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COMMON FIXED POINT THEOREMS FOR NEW CONTRACTION WITHOUT CONTINUITY COMPLETENESS AND COMPATIBILITY PROPERTY IN PARTIALLY ORDERED FUZZY METRIC SPACES

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Abstract. After the inspirational innovation of fuzzy set by Zadeh in 1965, Kramosil and Michàlek in 1975 pioneered the concept of fuzziness in metric spaces and very first they formulated the notion of fuzzy metric spaces. Jungck introduced the idea of commutativity (in 1976) and compatibility (1986) in metric spaces and same are utilized by Subrahmanyam (in 1995) in fuzzy metric spaces to prove an analogues version of Jungck result. In this paper, we prove common fixed point theorems for a pair of self-maps by introducing a new contraction which neither requires completeness of spaces nor continuity and compatible property of maps. An example is given in support of our main result.

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1. Introduction and preliminaries

Fixed point theorems are one of the most productive and successful apparatus in Mathematics which has massive applications inside as well as the outer side of the Mathematics. Even with noted improvements in computer expertise and its extraordinary achievement in facilitating countless areas of research, there still stand some major shortcomings: Computers are not intended to knob the situations wherein uncertainties are concerned. So to tackle uncertainty, we call for techniques other than traditional ones wherein some specific logic are required. Zadeh [9] led the foundation of fuzzy set to overcome the above stated problem. This hypothesis offers an approach to symbolize the ambiguity in everyday life.

Fuzzy set logic was also utilized in sense of metric spaces by Kramosil and Michalek [7] and after 20 years it was again modified by George and Veeramani [1] for the purpose of introducing Hausdorff topology in the fuzzy metric space.

Grabiec [10] proved the Banach result as an analogue in fuzzy metric spaces. This result was a breakthrough in mounting the fixed point theorems in FMS. Grabiec [10] first defined the convergence in the fuzzy metric spaces and then proved a fixed point result.

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Definition 1.1. [10]

(1) A sequence $\{x_n\}$ in a fuzzy metric space $(Z, M, *)$ is said to be convergent to $x \in Z$, if $\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \forall\, t > 0$.

(2) A sequence $\{x_n\}$ in a fuzzy metric space $(Z, M, *)$ is called Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \forall\, t > 0$ and each $p > 0$.

(3) A fuzzy metric space $(Z, M*)$ is said to be complete if every Cauchy sequence in Z converges in Z.

Lemma 1.1. [10] *Let $\{y_n\}$ be a sequence in an FM-space Z. If there exists a positive number $k < 1$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \ \forall\, t > 0, \ n \in N,$$

then $\{y_n\}$ is a Cauchy sequence.

Lemma 1.2. [10] *If for two points x, y of Z and for a positive number $k < 1$,*

$$M\,(x, y, kt) \geq M\,(x, y, t)\,,$$

then $x = y$.

Lemma 1.3. [10] *For all, $x, y \in Z$, $M\,(x, y, .)$ is non-decreasing.*

Theorem 1.4. [10] *Let $(Z, M, *)$ be a complete fuzzy metric spaces such that*

(1) $\lim_{t\to\infty} M(x, y, t) = 1$,

(2) $M(Fx, Fy, kt) \geq M(x, y, t)$

for all $x, y, \in Z$, where $0 < k < 1$, then F has a unique fixed point.

In 1994, Mishra [15] gave the idea of compatible mappings in fuzzy metric space and proved a lemma and a common fixed point theorem. In 1995, Subrahmanyam[13] extended the result of Jungcks [6] from complete metric spaces to fuzzy metric spaces as follows:

Theorem 1.5. [13] *Let $(Z, M, *)$ be a complete metric space and let $F, G : Z \to Z$ be maps that satisfy the following condition :*

(i) $G(Z) \subseteq F(Z)$,

(ii) F is continuous,

(iii) $M(Gx, Gy, \alpha t) \geq M(Fx, Fy, t)$ for all x, y, in Z and $0 < \alpha < 1$.

Then F and G have a unique common fixed point provided F and G commute.

In 2009, Abbas et al. [11] proved a theorem which provides necessary conditions for the existence of common fixed point of four non-compatible maps in a fuzzy metric space. In 2010, Mihet[5] employing the notion of (EA)- property in fuzzy metric spaces and proved a common fixed point theorem. Gupta and Mani[21, 24] proved some fixed point theorems in fuzzy metric spaces through rational inequality. Some more results of fixed point theorem in fuzzy metric spaces can be found in ([12, 22, 23]) and references there in.

