Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 1. pp. 15 – 23

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.1.15

THE LINEAR MODEL FOR WAVE GENERATION OF A BUMP

AZIS S. SANI AND L.H. WIRYANTO

ABSTRACT. A channel with a bump over flat bottom will affect the profile of the water surface that flows over it. This phenomenon can be modeled by the nondimensional Boussinesq equation. The governing equation used is expressed in variable of speed, elevation, the height of bump and Froude number. Here, we will investigate the dependance of wave formed by a bump on Froude number and dimension of the bump. In this paper, we focus only on the case of supercritical flow or Froude number greater than 1. We will derive the analytical solution for the free surface prole over a bump. From the analytical solution, we conclude that a bump generate waves with 1 peak and 2 dales. The formed peak does not propagate anywhere, but the two dales propagate to the direction of the flow with different speed and amplitude. Forward Time Backward Space (FTBS) method is implemented to solve the equation numerically. The obtained numerical results confirm the analytical solution well.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 76B07, 76B15, 76M20.

KEYWORDS AND PHRASES. Boussinesq Equation, Froude Number, FTBS Method.

1. PROBLEM DESCRIPTION AND MODEL

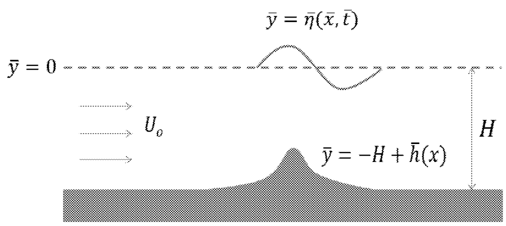

FIGURE 1. Side view of the channel

Consider a 2-D fluid flows into a channel which has a bump on its base as seen in Figure 1. In the figure we set the horizontal $\bar{x}$-axis along the undisturbed free surface and in line with dashed line $\bar{y} = 0$. The vertical $\bar{y}$-axis is perpendicular to the horizontal axis. The surface elevation is written as $\bar{y} = \bar{\eta}(\bar{x}, \bar{t})$. The base of channel forms a bump and follows the function $\bar{y} = -H + \bar{h}(\bar{x})$.

Far upstream, as $\bar{x} \to -\infty$, the flow is uniform with velocity U_0 and depth H. The fluid flow then disturbed by the bump. Our problem here is to describe the wave generation affected by the bump. An analytical approach had been done by Wiryanto and Mungkasi [1] to find the model. The fluid is assumed to be ideal and irrotational so that we can present the fluid flow in terms of potential function $\bar{\phi}$. This potential function is satisfying certain conditions, those are

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} = 0 \tag{1}$$

in the flow domain $-\infty < \bar{x} < \infty$, $-H + \bar{h}(\bar{x}) < \bar{y} < \bar{\eta}(\bar{x}, \bar{t})$; the condition

$$\bar{\phi}_{\bar{y}} - \bar{\eta}_{\bar{x}}\bar{\phi}_{\bar{x}} - \bar{\eta}_{\bar{t}} = 0, \tag{2}$$

$$\bar{\phi}_{\bar{t}} + \frac{1}{2}(\bar{\phi}^2_{\bar{x}} + \bar{\phi}^2_{\bar{y}}) + g\bar{\eta} = \frac{1}{2}U_0^2 \tag{3}$$

along the surface $\bar{y} = \bar{\eta}(\bar{x}, \bar{t})$, and

$$\bar{\phi}_{\bar{y}} - \bar{\phi}_{\bar{x}}\bar{h}_{\bar{x}} = 0, \tag{4}$$

along the bottom of channel $\bar{y} = -H + \bar{h}(\bar{x})$.

Since far upstream ($\bar{x} \to -\infty$) the flow is uniform, written as $\bar{\phi} = U_0\bar{x}$, we can write the potential function as a perturbation of the uniform flow, in terms of

$$\bar{\phi} = U_0\bar{x} + \bar{\Phi}(\bar{x}, \bar{y}, \bar{t}) \tag{5}$$

with $\bar{\Phi}$ is potential function of perturbation. Therefore, with (5), the governing equation (1)-(4) can be stated in terms of the $\bar{\Phi}$.

