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A NOTE ON r-WHITNEY TYPE CENTRAL FACTORIAL NUMBERS OF THE SECOND KIND

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Abstract. The central factorial numbers of the second kind are studied by several authors. In this paper, we study the r-Whitney type central numbers of the second kind and we give some identities and properties associated with special numbers.

1. Introduction

For $n \in \mathbb{N} \cup \{0\}$, the central factorial $x^{[n]}$ is defined by

$$x^{[0]} = 1, \;\; x^{[n]} = x\big(x + \frac{n}{2} - 1\big)\big(x + \frac{n}{2} - 2\big) \cdots \big(x - \frac{n}{2} + 1\big), \quad where \;\; (n \geq 1), \;\; (see[1,2,4,10]). \tag{1.1}$$

From (1.1), we note that the generating function of central factorial is given by

$$\Big(\frac{t}{2} + \sqrt{1 + \frac{t^2}{4}}\Big)^{2x} = \sum_{n=0}^{\infty} x^{[n]} \frac{t^n}{n!}, \quad (see[4]). \tag{1.2}$$

The central factorial number of the second kind is defined as

$$x^{[n]} = \sum_{k=0}^{n} T(n,k) x^{[k]}, \quad (see[1,2,4,7,10]). \tag{1.3}$$

The generating function of (1.3) is given by

$$\frac{1}{k!}\big(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\big)^k = \sum_{n=k}^{\infty} T(n,k) \frac{t^n}{n!}, \quad (k \in \mathbb{N} \cup \{0\}). \tag{1.4}$$

It is well known that the stirling polynomials of the second kind is given by the generating function to be

$$\frac{1}{k!}(e^t - 1)^k e^{xt} = \sum_{n=0}^{\infty} S_2(n,k|x) \frac{t^n}{n!}, \quad (see[3,5]). \tag{1.5}$$

From (1.5), we note that

$$(x+y)^n=\sum_{l=0}^{n}S_2(n,k|x)(y)_l,\quad (n\geq 0), \tag{1.6}$$

where $(y)_0=1$, $(y)_n=y(y-1)(y-2)\cdots(y-n+1)$, $(n\geq 1)$. For $n,r\in\mathbb{N}\cup\{0\}$, r-Whitney numbers of the second kind $W_{m,r}(n,k)$, $(m\in\mathbb{N})$, are defined by

$$(mx+r)^n=\sum_{k=0}^{n}m^kW_{m,r}(n,k)(x)_k,\quad (see[8]). \tag{1.7}$$

By (1.7), we easily get

$$W_{m,r}(n+1,k)=(r+mk)W_{m,r}(n,k)+W_{m,r}(n,k-1),\quad (1\leq k\leq n), \tag{1.8}$$

and

$$m^{n-k}\frac{1}{k!}\Delta^k\Big(\frac{r}{m}\Big)^n=\begin{cases} W_{m,r}(n,k), & \text{if}\, n\geq k,\\ 0, & \text{if}\, n<k,\end{cases} \tag{1.9}$$

where $\Delta f(x)=f(x+1)-f(x)$.

Recently, several authors have studied the stirling polynomials of the second kind and the central factorial numbers of the second kind (see[1-12]). In this paper, we study the r-Whitney type central factorial numbers of the second kind and investigate some properties for these numbers. In addition, we give some identities for the Whitney type central factorial numbers of the second kind associated with special numbers and polynomials.

2. r-Whitney type central factorial numbers of the second kind.

For $r\in\mathbb{N}$, in the point of view of (1.7), we consider the r-Whitney type central factorial numbers of the second kind which are given by

$$(mx+r)^n=\sum_{k=0}^{n}m^kT_{m,r}(n,k)x^{[k]}. \tag{2.1}$$

Now, we observe that

$$\begin{aligned} e^{(mx+r)t}&=\sum_{n=0}^{\infty}(mx+r)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}m^kT_{m,r}(n,k)x^{[k]}\Big)\frac{t^n}{n!}\\ &=\sum_{k=0}^{\infty}m^k\Big(\sum_{n=k}^{\infty}T_{m,r}(n,k)\frac{t^n}{n!}\Big)x^{[k]}. \end{aligned} \tag{2.2}$$