In 2004-05, Ran and Reurings [2] and Nieto and Lopez [8] proved some new results for contractions in partially ordered metric spaces. The main idea was to depict iterative technique in the contraction mapping principle with those in the monotone technique. Agarwal et al. [14], Bhaskar and Lakshmikantham [19], Altun et al [3] and Lakshmikantham and Ciric [20] further investigated some fixed points results in the area of ordered metric spaces. In 2010, Shakeri et al. [16] introduced partially ordered $\mathcal{L}$ -fuzzy metric spaces and proved a common fixed point theorem in these spaces. Vaezzadeh and Vaezpour[17] gave a notion of generalized weakly contraction mappings in partially ordered fuzzy metric spaces and established a coincidence point theorem on this spaces. Recently, Gupta et al. [25] proved a new fixed point theorem satisfying generalized Geraghty type contraction for pair of self mappings in partially ordered metric spaces. Choudhury et al. [4] and Wang [18] also established some coupled coincidence point and fixed point results for compatible mappings in partially ordered fuzzy metric spaces under weaker contractions.

Main aim of this paper is to replace a new and a special type of contraction to get a unique common fixed point result for pair of self maps in partially ordered fuzzy metric spaces without using the hypothesis of completeness of space, and without continuity and compatible property of maps.

2. Main results

Definition 2.1. [3] *Let $(E, \preceq)$ be a partially ordered set. Two mappings $F, G : E \to E$ are said to be weakly increasing if $Fx \preceq GFx$ and $Gx \preceq FGx$ for all $x \in E$.*

Theorem 2.1. *Let $(M, E, *, \preceq)$ be a partially ordered fuzzy metric space with continuous t -norms $a * b = ab$. Let $F, G : E \to E$ be weakly increasing mappings of E. Also, suppose that for every comparable pair $(x, y) \in E$*

$$\pi(M(Fx, Gy, t)) \geq \theta\Big(\frac{M(y, Gy, t)M(x, Fx, t)}{M(x, y, t)}\Big), \tag{1}$$

where $\pi, \theta : [0, 1] \to [0, 1]$ are continuous functions such that $\pi(1) = \theta(1) = 1$, $\pi(0) = \theta(0) = 0$, $\pi(r) > r$, $\theta(r) > r$ and are such that

$$\theta(r) > \pi(r), \quad \forall \quad 0 < r < 1. \tag{2}$$

Then F and G have a common fixed point.

Proof. Let $x_0 \in E$ be any arbitrary point. Define $Fx_0 = x_1$ and $Gx_1 = x_2$. Continuing this way, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in E, such that

$$x_{2n+1} = Fx_{2n} = y_{2n}, \quad x_{2n+2} = Gx_{2n+1} = y_{2n+1}, \quad \forall\, n \in N. \tag{3}$$

Since F and G are weakly increasing function

$$y_0 = x_1 = Fx_0 \preceq GFx_0 = Gx_1 = y_1 = x_2 \cdots.$$

Continuing this process, we obtain

$$y_0 \preceq y_1 \preceq y_2 \cdots \preceq y_{2n} \preceq y_{2n+1} \preceq \cdots.$$

Suppose that,

$$y_{2n} = y_{2n+1} \; for \; no \; n \in N. \tag{4}$$

Since x_{2n} and x_{2n+1} are comparable, we can use (1) and (2) to write

$$\begin{aligned}\pi(M(y_{2n}, y_{2n+1}, t)) &= \pi(M(Fx_{2n}, Gx_{2n+1}, t))\\ &\geq \theta\Big(\frac{M(x_{2n+1}, Gx_{2n+1}, t)M(x_{2n}, Fx_{2n}, t)}{M(x_{2n}, x_{2n+1}, t)}\Big)\\ &\geq \theta\Big(\frac{M(x_{2n+1}, x_{2n+2}, t)M(x_{2n}, x_{2n+1}, t)}{M(x_{2n}, x_{2n+1}, t)}\Big)\\ &\geq \theta(M(y_{2n}, y_{2n+1}, t))\\ &> \pi(M(y_{2n}, y_{2n+1}, t)),\end{aligned}$$

which is a contradiction. Thus (4) is false and so $y_{2n} = y_{2n+1}$ for some $n \in N$, say $n = k$. Consequently, with $y_{2k} = y_{2k+1}$ and $u = x_{2k+1}$, we get $Gu = u$, by (3). This proves that u is a fixed point of G.
Next, we prove that any fixed point of G is also a fixed point of F.
Suppose not, *i.e* $Fu \neq u$.
If we take $x = y = u$ in (1), we get

$$\begin{aligned}\pi(M(Fu, u, t)) = \pi(M(Fu, Gu, t)) &\geq \theta(\frac{M(u, Gu, t)M(u, Fu, t)}{M(u, u, t)})\\ &\geq \theta(\frac{M(u, u, t)M(u, Fu, t)}{M(u, u, t)})\\ &= \theta(M(u, Fu, t)) > \pi(M(u, Fu, t)),\end{aligned}$$

which is a contradiction. This means that $Fu = u$ for all $u \in E$. Therefore, we get $Fu = Gu = u$, that is, u is a common fixed point of F and G.

