R, so $ur \in (N : M)$ or $s \in (N : M)$. This implies $\frac{r}{p} = \frac{ur}{up} \in (N : M)_P$ or $\frac{s}{q} \in (N : M)_P$. Hence $(N : M)_P$ is an almost prime ideal in R_P.
(2) Suppose $(N : M)_P$ is almost prime ideal in R_P. It is clear that $(N : M)$ is a proper subset of R. If $(N : M) = R$ we have $(N : M)_P = R_P$, contradiction. Let $r, s \in (N : M)$ such that $rs \in (N : M) - (N : M)^2$. Since $1_R \notin P$, then $\frac{rs}{1_R} \in (N : M)_P$.

We will show that $\frac{rs}{1_R} \notin (N:M)_P^2$. If $\frac{rs}{1_R} \in (N:M)_P^2$,then $\exists q \notin P$ such that $qrs \in (N:M)^2$. By Lemma 2.6 $qrs \in (N:M)^2 \subseteq ((N:M)N:M)$.
Claim: $q \in S((N:M)N)$. Since $rs \notin (N:M)^2$ and M multiplication module, then $rsM \nsubseteq (N:M)^2M = (N:M)N$. It means $\exists m \notin (N:M)N$ such that $rsm \notin (N:M)N$, but $qrsm \in (N:M)N$. So $q \in S((N:M)N)$ this prove the claim. Furthermore, $q \in S((N:M)N) \subseteq P$, contradict the fact that $q \notin P$. Hence $\frac{rs}{1_R} \in (N:M)_P - (N:M)_P^2$. Since $(N:M)_P$ is an almost prime ideal in R_P, then $\frac{r}{1_R} \in (N:M)_P$ or $\frac{s}{1_R} \in (N:M)_P$. Claim that if $\frac{r}{1_R} \in (N:M)_P$, then $r \in (N:M)$. Assume the contrary that $r \notin (N:M)$, it means $\frac{r}{1_R} \notin (N:M)_P$, contradiction. So $r \in (N:M)$. By similar argument we have $\frac{s}{1_R} \in (N:M)_P$ implies $s \in (N:M)$. So $(N:M)$ is an almost prime ideal in R.

□

It is well known that R-module M is called finitely generated if it has a finite generating set, and M is called faithful if $ann(M) = \{0_R\}$, where $ann(M)$ is annihilator set of M. In the next theorem we will give the relation between faithful, multiplication and finitely generated module to its localization.

Theorem 2.8. *Let M be an R-module and P be any maximal ideal of R.*

(1) *If M is a faithful R-module, then M_P is a faithful R_P-module.*
(2) *If M is a multiplication R-module, then M_P is a multiplication R_P-module.*
(3) *If M is a finitely generated module R-module, then M_P is a finitely generated R_P-module.*

Proof. Let M be an R-module.

(1) Suppose M is faithful R-module. To prove M_P is faithful, we need to show $ann(M_P) = (\{0_{M_P}\} : M_P) = \{0_{R_P}\}$. Take arbitrary $\frac{r}{p} \in (\{0_{M_P}\} : M_P)$. Claim: If $\frac{r}{p} \in ann(M_P)$, then $r \in ann(M)$. To prove the claim, assume the contrary that $r \notin ann(M)$, it means $\exists m \in M$ such that $rm \neq 0_M$. Since $\frac{0_M}{1_R} \in M_P$, then $\frac{rm}{p \cdot 1_R} \neq \frac{0_M}{p}$, thus $\frac{r}{p} \notin ann(M_P)$, contradiction. This proves the claim. Furthermore, since $r \in ann(M)$ and M is faithful, it means $r = 0_R$. So $\frac{r}{p} = 0_{R_P}$. Hence M_P is faithful R_P-module.
(2) Suppose M is a multiplication module. Let $\bar{N}$ be arbitrary submodule of M_P. Note that M is a multiplication module. By Theorem 2.5, $\bar{N} = N_P$ with $N = (N:M)M$ submodule of M. Moreover, by [7, Lemma 2.19] and Lemma 2.2 $\bar{N} = N_P = ((N:M)M)_P \subseteq (N_P : M_P)M_P$. Furthermore, it is clear that $(N_P : M_P)M_P \subseteq N_P = \bar{N}$. So, M_P is a multiplication module.
(3) Suppose M is finitely generated with $B = \{b_i\}_{i=1}^n$ be the generator of M for $n \in \mathbb{N}$. We will prove that $B' = \{\frac{b_i}{1_R}\}_{i=1}^n$ is the generator of M_P. Take arbitrary $\frac{m}{p} \in M_P$, with $m \in M, p \notin P$. Since $m \in M$,

then $m = \sum_{i=1}^{n} r_i b_i$ with $r_i \in R, b_i \in M \ \forall i \in \{1, 2, ..., n\}$. So $\frac{m}{p} = \frac{1_R}{p} \sum_{i=1}^{n} r_i b_i = \sum_{i=1}^{n} \frac{r_i}{p} \frac{b_i}{1_R}$. So B' is the generator of M_P. Hence M_P is finitely generated.

□

Theorem 2.9. *Let M be a finitely generated faithful multiplication R-module and N be a proper submodule of M with $S((N : M)N) \subseteq P$. If N is an $S(N)$-almost prime submodule of M, then $(N : M)$ is an almost prime ideal of R.*

Proof. Let P be a maximal ideal of R with $S(N) \subseteq P$. By Theorem 2.8 M_P is a finitely generated faithful multiplication R_P-module. Since N is an $S(N)$-almost prime submodule, then N_P is an almost prime submodule of M_P. By [9, Theorem 3.5] $(N_P : M_P)$ is an almost prime ideal. Furthermore, $(N_P : M_P) = (N : M)_P$ by [7, Theorem 2.21]. So by Theorem 2.7, $(N : M)$ is an almost prime ideal of R. □

In the next two theorems, we will give some characterizations of $S(N)$-almost prime submodules related to its finite direct sum and tensor product.