Let $f(x)=2\log\big(\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}\big)$. Then we note that the inverse function of $f(t)$ is given by

$$f^{-1}(t)=e^{\frac{t}{2}}-e^{-\frac{t}{2}}. \tag{2.3}$$

So, we have

$$f\big(f^{-1}(mt)\big)=f^{-1}\big(f(mt)\big)=mt. \tag{2.4}$$

From (2.4), we can derive the following equation (2.5).

$$\begin{aligned} e^{(mx+r)t} &= e^{rt}e^{mxt}=e^{rt}e^{x\left(2\log\left(\frac{e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}}{2}+\sqrt{1+\frac{\left(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\right)^2}{4}}\right)\right)} \\ &= e^{rt}e^{\log\left(\frac{e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}}{2}+\sqrt{1+\frac{\left(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\right)^2}{4}}\right)^{2x}} \\ &= e^{rt}\left(\frac{e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}}{2}+\sqrt{1+\frac{\left(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\right)^2}{4}}\right)^{2x} \\ &= e^{rt}\sum_{k=0}^{\infty}x^{[k]}\frac{1}{k!}\big(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\big)^k. \end{aligned} \tag{2.5}$$

Therefore, by (2.2) and (2.5), we obtain the following generating function of the r-Whitney type central factorial numbers of the second kind.

$$\frac{1}{m^k k!}e^{rt}\big(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\big)^k=\sum_{n=k}^{\infty}T_{m,r}(n,k)\frac{t^n}{n!}, \tag{2.6}$$

where $m,r\in\mathbb{N}$ and $n,k\in\mathbb{N}\cup\{0\}$.

From (1.4), we have

$$\begin{aligned} \frac{1}{m^k k!}e^{rt}\big(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\big)^k &= \Big(\frac{1}{m^k}\sum_{l=0}^{\infty}r^l\frac{t^l}{l!}\Big)\Big(\sum_{i=k}^{\infty}T(i,k)\frac{m^i t^i}{i!}\Big) \\ &= \frac{1}{m^k}\sum_{n=k}^{\infty}\Big(\sum_{i=k}^{n}T(i,k)\binom{n}{i}r^{n-i}m^i\Big)\frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we get

$$T_{m,r}(n,k)=\sum_{i=k}^{n}T(i,k)\binom{n}{i}r^{n-i}m^{i-k}, \tag{2.8}$$

where $n,k\in\mathbb{N}\cup\{0\}$ and $m,r\in\mathbb{N}$.

From (1.5), we note that

$$\begin{aligned}\frac{1}{m^k k!}e^{rt}\big(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\big)^k &= \frac{1}{m^k k!}e^{(r-\frac{m}{2})t}(e^{mt}-1)^k \\ &= \frac{1}{m^k}\sum_{n=k}^{\infty}S_2\big(m,k|\frac{r-\frac{m}{2}}{m}\big)\frac{m^n t^n}{n!} = \sum_{n=k}^{\infty}S_2\big(n,k|\frac{r-\frac{m}{2}}{m}\big)m^{n-k}\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Therefore, by (2.6) and (2.9), we get

$$T_{m,r}(n,k) = S_2\big(n,k|\frac{r-\frac{m}{2}}{m}\big)m^{n-k}, \quad (n,k\geq 0). \tag{2.10}$$

The central difference operator δ is defined by

$$\delta f(x) = f\big(x+\frac{1}{2}\big) - f\big(x-\frac{1}{2}\big). \tag{2.11}$$

From (2.11), we can easily derive the following equation (2.12).

$$\delta^k f(x) = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}f\big(x+l-\frac{k}{2}\big), \quad (k\in\mathbb{N}). \tag{2.12}$$

Now, we observe that

$$\begin{aligned}\frac{1}{m^k k!}e^{rt}\big(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\big)^k &= \frac{1}{m^k k!}e^{(r-\frac{m}{2})t}(e^{mt}-1)^k \\ &= \frac{1}{m^k k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{\left(r-\frac{m}{2}+lm\right)t} \\ &= \frac{1}{m^k k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{n=0}^{\infty}\Big(r+\big(l-\frac{k}{2}\big)m\Big)^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\Big(\frac{m^{n-k}}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(\frac{r}{m}+\big(l-\frac{k}{2}\big)\big)^n\Big)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