The next step is to do a scaling to all variables so that we will get some small parameters. In other side, we can expand the potential function of perturbation Φ in series of those small parameters. The solution of the series is obtained in terms of those small parameters. With using depth average velocity u we then obtain the Boussinesq-type model :

$$\eta_t + F(\eta_x + u_x - h_x) + \epsilon F(\eta_x u_{xx} + u_x\eta - u_x h) = 0, \tag{6}$$

$$u_t + Fu_x + \frac{1}{F}\eta_x + \epsilon F u u_x = 0. \tag{7}$$

with Froude number $F = U_0/\sqrt{gH}$ is parameter that describes the strength of fluid flowing into the channel and g is gravitational acceleration. Variable η is the elevation of surface, u is the velocity of flow, and h is the height of bump.

In this model all variables are given in non-dimensional and scaled. Parameter ϵ presents the ratio between amplitude and uniform depth H. In case that the amplitude is much smaller than the uniform depth, $\epsilon \to 0$, we will get the linear model

$$\eta_t + F(\eta_x + u_x - h_x) = 0, \tag{8}$$

$$u_t + Fu_x + \frac{1}{F}\eta_x = 0 \tag{9}$$

which can be solved analytically and numerically.

2. Analytical Solution

The analytical solution consists both of steady and transient solution. The steady solution is related to the condition of elevation and velocity which are not change with time while the transient solution is change. Suppose that the complete analytical solution for both elevation and velocity is :

$$\eta(x,t) = \eta_0(x,t) + \eta_1(x,t) \tag{10}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) \tag{11}$$

2.1. **Steady Solution.**

The steady solution can be obtained with substituting $\eta_t = 0$ and $u_t = 0$ to (8)-(9) so we have :

$$\eta_x + u_x - h_x = 0 \tag{12}$$

$$Fu_x + \frac{1}{F}\eta_x = 0 \tag{13}$$

The value $\eta_t = 0$ and $u_t = 0$ is valid for this case because the steady solution states the condition of elevation and velocity that do not change with time after long run. With applying ordinary substitution and elimination technique to (12)-(13) we get the steady solution as below :

$$\eta_0(x,t) = \frac{F^2}{F^2-1}h(x), \tag{14}$$

$$u_0(x,t) = \frac{-1}{F^2-1}h(x). \tag{15}$$

2.2. **Transient Solution.**

To find the transient solution we must substitute the complete solution (10)-(11) and steady solution (14)-(15) to the model (8)-(9) so we have :

$$\eta_{1t} + (F\eta_{1x} + u_{1x}) = 0, \tag{16}$$

$$u_{1t} + Fu_{1x} + \frac{1}{F}\eta_{1x} = 0. \tag{17}$$

We can simplify (16)-(17) to be $s_t = As_x$ with :

$$s = [\eta_1, u_1]^T, \text{ and}$$

$$A = \begin{bmatrix} -F & -F \\ \frac{-1}{F} & -F \end{bmatrix}.$$

Note that the equation $s_t = As_x$ is still in coupled condition so we must change it to be uncoupled first to ease our process in finding the solution.

We know that matrix A has eigenvalues $\lambda_1 = 1-F$ and $\lambda_2 = -1-F$ with corresponding eigenvectors respectively $X_1 = [-F,1]^T$ and $X_2 = [F,1]^T$. Therefore, we can make a relation $s = Py$ with P is a 2x2 matrix consists of eigenvectors and y is a 2x1 matrix consists variable y_1 dan y_2 that we will find later :

$$P = \begin{bmatrix} -F & F \\ 1 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Afterwards we substitute $s = Py$ to $s_t = As_x$:

$$\begin{aligned} (Py)_t &= A(Py)_x \\ Py_t &= APy_x \\ y_t &= P^{-1}APy_x \\ y_t &= Dy_x \end{aligned}$$

with D is a diagonal matrix that consists of eigenvalues of A :

$$D = \begin{bmatrix} 1-F & 0 \\ 0 & -1-F \end{bmatrix}.$$

Note that $y_t = Dy_x$ states uncoupled equations :

(18) $$y_{1t} = (1-F)y_{1x},$$

(19) $$y_{2t} = (-1-F)y_{2x}$$

that have solution respectively :

(20) $$y_1(x,t) = f(\frac{1}{F-1}x - t),$$

(21) $$y_2(x,t) = g(\frac{1}{F+1}x - t)$$

with f and g are arbitrary functions that we will find next.