Theorem 2.10. *Let M, M' are R-modules and N, N' are proper submodules of M and M' respectively. If $N \oplus N'$ is an $S(N \oplus N')$-almost prime submodule of $M \oplus M'$ and $S(N \oplus N') \subseteq S(N)$, then N is $S(N)$-almost prime submodule of M.*

Proof. Suppose P is any maximal ideal of R such that $S(N) \subseteq P$. Take arbitrary $\frac{r}{p}\frac{m}{s} \in N_P - (N_P : M_P)N_P$. We will prove that $\frac{r}{p} \cdot \frac{(m,0)}{s} \in (N \oplus N')_P - ((N \oplus N')_P : (M \oplus M')_P)(N \oplus N')_P$. Since $\frac{rm}{ps} \in N_P$ and $r \cdot 0 \in N'$, then $\frac{r}{p} \cdot \frac{(m,0)}{s} \in (N \oplus N')_P$.

We will prove $\frac{r}{p} \cdot \frac{(m,0)}{s} \notin ((N \oplus N')_P : (M \oplus M')_P)(N \oplus N')_P$. Assume the contrary that $\frac{r}{p} \cdot \frac{(m,0)}{s} \in ((N \oplus N')_P : (M \oplus M')_P)(N \oplus N')_P$. From Lemma 2.2, we have $((N \oplus N')_P : (M \oplus M')_P)(N \oplus N')_P = ((N \oplus N' : M \oplus M')(N \oplus N'))_P$. It means there exists $q \notin P$ such that $q(rm, 0) \in (N \oplus N' : M \oplus M')(N \oplus N')$. By [12, Lemma 3.2] and [12, Lemma 3.3], we have $(N \oplus N' : M \oplus M')(N \oplus N') \subseteq (N \oplus N' : M \oplus M')N \oplus (N \oplus N' : M \oplus M')N') \subseteq ((N : M)N \oplus (N' : M')N')$. So $(qrm, 0) = q(rm, 0) \in ((N : M)N \oplus (N' : M')N')$, which means $qrm \in (N : M)N$. It follows $\frac{rm}{ps} = \frac{qrm}{qps} \in ((N : M)N)_P = (N_P : M_P)N_P$, contradiction. Hence $\frac{r}{p} \cdot \frac{(m,0)}{s} \in (N \oplus N')_P - ((N \oplus N')_P : (M \oplus M')_P)(N \oplus N')_P$.

Since $N \oplus N'$ is $S(N \oplus N')$-almost prime submodule, then $\frac{r}{p} \in ((N \oplus N')_P : (M \oplus M')_P)$ or $\frac{(m,0)}{s} \in (N \oplus N')_P$.

Case 1: Suppose $\frac{r}{p} \in ((N \oplus N')_P : (M \oplus M')_P)$. By [12, Lemma 3.2] we have $\frac{r}{p} \in (N \oplus N' : M \oplus M')_P \subseteq (N : M)_P = (N_P : M_P)$.

Case 2: Suppose $\frac{(m,0)}{s} \in (N \oplus N')_P$. It means $\exists q \notin P$ such that $q(m, 0) \in N \oplus N'$. This implies $qm \in N$. Since $q \notin P$ and $S(N) \subseteq P$, then $m \in N$. Thus $\frac{m}{s} \in N_P$.

From case 1 and case 2, N is $S(N)$-almost prime submodule of M. □

In [6], David Einsenbud gave a relation of a localization with a tensor product. In the next theorem, we will give a relation between an $S(N)$-almost prime submodules with the tensor product. Proving this relation needs these following lemmas.

Lemma 2.11. *Suppose N is a proper submodule of R-module M and P is any maximal ideal of R such that $S(N) \subseteq P$ and $S((N:M)N) \subseteq P$. Then N_P is an almost prime submodule of R-module M_P. In other words, if N_P is an almost prime submodule of R_P-module M_P, then N_P is an almost prime submodule of R-module M_P.*

Proof. Take arbitrary $r \in R$, $\frac{m}{p} \in M_P$ with $m \in M$ and $p \notin P$ such that $r\frac{m}{p} \in N_P - (N_P : M_P)N_P$. Noted that $\frac{r}{1_R} \in R_P$. So $\frac{rm}{p} = \frac{r}{1_R}\frac{m}{p}$. We will prove that $\frac{rm}{p} \notin (N_P : M_P)N_P$. Assume the contrary that $\frac{rm}{p} \in (N_P : M_P)N_P = ((N : M)N)_P$. So $\exists q \notin P$ such that $qrm \in (N : M)N$. Since $q \notin P$ and $S((N : M)N)$ contained in P, then $rm \in (N : M)N$, a contradiction. Hence $\frac{rm}{p} \in N_P - (N_P : M_P)N_P$.
Since N is an $S(N)$-almost prime submodule of M, then N_P is an almost prime submodule of M_P as R_P-module. So $\frac{r}{1_R} \in (N_P : M_P)$ or $\frac{m}{p} \in N_P$.
We will prove that $\frac{r}{1_R} \in (N_P : M_P)$ (with N_P as submodule of R_P-module M_P) implies $r \in (N_P : M_P)$ (with N_P as a submodule of R-module M_P).
Assume the contrary that $r \notin (N_P : M_P)$. It means $\frac{r}{1_R} \notin (N : M)_P \subseteq (N_P : M_P)$, contradiction. So we have $r \in (N : M)$ or $m \in N_P$. Hence N_P is an almost prime submodule of R-module M_P. □

Corollary 2.12. *Suppose N is an $S(N)$-almost prime submodules of M. Then $N \otimes R_P$ is an almost prime submodule of $M \otimes R_P$ for all P maximal ideals of R with $S(N) \subseteq P$ and $S((N : M)N) \subseteq P$.*

Proof. By lemma 2.11, N_P is an almost prime submodule of R-module M_P. By [6, Lemma 2.4], $N_P \cong N \otimes R_P$ and $M_P \cong M \otimes R_P$. Hence $N_P \cong N \otimes R_P$ is an almost prime submodule of R-module $M_P \cong M \otimes R_P$. □

Next characterization of tensor product is hold for almost prime submodules but it is failed for $S(N)$-almost prime submodules. Proving this fact needs two following lemmas.