□

Remark 1. *It should be noted from (1) that if v is a fixed point of G, then $Fu = v$ for all u comparable with v.*

In order to obtain uniqueness of fixed point, we add following condition in Theorem 2.1.

(5)
For each $x, y \in E$, there exists $z \in E$ which is comparable to x and y.

Theorem 2.2. *Adding the condition (5) to the hypotheses of Theorem 2.1 and further assume that*

(6) $$M(x, y, t) \cdot M(x, y, t) \leq 1.$$

Then the self maps F and G have a unique common fixed point.

Proof. Suppose on contrary that $u \neq v$ are two fixed points of F and G such that $Fu = Gu = u$ and $Fv = Gv = v$. Now we discuss here two different cases.

Case 1. If u is comparable to v, then $Fu = u$ is comparable to $v = Gv$. Also from above Remark 1, $Fu = v$.

Hence, from (1)

$$\begin{aligned}\pi(M(u,v,t)) &= \pi(M(Fu,Gv,t))\\ &\geq \theta\Big(\frac{M(v,Gv,t)M(u,Fu,t)}{M(u,v,t)}\Big)\\ &= \theta\Big(\frac{M(v,v,t)M(u,v,t)}{M(u,v,t)}\Big) = \theta(1) = 1.\end{aligned}$$

We therefore have $M(u,v,t) = 1$, a contradiction to our assumption, thus $u = v$.

Case 2. If u is not comparable to v, then there exists $w \in X$ comparable to u and v such that $Gw = w$ is comparable to $u = Gu$ and $Fv = v$, so from (1)

$$\begin{aligned}\pi(M(v,Gw,t)) &= \pi(M(Fv,Gw,t))\\ &\geq \theta\Big(\frac{M(w,Gw,t)M(v,Fv,t)}{M(v,w,t)}\Big)\\ &= \theta\Big(\frac{M(w,w,t)M(v,v,t)}{M(v,w,t)}\Big)\\ &> \frac{1}{M(v,w,t)}. \end{aligned} \tag{7}$$

Since $\pi(r) > r$, implies

$$\pi(M(v,Gw,t)) > M(v,Gw,t) = M(v,w,t). \tag{8}$$

From (7) and (8), we get

$$M(v,w,t) > \frac{1}{M(v,w,t)}. \implies M(v,w,t)\cdot M(v,w,t) > 1 \tag{9}$$

which is contradiction to our assumption (see eq. 6). Thus $M(v,Gw,t) = 1$, *i.e.* $Gw = v$. In similar manner, we can prove that $Gw = u$.

Since the function G is well defined function therefore, $u = v$. This finishes the proof of Theorem 2.2. □

We give an example in support of our finding.

Example 1. *Let $E = \mathcal{W}$ (set of Whole number) and d be the usual metric $d(x,y) = |x-y|$ for all $x,y \in E$. We define partial order $\preceq$ on E as $x \preceq y$ if and only if $y \leq x$ for all $x,y \in E$.*
Define fuzzy metric as

$$M(x,y,t) = \frac{t}{t+d(x,y)}$$

Then clearly, $(E,M,*,\preceq)$ is a partially ordered fuzzy metric spaces.
Consider two maps $F,G : E \to E$, defined as

$$Fx = x; \qquad Gx = 0,\ \forall\ x \in E.$$

Clearly F and G are weakly increasing mappings.
Let $\pi,\theta : [0,1] \to [0,1]$ be defined as:

$$\pi(r) = r; \qquad \theta(r) = r+1 \quad \pi(1) = \theta(1) = 1, \pi(0) = \theta(0) = 0.$$

Then for all $r \in (0,1)$, $\theta(r) > \pi(r)$, *Without loss of generality, if we assume that* $x > y$, *then clearly all the condition of Theorem 2.1 are satisfied. Also,* $F(0) = 0 = G(0)$ *and thus* 0 *is the only one common fixed point of* F *and* G.

3. Conclusion

In this paper, the contraction given in Theorem 2.1 is particularly constructed for pair of maps so that on omitting the completeness, continuity and compatibility property, still we get a common fixed point. Moreover, on adding an additional condition (5) and (6) in the hypothesis of Theorem 2.1, we get a unique common fixed point. An Example is given to justify the result.

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