From relation $s = Py$, (20)-(21), and steady solution (14)-(15), now we have complete solution as below :

(22) $$\eta(x,t) = \frac{F^2}{F^2-1}h(x) - Ff(\frac{1}{F-1}x - t) + Fg(\frac{1}{F+1}x - t)$$

(23) $$u(x,t) = \frac{-1}{F^2-1}h(x) + f(\frac{1}{F-1}x - t) + g(\frac{1}{F+1}x - t)$$

Substitute the initial value $\eta(x,0) = 0$ and $u(x,0) = 0$ to (22)-(23). These initial value is valid because in $t = 0$ there is no stream entering the channel so the surface is still flat and the velocity is zero. With this step we get :

(24) $$-Ff(\frac{1}{F-1}x) + Fg(\frac{1}{F+1}x) = -\frac{F^2}{F^2-1}h(x),$$

(25) $$f(\frac{1}{F-1}x) + g(\frac{1}{F+1}x) = \frac{1}{F^2-1}h(x).$$

From (24)-(25) we have :

$$f(\frac{1}{F-1}x) = \frac{1}{2(F-1)}h(x), \tag{26}$$

$$g(\frac{1}{F-1}x) = \frac{-1}{2(F+1)}h(x). \tag{27}$$

With transformation $x \to (F-1)x$ for (26) and $x \to (F+1)x$ for (27) we get :

$$f(x) = \frac{1}{2(F-1)}h((F-1)x), \tag{28}$$

$$g(x) = \frac{-1}{2(F+1)}h((F+1)x). \tag{29}$$

Next, substitute $f(x)$ and $g(x)$ to the relation $s = Py$ to get the transient solution :

$$\eta_1(x,t) = -\frac{F}{2(F-1)}h(x-(F-1)t) - \frac{F}{2(F+1)}h(x-(F+1)t), \tag{30}$$

$$u_1(x,t) = \frac{1}{2(F-1)}h(x-(F-1)t) - \frac{1}{2(F+1)}h(x-(F+1)t). \tag{31}$$

In two subsection above, we obtain the steady solution (14)-(15) and the transient solution (26)-(27). Thus, our complete analytical solution is :

(32)

$$\eta(x,t) = \frac{F^2}{F^2-1}h(x) - \frac{F}{2(F-1)}h(x-(F-1)t) - \frac{F}{2(F+1)}h(x-(F+1)t),$$

(33)

$$u(x,t) = \frac{-1}{F^2-1}h(x) + \frac{1}{2(F-1)}h(x-(F-1)t) - \frac{1}{2(F+1)}h(x-(F+1)t).$$

3. Numerical Solution and Stability Analysis

In this section we will use numerical approximation to obtain the solution of model. As usual, in the first step we make space and time discretization. We divide space domain [0,L] in width of Δx so we have Nx sub-intervals with end points $x_j = (j-1)\Delta x$, for $j = 1, 2, ..., Nx + 1$. Similarly for time discretization, we divide time domain [0,T] in width of Δt so we have Nt time steps with each time is defined by $t_n = (n-1)\Delta t$, for $n = 1, 2, ..., Nt+1$.

We will apply The Forward Time Backward Space (FTBS) method to this model. This method is appplied to the transport equation (18)-(19) and we use absorbing wall type to define the right boundary condition. The left wall is assumed to be far enough from the bump so the profile of fluid surface in the left boundary is not affected by the existance of bump. Therefore, we apply fixed end wall type to the left boundary.

With FTBS method, we define :

$$\frac{\partial y}{\partial t}\Big|_j^n = \frac{y_j^{n+1} - y_j^n}{\Delta t}, \tag{34}$$

$$\frac{\partial y}{\partial x}\Big|_j^n = \frac{y_j^n - y_{j-1}^n}{\Delta x}. \tag{35}$$