Lemma 2.13. *Suppose M, M' are R-modules and I is an arbitrary ideal of R. Then $I(M \otimes M') = (IM \otimes M') = (M \otimes IM')$.*

Proof. We will prove $I(M \otimes M') = (IM \otimes M')$, as for $I(M \otimes M') = (M \otimes IM')$ by using similar argument.
First, we will prove $I(M \otimes M') \subseteq (IM \otimes M')$. Take arbitrary $m \in I(M \otimes M')$, it means $m = \sum_{i=1}^{n} r_i(m_i \otimes n_i)$ with $r_i \in I$ and $m_i \in M, n_i \in M'$ $\forall i = 1, 2, ..., n$ and $n \in \mathbb{N}$. Then $m = \sum_{i=1}^{n} (r_i m_i) \otimes n_i \in (IM \otimes M')$. Hence $I(M \otimes M') \subseteq (IM \otimes M')$.
For other inclusion, take arbitrary $m \in (IM \otimes M')$, it means $m = \sum_{i=1}^{n} (r_i m_i) \otimes n_i$ with $r_i \in I$ and $m_i \in M, n_i \in M'$ $\forall i = 1, 2, ..., n$. By definition of tensor

product, we have $m = \sum_{i=1}^{n}(r_i m_i) \otimes n_i = \sum_{i=1}^{n} r_i(m_i \otimes n_i) \in I(M \otimes M')$. So $(IM \otimes M') \subseteq I(M \otimes M')$. □

Lemma 2.14. *Suppose M, M' are R-modules and N, N' are submodules of M and M' respectively. Then $(N \otimes N' : M \otimes M') \subseteq (N : M)$.*

Proof. Take arbitrary $r \in (N \otimes N' : M \otimes M')$ and $m \in M$. Note that $(m, 0) \in M \otimes M'$. So $rm \otimes 0 = r(m \otimes 0) \in N \otimes N'$. This implies $r \in (N : M)$. Hence $(N \otimes N' : M \otimes M') \subseteq (N : M)$. □

Theorem 2.15. *Suppose M, M' are R-modules and N, N' are proper submodules of M and M' respectively. If $N \otimes N'$ is an almost prime submodule of $M \otimes M'$, then N is an almost prime submodule of M.*

Proof. Take arbitrary $r \in R$ and $m \in M$ such that $rm \in N - (N : M)N$. Since $m \otimes 0 \in M \otimes M'$ and $0 \in N'$, then $rm \otimes 0 \in N \otimes N'$. We will prove that $rm \otimes 0 \notin (N \otimes N' : M \otimes M')(N \otimes N')$.
Assume the contrary that $rm \otimes 0 \in (N \otimes N' : M \otimes M')(N \otimes N')$. Then by Lemma 2.13 and Lemma 2.14, we have $rm \otimes 0 = r(m \otimes 0) \in (N \otimes N' : M \otimes M')(N \otimes N') \subseteq (N \otimes N' : M \otimes M')N \otimes N' \subseteq (N : M)N \otimes N'$, which implies $rm \in (N : M)N$, contradiction. Hence $rm \otimes 0 \in N \otimes N' - (N \otimes N' : M \otimes M')(N \otimes N')$. Since $N \otimes N'$ is an almost prime submodule, we have $r \in (N \otimes N' : M \otimes M')$ or $m \otimes 0 \in N \otimes N'$.
If $r \in (N \otimes N' : M \otimes M')$, then it is clear that $r \in (N \otimes N' : M \otimes M') \subseteq (N : M)$.
If $m \otimes 0 \in N \otimes N'$, then $m \in N$. Hence N is an almost prime submodule of M. □

3. S(N)-Almost Prime Submodule and Primal Submodule

In this section, we will give some characterizations of a submodule which both $S(N)$-almost prime and primal submodule. We will give the result of our observation about behavior of submodule which both $S(N)$-almost prime and primal including its relation to an almost prime submodule. We begin this section by proving this trivial but useful lemma.

Lemma 3.1. *Let M be an R-module and N be a primal submodule of M. Then $S(N)$ is a proper ideal of R.*

Proof. Assume the contrary that $S(N) = R$. It means $1_R \in S(N)$. Then $\exists m \in M - N$ such that $m = 1_R m \in N$, contradiction. □

Theorem 3.2. *Let M be an R-module and N is a primal submodule of M with $S((N : M)N) \subseteq S(N)$.*

(1) *If for any ideal I of R and any submodule K of M such that $IK \subseteq N - (N : M)N$ implies $I \subseteq (N : M)$ or $K \subseteq N$, then N is an $S(N)$-almost prime submodule of M.*
(2) *If M be a multiplication module and N is an $S(N)$-almost prime submodule of M, then for any ideal I of R and any submodule K of M such that $IK \subseteq N-(N : M)N$ implies $I \subseteq (N : M)$ or $K \subseteq N$.*

Proof. Suppose N is a primal submodule of M.

(1) Suppose the assumption in first point in theorem is true and let P be any maximal ideal of R such that $S(N) \subseteq P$. Since N is a proper submodule of M, by [7, Theorem 2.17] N_P is a proper submodule of M_P. Take arbitrary $\bar{I}$ ideal of R_P and $\bar{K}$ submodule of M_P such that $\bar{I}\bar{K} \subseteq N_P - (N_P : M_P)N_P$. By [7, Theorem 2.16] $\bar{I} = I_P$ and $\bar{K} = K_P$. We will prove $IK \subseteq N$. Take arbitrary $rk \in IK$, with $r \in I, k \in K$. It is clear that $\frac{r}{1_R} \in I_P$ and $\frac{k}{1_R} \in K_P$. Then $\frac{rk}{1_R} \in I_P K_P \subseteq N_P$. It means $\exists q \notin P$ such that $qrk \in N$. Since $q \notin P$ implies $q \notin S(N)$. Then $rk \in N$. So $IK \subseteq N$.
We will prove $IK \nsubseteq (N : M)N$. Assume the contrary that $IK \subseteq (N : M)N$, by [7, Proposition 2.1] and Lemma 2.2 $(IK)_P \subseteq ((N : M)N)_P = (N_P : M_P)M_P$, contradiction. So we have proven $IK \subseteq N - (N : M)N$. By the assumption of theorem, $I \subseteq (N : M)$ or $K \subseteq N$. Then by [7, Proposition 2.1], $I_P \subseteq (N : M)_P = (N_P : M_P)$ or $K_P \subseteq N_P$. Furthermore, by [9, Theorem 3.1] N_P is an almost prime submodule of M_P. It means N is an $S(N)$-almost prime submodule.