Let us take $f(x)=x^n$ in (2.12). Then we have

$$\delta^k x^n = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(x+l-\frac{k}{2}\big)^n, \quad (n,k\geq 0). \tag{2.14}$$

From (2.13) and (2.14), we note that

$$\frac{1}{m^k k!}e^{rt}\big(e^{\frac{mt}{2}}-e^{-\frac{mt}{2}}\big)^k = \sum_{n=0}^{\infty}\frac{m^{n-k}}{k!}\Big(\delta^k\Big(\frac{r}{m}\Big)^n\Big)\frac{t^n}{n!}. \tag{2.15}$$

Therefore, by (2.6) and (2.15), we get

$$m^{n-k}\Big(\frac{1}{k!}\delta^k\Big(\frac{r}{m}\Big)^n\Big) = \begin{cases} T_{m,r}(n,k), & \text{if } n\geq k, \\ 0, & \text{if } n<k. \end{cases} \tag{2.16}$$

Now, we observe that

$$\begin{aligned}\delta^k\Big(\frac{r}{m}\Big)^{n+1} &= \sum_{l=0}^{k}\binom{k}{l}\big(\frac{r}{m}+l-\frac{k}{2}\big)^{n+1}(-1)^{k-l}\\&= \sum_{l=0}^{k}\binom{k}{l}\big(\frac{r}{m}+l-\frac{k}{2}\big)^{n}\big(\frac{r}{m}+l-\frac{k}{2}\big)(-1)^{k-l}\\&= \big(\frac{r}{m}-\frac{k}{2}\big)\delta^k\Big(\frac{r}{m}\Big)^n+k\sum_{l=1}^{k}\binom{k}{l}l\big(\frac{r}{m}+l-\frac{k}{2}\big)^{n}(-1)^{k-l}\\&= \big(\frac{r}{m}-\frac{k}{2}\big)\delta^k\Big(\frac{r}{m}\Big)^n+k\sum_{l=1}^{k}\Big\{\binom{k}{l}-\binom{k-1}{l-1}\Big\}\\&\times\big(\frac{r}{m}+l-\frac{k}{2}\big)^{n}(-1)^{k-l}\\&= \big(\frac{r}{m}-\frac{k}{2}\big)\delta^k\Big(\frac{r}{m}\Big)^n+k\sum_{l=0}^{k}\Big\{\binom{k}{l}-\binom{k-1}{l-1}\Big\}\\&\times\big(\frac{r}{m}+l-\frac{k}{2}\big)^{n}(-1)^{k-l}\\&= \big(\frac{r}{m}-\frac{k}{2}\big)\delta^k\Big(\frac{r}{m}\Big)^n+k\Big(\delta^k\big(\frac{r}{m}\big)^n+\delta^{k-1}\big(\frac{r}{m}-\frac{1}{2}\big)^n\Big).\end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we note that

$$\begin{aligned}T_{m,r}(n+1,k) &= m^{n+1-k}\frac{1}{k!}\delta^k\big(\frac{r}{m}\big)^{n+1}\\&= m^{n+1-k}\frac{1}{k!}\Big\{\Big(\frac{r}{m}-\frac{k}{2}\Big)\delta^k\Big(\frac{r}{m}\Big)^n+k\Big(\delta^k\Big(\frac{r}{m}\Big)^n\\&+\delta^{k-1}\Big(\frac{r}{m}-\frac{1}{2}\Big)^n\Big)\Big\}\\&= m\Big(\frac{r}{m}-\frac{k}{2}\Big)T_{m,r}(n,k)+mkT_{m,r}(n,k)\\&+T_{2m,2r-m}(n,k-1)\Big(\frac{1}{2}\Big)^{n+1-k}\\&= \big(r+\frac{mk}{2}\big)T_{m,r}(n,k)+\Big(\frac{1}{2}\Big)^{n+1-k}T_{2m,2r-m}(n,k-1).\end{aligned} \tag{2.18}$$

Therefore, by (2.18), we get

$$T_{m,r}(n+1,k)=\big(r+\frac{mk}{2}\big)T_{m,r}(n,k)+\Big(\frac{1}{2}\Big)^{n+1-k}T_{2m,2r-m}(n,k-1), \tag{2.19}$$

where $m,r\in\mathbb{N}$ with $r\geq[\frac{m+1}{2}]$ and $k,n\in\mathbb{Z}$ with $1\leq k\leq n$.

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