Thus, with applying (34)-(35) to (18)-(19) we have difference equations

$$\frac{y1_j^{n+1} - y1_j^n}{\Delta t} = (1-F)\frac{y1_j^n - y1_{j-1}^n}{\Delta x}$$

$$y1_j^{n+1} = (1+C_1)y1_j^n - C_1 y_{j-1}^n \tag{36}$$

and

$$\frac{y2_j^{n+1} - y2_j^n}{\Delta t} = -(1+F)\frac{y2_j^n - y2_{j-1}^n}{\Delta x}$$

$$y2_j^{n+1} = (1+C_2)y2_j^n - C_2 y2_{j-1}^n \tag{37}$$

as the numerical solution for the uncoupled equations (18)-(19) with defining courant number $C_1 = \frac{\Delta t}{\Delta x}(1-F)$ and $C_2 = -\frac{\Delta t}{\Delta x}(1+F)$. In general, for $m = 1, 2$, (36)-(37) can be written as :

$$ym_j^{n+1} = (1+C_m)ym_j^n - C_m ym_{j-1}^n. \tag{38}$$

With relation $s = Py$ we can obtain the numerical solution for the transient part. Furthermore, the complete numerical solution can be obtained by adding this transient numerical solution to the steady solution (14)-(15) with same discretization.

Next, we evaluate the stability of the scheme using von Neumann method. Substitute $ym_j^n = r_m^n exp(ia\Delta xj)$ to (38) and the scheme is stable if norm of the amplification vector r_m, $|r_m|$, is not greater than 1, for $m = 1, 2$. Thus, we have :

$$r_m^{n+1}exp(ia\Delta xj) = (1+C_m)r_m^n exp(ia\Delta xj) - C_m r_m^n exp(ia\Delta x(j-1))$$

$$r_m = (1+C_m) - C_m exp(-ia\Delta x)$$

$$r_m = (1+C_m) - C_m(cos(a\Delta x) - isin(a\Delta x))$$

$$r_m = (1+C_m) - C_m cos(a\Delta x) - iC_m sin(a\Delta x))$$

$$|r_m| = \sqrt{((1+C_m) - C_m cos(a\Delta x))^2 + (C_m sin(a\Delta x)))^2} \leq 1$$

$$2C_m(1 - cos(a\Delta x)) + C_m^2(1 - 2cos(a\Delta x) + (cos(a\Delta x))^2) + C_m^2(sin(a\Delta x))^2 \leq 0$$

$$2C_m(C_m+1)(1 - cos(a\Delta x)) \leq 0$$

$$-1 \leq C_m \leq 0$$

For $m = 1$ we get $\Delta t \leq \Delta x/(F-1)$ and this valid for $F > 1$ while for $m = 2$ we have $\Delta t \leq \Delta x/(F+1)$ that valid for all real of F. With this result, we must choose $\Delta t \leq \Delta x/(F+1)$ to make sure that $|r_m| \leq 1$ for $m = 1, 2$ and guarantee the stability of the scheme. Beside that, we can only set $F > 1$ to make sure the stability condition satisfied in case of $m = 1$. This means that the FTBS method is stable only for supercritical flow.

4. Simulation

4.1. Simulation for Analytical Solution.

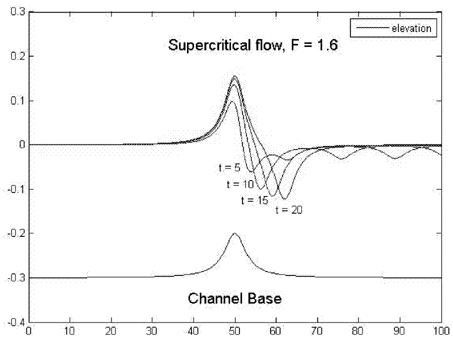

Figure 2. Side view for supercritical flow

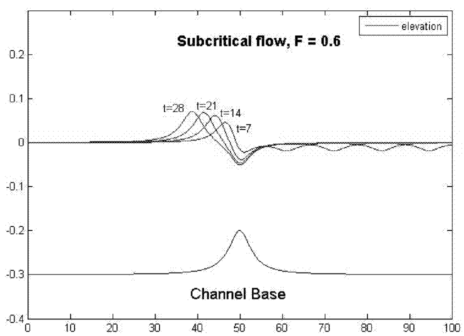

Figure 3. Side view for subcritical flow

Here, we simulate the analytical solution (32). We also compare the simulation result to the analytical solution itself to see how the analytical solution is shown in the simulation. In this case we use $h(x) = 01/(1+0.1(x-50)^2)$ as function for the bump. We also set $\Delta x = 0.1$ and $\Delta t = \Delta x/(1+F)$ for discretization.