(2) Suppose N is $S(N)$-almost prime submodule and M be a multiplication module. Take arbitrary ideal I of R and K any submodule of M such that $IK \subseteq N-(N : M)N$. By Lemma 3.1, $S(N)$ is a proper ideal of R. It means $\exists P$ a maximal ideal of R such that $S(N) \subseteq P$. We will prove $I_P K_P \subseteq N_P - (N_P : M_P)N_P$. Since N is an $S(N)$-almost prime submodule, then N_P is an almost prime submodule of M_P. Note that $IK \subseteq N$ implies $I_P K_P = (IK)_P \subseteq N_P$. So it is suficient to show $I_P K_P \nsubseteq (N_P : M_P)N_P$. If $I_P K_P \subseteq (N_P : M_P)N_P$, then $\forall r \in I, k \in K$ we have $\frac{rk}{1_R} \in (N_P : M_P)N_P$. By Lemma 2.2, $\frac{rk}{1_R} \in (N_P : M_P)N_P = ((N : M)N)_P$. It means $\exists q \notin P$ such that $qrk \in (N : M)N$. Note that $q \notin P$ implies $q \notin S((N : M)N)$, so $rk \in (N : M)N$. Hence $IK \subseteq (N : M)N$, contradiction. Then we have proven that $I_P K_P \subseteq N_P - (N_P : M_P)N_P$.
By Theorem 2.9 and [9, Theorem 3.1], we have $I_P \subseteq (N_P : M_P)$ or $K_P \subseteq N_P$. If $K_P \subseteq N_P$, then by [7, Proposition 2.1] $K \subseteq N$. If $I_P \subseteq (N_P : M_P) = (N : M)_P$, then $I \subseteq (N : M)$.

□

The notion of multiplication between submodules that we have in multiplication module gives us two following theorem which have similar proof to Theorem 3.2.

Theorem 3.3. *Let M be a finitely generated faithfully multiplication R-module and N is a primal submodule of M with $S((N : M)N) \subseteq S(N)$. Then the following are equivalent.*

(1) *For any K, L submodules of M such that $KL \subseteq N - (N : M)N$ implies $K \subseteq N$ or $L \subseteq N$.*
(2) *N is an $S(N)$-almost prime submodules of M.*

Theorem 3.4. *Let M be a finitely generated faithfully multiplication R-module and N is a primal submodule of M with $S((N:M)N) \subseteq S(N)$.*

(1) *For any $m, m' \in M$ such that $mm' \subseteq N - (N:M)N$ implies $m \in N$ or $m' \in M$.*
(2) *N is an $S(N)$-almost prime submodules of M.*

Lemma 3.5. *Let M be an R-module and N be a proper submodule of M. Suppose T be a multiplicatively closed set of R. If N is a prime submodule of M, then $M_T/N_T \cong (M/N)_T$.*

Proof. Define $f : M_T \to (M/N)_T$ with $f(\frac{m}{t}) = \frac{m+N}{t}$.
We will prove that f is a mapping. Take arbitrary $\frac{m}{t}, \frac{m'}{t'} \in (M/N)_T$ with $\frac{m}{t} = \frac{m'}{t'}$. It means $\exists s \in T$ such that $s(t'm - tm') = 0_M$. Moreover, $st'm = stm'$. Then $f(\frac{m}{t}) = \frac{m+N}{t} = \frac{st'}{st'}\frac{m+N}{t} = \frac{st'm+N}{st't} = \frac{stm'+N}{stt'} = \frac{m'+N}{t'} = f(\frac{m'}{t'})$.
We will prove f is surjective. Take an arbitrary $\frac{m+N}{t}$. It means $\exists s \in T$ such that $(sm) + N = s(m+N) \in M/N$. It follows that $sm \in M$. Then $\exists \frac{m}{t} = \frac{sm}{st} \in M_T$.
It is trivial to prove that f is a module homomorphism. We will prove $ker(f) = N_T$. Suppose $\frac{n}{t} \in N_T$. Then $f(\frac{n}{t}) = \frac{n+N}{t} = \frac{N}{t} = 0_{(M/N)_T}$ which means $\frac{n}{t} \in ker(f)$. Hence, $N_T \subseteq ker(f)$. For another containment, suppose $\frac{m}{t} \in ker(f)$. Then $\frac{m}{t} = 0_{(M/N)_T}$. It means $\frac{m+N}{t} = \frac{N}{t'}$, for some $t \in T$. It follows there exists $s \in T$ such that $s(m+N) = N$. It means $s(m+N) = N$, then $sm \in N$. Since N is a prime submodule of M, then $r \in (N:M)$ or $m \in N$. Hence $\frac{m}{t} = \frac{sm}{st} \in N_T$. So both cases give us $\frac{m}{t} \in N_T$. Then $ker(f) = N_T$.
So, by first isomorphism theorem $M_T/N_T \cong (M/N)_T$. □

Theorem 3.6. *Let N both primal and almost prime submodule of R-module M with $S((N:M)N) \subseteq S(N)$. If K be a prime submodule of M with $K \subseteq N$ and $S(K) \subseteq S(N)$, then N/K is an $S(N/K)$-almost prime submodule of M/K.*

Proof. Since N is almost prime submodule of M, then N is an $S(N)$-almost prime submodule. Take arbitrary $r \in R, m \in M$ such that $rm \in N - (N : M)N$. Because N is a primal submodule, it means $S(N)$ is an ideal of R. Note that $S(N)$ is a proper ideal of R. If $S(N) = R$, then $1_R \in S(N)$. It means $\exists m \notin N$ such that $m = 1_R m \in M$, contradiction. Since $S(N)$ is a proper ideal of R, then $\exists P$ a maximal ideal of R such that $S(N) \subseteq P$. By [7, Proposition 2.1], $K \subseteq N$ implies $K_P \subseteq N_P$. Since N is an $S(N)$-almost prime submodule, then N_P is an almost prime submodule of M_P. By [9, Theorem 2.4], N_P/K_P is an almost prime submodule. By Lemma 3.5, $(N/K)_P$ is isomorphic to N_P/K_P which an almost prime submodule. Hence, N/K is an $S(N/K)-$almost prime submodule. □

Finally, by observing the behavior of $S(N)$-weakly prime and $S(N)$-almost prime submodule which primal, we have two following theorems.

Theorem 3.7. *Let M be an R-module and N is both $S(N)$-weakly prime submodule and primal submodule of M with $S(\{0_M\}) \subseteq S(N)$. Then N is a weakly prime submodule of M.*

Proof. Suppose N be an $S(N)$-weakly prime and primal submodule of M. Take arbitrary $r \in R, m \in M$ such that $0_M \neq rm \in N$. Because N is a primal submodule, then $S(N)$ is an ideal of R. This means $S(N)$ is a proper ideal of R. If $S(N) = R$, then $1_R \in S(N)$. It means $\exists m \notin N$ such that $m = 1_R m \in M$, contradiction. Since $S(N)$ is a proper ideal of R, then $\exists P$ a maximal ideal of R such that $S(N) \subseteq P$.
We need to show that $0_M \neq rm \in N$ implies $0_{M_P} \neq \frac{rm}{1_R} \in N_P$. It is clear that $\frac{rm}{1_R} \in N_P$. Assume the contrary that $\frac{r}{1_R} = 0_M$, it means $\exists q \notin P$ such that $qrm = 0_M$. Note that $q \notin P$ implies $q \notin S(\{0_M\})$. So $rm = 0_M$, contradiction. Hence, $0_{M_P} \neq \frac{rm}{1_R} \in N_P$.
Since N is an $S(N)$-weakly prime submodule of M, then N_P is a weakly prime submodule of M_P. Because $0_{M_P} \neq \frac{rm}{1_R} \in N_P$, then $\frac{r}{1_R} \in (N_P : M_P)$ or $\frac{m}{1_R} \in N_P$. If $\frac{m}{1_R} \in N_P$, then $\exists s \notin P$ such that $sm \notin N$. Noted that $s \notin P$ implies $s \notin S(N)$. It means $m \in N$.
If $\frac{r}{1_R} \in (N_P : M_P)$, then $\frac{r}{1_R} \in (N_P : M_P) = (N : M)_P$ by [7, Theorem 2.21]. Consequently $\exists t \notin P$ such that $tr \in (N : M)$. We will prove that $tr \in (N : M)$ implies $r \in (N : M)$. Assume the contrary that $r \notin (N : M)$. Then $\exists x \notin N$ such that $rx \notin N$, but $trx \in N$. It means $t \in S(N) \subseteq P$, contradiction. Hence $r \in (N : M)$. Then we have $m \in N$ or $r \in (N : M)$. So, N is a weakly prime submodule. $\square$

Theorem 3.8. *Let M be an R-module and N is both $S(N)$-almost prime submodule and primal submodule of M with $S((N : M)N) \subseteq S(N)$. Then N is an almost prime submodule of M.*

Proof. Suppose N be an $S(N)$-almost prime and primal submodule. Take arbitrary $r \in R, m \in M$ such that $rm \in N - (N : M)N$. To simplify it, we give the idea of the proof. Firstly, we will show that $S(N)$ is a proper ideal of R which means $\exists P$ a maximal ideal of R such that $S(N) \subseteq P$. Moreover, we will prove that $\frac{rm}{1_R} \in N_P - (N_P : M_P)N_P$. Since N is $S(N)$ almost prime we will have $\frac{r}{1_R} \in (N_P : M_P)$ or $\frac{m}{1_R} \in N_P$. Lastly, we will prove $r \in (N : M)$ or $m \in N$.
Since N is a primal submodule, then $S(N)$ is an ideal of R. We will prove $S(N)$ is a proper ideal of R. If $S(N) = R$, then $1_R \in S(N)$. So $\exists m \notin N$ such that $m = 1_R m \in N$, contradiction. So we have $S(N)$ is a proper ideal of R which implies $\exists P$ a maximal ideal of R such that $S(N) \subseteq P$. Furthermore, since N is an $S(N)$-almost prime submodule of M, then N_P is an almost prime submodule of M_P.
We will prove that $\frac{r}{1_R}\frac{m}{1_R} \in N_P - (N_P : M_P)N_P$. It is clear that $\frac{r}{1_R}\frac{m}{1_R} \in N_P$. Assume the contrary that $\frac{r}{1_R}\frac{m}{1_R} \in (N_P : M_P)N_P$. By Lemma 2.2, $\frac{rm}{1_R} = \frac{r}{1_R}\frac{m}{1_R} \in (N_P : M_P)N_P = ((N : M)N)_P$. It means $\exists q \notin P$ such that $qrm \in (N : M)N$. Since $q \notin P$, then $q \notin S((N : M)N)$. So $rm \in (N : M)N$, contradiction. This implies $\frac{r}{1_R}\frac{m}{1_R} \in N_P - (N_P : M_P)N_P$.
Since N_P is an almost prime submodule of M_P, then $\frac{r}{1_R} \in (N_P : M_P)$ or $\frac{m}{1_R} \in N_P$. If $\frac{m}{1_R} \in N_P$, then $\exists q \notin P$ such that $qm \in N$. Since $q \notin P$, it follows $m \in N$.
If $\frac{r}{1_R} \in (N_P : M_P)$, then by [7, Theorem 2.21] $\frac{r}{1_R} \in (N_P : M_P) = (N : M)_P$. It means $\exists t \notin P$ such that $tr \in (N : M)$. Now, we need to prove $r \in (N :$

M). If $r \notin (N : M)$, then $\exists x \notin N$ such that $rx \notin N$, but $trx \in N$. It means $t \in S(N) \subseteq P$, leads to a contradiction. So $r \in (N : M)$. Hence we have $r \in (N : M)$ or $m \in N$. It means N is an almost prime submodule. □

In his article, Hani A. Khasan showed a characterization of almost prime submodule as in following theorem. We will present his theorem and prove the submodule version of that theorem.