Both of the simulation in Figure 2 and Figure 3 show that there are three waves formed, those are two dales and one peak. In Figure 2 the peak does not move anywhere with time. This indicate that the peak is the representation for the steady solution and the two dales are representation for the transient solution. The middle dale is related to the first part of the transient solution of elevation η_1 because the middle-dale moves slower than the right-middle and this is confirmed in (32) by the coefficient $F - 1$ of t in the second part of η_1 which is smaller than the third part.

In Figure 3 the middle dale does not move with time, so this is the representation for the steady solution. In other side, the peak moves to the left with time. This shows that the peak is related to the second part of (32). Note that the value of F which is less than 1 causes the coefficient of t in function h to be positive. This indicates the movement to the left.

4.2. Simulation for The FTBS Method.

4.2.1. *Supercritical Flow.*

In Figure 4 we see that the simulation result for the FTBS method confirms the analytical solution well. The different result we can see in the

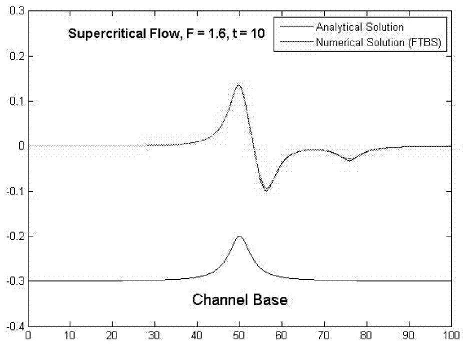

FIGURE 4. FTBS, $\Delta t = 0.01$, $\Delta x = 0.1$

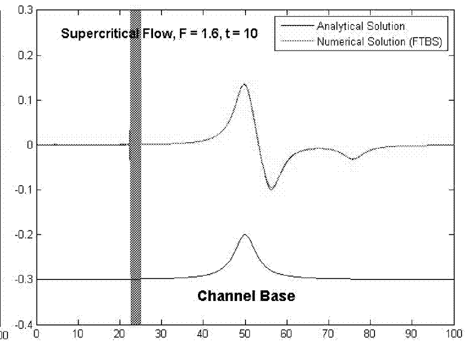

FIGURE 5. FTBS, $\Delta t = 0.04$, $\Delta x = 0.1$

next image, Figure 5. The result shows unstable simulation indicated by red vertical-line. Note that the discretization with $\Delta t = 0.04$ and $\Delta x = 0.1$ yield $\Delta t = 0.04 \geq \Delta x/(F+1) = 0.038$. This shows that Δt we choose is out of the stability boundary we have obtained in previous section. This causes the unstability.

4.2.2. *Subcritical Flow.*

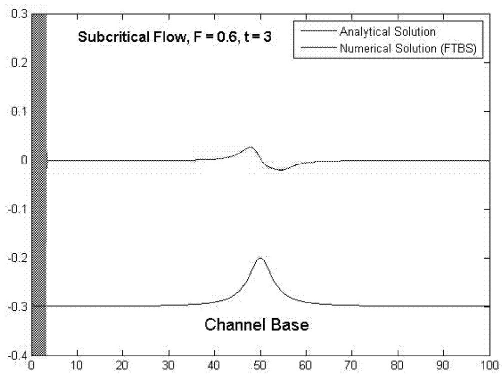

FIGURE 6. FTBS method, $\Delta t = 0.01$, $\Delta x = 0.1$

As we obtained in the previous chapter, the FTBS method will not show the stability for subcritical flow. This is confirmed by the simulation result in Figure 6 above that we can see a red vertical-line in the image.

5. CONCLUSION

The existance of a bump in the base of a channel will affect the profile of surface that flowing on the channel. This problem can be modeled by

Boussinesq equation and solved numerically by The Forward Time Backward Space method. But, this method only can be used for the case of supercritical fluid flow. In this case the bump causes the formation of one peak and two dales on the surface.

References

[1] L. H. Wiryanto and Sudi Mungkasi, *A Boussinesq-type Model for Waves Generated by Flow Over A Bump*, Appl. Math. Sci. 8 (2014), 5293-5302.

Department of Mathematics, Bandung Institute of Technology
E-mail address: azisssani327@gmail.com

Department of Mathematics, Bandung Institute of Technology
E-mail address: leo@math.itb.ac.id