Theorem 3.9 (Theorem 2.5 in [9])**.** *Let M be an R-module and N be a proper submodule of M. The following are equivalent:*

(1) *N is an almost prime submodule*
(2) *For $r \in R - (N : M)$, $(N :< r >) = N \cup ((N : M)N :< r >)$*
(3) *For $r \in R - (N : M)$, $(N :< r >) = N$ or $((N : M)N :< r >)$*

Theorem 3.10. *Let M be an R-module and N be a proper submodule of M. The following are equivalent:*

(1) *N is an almost prime submodule.*
(2) *For $m \in M - N$, $(N : Rm) = (N : M) \cup ((N : M)N : Rm)$.*
(3) *For $m \in M - N$, $(N : Rm) = (N : M)$ or $((N : M)N : Rm)$.*

Proof. $1 \Rightarrow 2$. It is clear that $(N : M) \cup ((N : M)N : Rm) \subseteq (N : Rm)$. For other containment, take arbitrary $m \in M - N$. If $rm \in (N : M)N$, then $r \in ((N : M)N : Rm)$. If $rm \notin (N : M)N$, then $rm \in N - (N : M)N$. Since N is an almost prime submodule and $m \notin N$, it means $r \in (N : M)$. Hence $(N : Rm) \subseteq (N : M) \cup ((N : M)N : Rm)$. For $2 \Rightarrow 3$ it is well known that if an ideal is union of two ideals, then it is equal to one of them. For $3 \Rightarrow 1$. Take arbitrary $r \in R, m \in M$ such that $rm \in N - (N : M)N$ with $m \in M - N$. To prove N is almost prime submodule, it is sufficient to show $r \in (N : M)$. Since $rm \in N$, then $r \in (N : Rm)$ with $r \notin ((N : M)N : Rm)$. It follows that $((N : M)N : Rm) \neq (N : Rm)$. By the theorem assumption, $(N : Rm) = (N : M)$. So, $r \in (N : Rm) = (N : M)$. Then, N is an almost prime submodule of M. □

By combining Theorem 3.8, Theorem 3.10, Theorem 3.2, and [9, Theorem 2.5] the following theorem is obtained.

Theorem 3.11. *Let M be an R-module and N be a primal submodule of M. Then the following are equivalent:*

(1) *N is an $S(N)$-almost prime submodule of M.*
(2) *For $m \in M - N$, $(N : Rm) = (N : M) \cup ((N : M)N : Rm)$.*
(3) *For $r \in R - (N : M)$, $(N :< r >) = N \cup ((N : M)N :< r >)$.*
(4) *N is an almost prime submodule of M.*

4. S(I)-Almost Prime Ideals

In this section, we introduce similar notion of $S(N)$-almost prime submodules in ideals and we call it $S(I)$-almost prime ideals. It is well known that an ideal I of R is called prime ideal if $rs \in I$ implies $r \in I$ or $s \in I$. In 2003, D. D Anderson and Eric Smith gave a generalization of prime ideal that is: an ideal I of R is called weakly prime ideal if $0 \neq rs \in I$ implies $r \in I$ or $s \in I$ [3]. In 2005, Bhatwadekar and Sharma gave a more generalized notion that is: an ideal I of R is called almost prime if $rs \in I - I^2$ implies $r \in I$

or $s \in I$ [11]. In this section, we give some characterizations of $S(I)$-almost prime ideals and some remarks of proving idea of the characterization since it is analogue to prove in submodule version.

In this section we need some similar notions as in submodule. Let I be an ideal of ring R, $r \in R$ is called prime to I if $rs \in I$ implies $s \in I$. Furthermore, we can construct a set $S(I) = \{r \in R | \exists s \notin I \text{ such that } rs \in I\}$ which is the set of all elements in R that not prime to I. Moreover, I is called a primal ideal of R if $S(I)$ is an ideal of R [4].

Definition 4.1. *Let R be a ring. A proper ideal I of R is called $S(I)$-locally prime ideal if I_P is a prime ideal of R_P for each maximal ideal P with $S(I) \subseteq P$.*

Definition 4.2. *Let R be a ring. A proper ideal I of R is called $S(I)$-weakly prime submodule if I_P is a weakly prime ideal of R_P for each maximal ideal P with $S(I) \subseteq P$.*

Definition 4.3. *Let R be a ring. A proper ideal I of R is called $S(I)$-almost prime submodule if I_P is an almost prime ideal of R_P for each maximal ideal P with $S(I) \subseteq P$.*

Remark 4.1. *Let R be a ring and I be an ideal of R.*

(1) *If I is $S(I)$-locally prime ideal, then I is an $S(I)$-weakly prime ideal.*
(2) *If I is an $S(I)$-weakly prime ideal, then I is an $S(I)$-almost prime ideal.*
(3) *If I is an almost prime ideal, then I is an $S(I)$-almost prime ideal.*

Next remark is giving a relation of I as an ideal of R compared to I as a submodule when we consider R as an R-module. Using the remark, we can prove the characterization of $S(I)$-almost prime ideal which similar to submodule version in section 2 and section 3.

Remark 4.2. *Let R be a ring. Then the following statements are hold:*

(1) *$(I : R) = I$ for I any ideal of R.*
(2) *R is a multiplication R-module.*
(3) *If R is a faithful R-module, then R is an integral domain.*

Theorem 4.3. *Let I be a proper ideal of R and $S(I^2) \subseteq I$. Then I is prime ideal of R if and only if I is almost prime ideal.*

Theorem 4.4. *Let R be a ring. Suppose P is a maximal ideal of R with $S(I^2) \subseteq P$ and $S(I) \subseteq P$. If $\bar{I}$ is an ideal of R_P, then $I = \{a \in R | \frac{a}{1_R} \in \bar{I}\}$ is an ideal of R and $\bar{I} = I_P$. Furthermore, if $\bar{I}$ is an almost prime ideal of R_P, then I is an almost prime ideal of R.*

Theorem 4.5. *Let I be an ideal of R and P be a maximal ideal of R with $S(I) \subseteq P$.*

(1) *If I is an almost prime ideal of R, then I_P is an almost prime ideal of R_P.*
(2) *If I_P is an almost prime ideals of R_P and $S(I^2) \subseteq P$, then I is an almost prime ideal of R.*

Theorem 4.6. *Let R, R' be a ring and I, I' are ideals of R and R' respectively. If $I \oplus I'$ is an $S(I \oplus I')$-almost prime ideal of $R \oplus R'$ and $S(I \oplus I') \subseteq S(I)$, then I is an $S(I)$-almost prime ideal of R.*

Theorem 4.7. *Let R be a ring and I is both $S(I)$-almost prime and primal ideal of R with $S(I^2) \subseteq S(I)$. Then I is an almost prime ideal of R.*

5. S(N)-Almost Prime Submodules and S((N:M))-Almost Prime Ideals

In this section we gives a relation between elements that prime to a submodule N and element that prime to its residual ideal $(N : M)$. In addition, we give some characterizations related to $S(N)$-locally prime submodules with $S(I)$-locally prime ideals, $S(N)$-almost prime submodules and $S(I)$-almost prime ideals with $I = (N : M)$.

Lemma 5.1. *Let M be an R-module and N be a submodule of M. If $r \in R$ is prime to N, then r is prime to $(N : M)$. Equivalently $S((N : M)) \subseteq S(N)$.*

Proof. Suppose $r \in R$ is prime to N. Take arbitrary $s \in R$ such that $rs \in (N : M)$. To prove r is prime to $(N : M)$, we need to prove $s \in (N : M)$. Note that $rs \in (N : M)$ implies $rsm \in N$ for any $m \in M$. Since r is prime to N, we have $sm \in N$ for any $m \in M$, which means $sM \subseteq N$. So, $s \in (N : M)$. Hence r is prime to $(N : M)$. □

Note that the inclusion of $S(N) \subseteq S((N : M))$ is not generally true. In the next Lemma, we give two characterizations for having the true inclusion. For doing this, we need a notion of cancellation module that is: M an R-module is called cancellation if for all ideal I, J of R such that $IM = JM$ implies $I = J$.

Lemma 5.2. *Suppose M is an R-module and N is any submodule of M.*

(1) *If M is a cyclic module and $r \in R$ is prime to $(N : M)$, then r is prime to N. Equivalently if M is cyclic, then $S(N) \subseteq S((N : M))$.*
(2) *If M is finitely generated faithfully multiplication module and r is prime to $(N : M)$, then r is prime to N. Equivalently if M is finitely generated faithfully multiplication module, then $S(N) \subseteq S((N : M))$.*

Proof. Suppose N be a submodule of M.

(1) Suppose M is cyclic with $M = \langle m \rangle$. We will prove $S(N) \subseteq S((N : M))$. Take arbitrary $r \in S(N)$. It means exists $x \in M - N$ such that $rx \in N$. Since M is cyclic, we have $x = sm$ for some $s \in R$. Then $rsm = rx \in N$. Note that $s \notin (N : M)$. If $s \in (N : M)$, then $sm \in N$, a contradiction. It is clear that $rs \in (N : M)$. Since m is generator of M and $rsm \in N$, then we have $s \notin (N : M)$, but $rs \in (N : M)$. So $r \in S((N : M))$.
(2) Take arbitrary $r \in S(N)$. It means there exist $m \notin M$ such that $rm \in N$. Note that to prove $r \in S((N : M))$, we need $s \notin (N : M)$ such that $rs \in (N : M)$.
Since M is a multiplication module and $rm \in N$, then $r\langle m \rangle = r(\langle m \rangle : M)M \subseteq (\langle rm \rangle : M)M = \langle rm \rangle \subseteq N = (N : M)M$. One can easily se that M is a cancellation module. Then, $r(\langle m \rangle : M)M \subseteq$

$N = (N : M)M$ implies $r(\langle m\rangle : M) \subseteq (N : M)$. If $(\langle m\rangle : M) \nsubseteq (N : M)$, we have $s \notin (N : M)$ such that $rs \in (N : M)$.
We claim that $(\langle m\rangle : M) \nsubseteq (N : M)$. To prove the claim, suppose $(\langle m\rangle : M) \subseteq (N : M)$. It means $m \in \langle m\rangle = (\langle m\rangle : M)M \subseteq (N : M)M = N$. Hence $m \in N$, contradiction. So $(\langle m\rangle : M) \nsubseteq (N : M)$.
It is clear $(\langle m\rangle : M) \neq \varnothing$. So, we have $s \in (\langle m\rangle : M) - (N : M)$ such that $rs \in (N : M)$. Hence, $r \in S((N : M))$ which implies $S(N) \subseteq S((N : M))$.

□

In these following theorem and lemma, we abbreviate finitely generated failthfully multiplication module as $fgfm$ module. Two last lemmas give us some relation between $S((N : M))$ ideal and $S(N)$ submodule.

Theorem 5.3. *Let M be an R-module and N be a proper submodule of M.*

(1) *If $(N : M)$ is an $S((N : M))$-locally prime ideal and $R/(N : M)$-module M/N is torsion-free, then N is an $S(N)$-locally prime submodules of M.*
(2) *Suppose M is a cyclic or $fgfm$ modules. If N is an $S(N)$-locally prime submodules, then $(N : M)$ is an $S((N : M))$-locally prime ideal and $R/(N : M)$-module M/N is torsion-free.*

Proof. Suppose N be a submodule of M.

(1) Let P be any maximal ideal of R such that $S(N) \subseteq P$. We will prove N_P is a prime submodule of M_P. From Lemma 5.1, $S((N : M)) \subseteq S(N) \subseteq P$ and $(N : M)$ is an $S((N : M))$-locally prime ideal implies $(N : M)_P$ is a prime ideal. We will prove that $R_P/(N : M)_P$-module M_P/N_P is torsion-free. Take arbitrary $\frac{r}{p} + (N : M)_P \in R_P/(N : M)_P$, $0_{M_P/N_P} \neq \frac{m}{q} + N_P \in M_P/N_P$ such that $\frac{r}{p}\frac{m}{q} + N_P = (\frac{r}{p} + (N : M)_P)(\frac{m}{q} + N_P) = 0_{M_P/N_P} = N_P$. It means $\frac{r}{p}\frac{m}{q} + \in N_P$, so $\exists t \notin P$ such that $trm \in N$. Since $t \notin P$ implies $t \notin S(N)$, we have $(r + (N : M))(m + N) = (rm) + N = N$. Since $R/(N : M)$-module M/N is torsion-free and $\frac{m}{p} \neq 0_{M_P/N_P} = N_P$, then $r \in (N : M)$. It means $\frac{r}{p} \in (N : M)_P$ and M_P/N_P is torsion-free. From [4, Theorem 2.2], N_P is prime. Hence N is an $S(N)$-locally prime submodule of M.
(2) Let M be a cyclic or $fgfm$ R-module. Take arbitrary P maximal ideal of R such that $S(N) \subseteq R_P$. By Lemma 5.2 , we have $S(N) \subseteq S((N : M)) \subseteq P$. Then N_P is a prime submodule of R_P-module M_P. By [4, Theorem 2.2], $(N : M)_P$ is prime ideal of R_P. So $(N : M)$ is an $S((N : M))$-locally prime ideal of R.
We need to prove $R/(N : M)$-module M/N is torsion-free. Take arbitrary $r + (N : M) \in R/(N : M)$ and $N = 0_{M/N} \neq m + N \in M/N$ such that $(rm) + N = (r + (N : M))(m + N) = 0_{M/N} = N$. It means $rm \in N$. Clearly $\frac{r}{1_R}\frac{m}{1_R} \in N_P$. Since N is an $S(N)$-locally prime submodule, we have $\frac{r}{1_R} \in (N : M)_P$ or $\frac{m}{1_R} \in N_P$. The case $\frac{m}{1_R} \in N_P$ implies $\exists t \notin P$ and $tm \in N$. Since $t \notin S(N)$, we have $m \in N$, that leads to a contradiction. So $\frac{r}{1_R} \in (N : M)_P$. It follows $\exists t \notin P$ such that $tr \in (N : M)$. Note that $t \notin S((N : M))$ implies $r \in (N : M)$.

It means $r + (N : M) = (N : M) = 0_{R/(N:M)}$. Hence M/N is torsion-free.

□

Note that every cyclic module is finitely generated multiplication module. Using this fact, we are able to give the relation between $S(N)$-weakly prime submodules with $S((N : M))$-weakly prime ideals and $S(N)$-almost prime submodules with $S((N : M))$-almost prime ideals which are represented in the following theorems. Since the proof are similar, we only give the proof for $S(N)$-almost prime submodule version. The theorem which support the proof of weakly prime version can be found in [2], [7] and [5].

Theorem 5.4. *Let M be an R-module and N be a proper submodule of M.*

(1) *If M is a multiplication module and N is an $S(N)$-weakly prime submodules of M, then $(N : M)$ is an $S((N : M))$-weakly prime ideal of R.*
(2) *If M is a faithfully cyclic or $fgfm$ module and $(N : M)$ is an $S((N : M))$-weakly prime ideal of R, then N is an $S(N)$-weakly prime submodules of M.*

Theorem 5.5. *Let M be an R-module and N be a proper submodule of M.*

(1) *If M is a multiplication module and N is an $S(N)$-almost prime submodules of M, then $(N : M)$ is an $S((N : M))$-almost prime ideal of R.*
(2) *If M is a faithfully cyclic or $fgfm$ module and $(N : M)$ is an $S((N : M))$-almost prime ideal of R, then N is an $S(N)$-almost prime submodules of M.*

Proof. Suppose N be a submodule of M.

(1) Suppose M be a multiplication module and N be a proper submodule of M. By Theorem 2.8, M_P is a multiplication R_P-module. Let P be any maximal ideal such that $S(N) \subseteq P$. By Lemma 5.1, $S((N : M)) \subseteq S(N) \subseteq P$. Since $(N : M)$ is an $S((N : M))$-almost prime ideals of R, we have $(N : M)_P$ is an almost prime ideal of R_P. From [7, Theorem 2.21], $(N_P : M_P) = (N : M)_P$. Furthermore, by [9, Theorem 3.5] N_P is an almost prime submodule. Hence, N is an $S(N)$-almost prime submodule.
(2) Let M be a faithfully cyclic or $fgfm$ R-module. Both of them implies M is finitely generated faithfully multiplication R-module. Based on Lemma 2.8, M_P is finitely generated faithfully multiplication R_P-module. Suppose P is any maximal ideal of R such that $S((N : M)) \subseteq P$. By Lemma 5.2, $S(N) \subseteq S((N : M)) \subseteq P$. Since N is an $S(N)$-almost prime submodule, then N_P is almost prime submodule. By [9, Theorem 3.5], $(N_P : M_P)$ is an almost prime ideal. By [7, Theorem 2.21], $(N_P : M_P) = (N : M)_P$ is an almost prime ideals. Hence, $(N : M)$ is an $S(N : M)$-almost prime ideal.

□

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