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Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials

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Abstract. In this paper, we derive generating functions for the partially degenerate Hermite poly-Bernoulli polynomials and investigate some properties of these polynomials related to the Stirling numbers of the second kind. Further, we derive the summation formulae and general symmetry identities for that polynomials by using different analytical means on its generating function. Also, generating functions and summation formulae for the polynomials related to partially degenerate Hermite poly-Bernoulli polynomials are obtained as applications of main results.

Keywords: Hermite polynomials, degenerate poly-Bernoulli polynomials, partially degenerate Hermite poly-Bernoulli polynomials, summation formulae, symmetric identities.

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1. Introduction

In recent years, many authors have been studied poly-Bernoulli polynomials (see [5], [6], [8], [9], [10], [11], [12]). The poly-Bernoulli polynomials have wide-ranging application in number theory, combinatorics, Zeta function theory and other fields of applied mathematics. The aim of this paper is to provide some answers to the problems arising in the study of the development of degenerate Hermite-poly-Bernoulli polynomials. This approach will permit the introduction of the Hermite polynomials to establish the basis of degenerate Hermite-poly-Bernoulli polynomials. Furthermore, we derive a summation formula and general symmetry identities of degenerate Hermite-poly-Bernoulli polynomials. In these introductory remarks, we discuss the properties of the partially degenerate poly-Bernoulli polynomials and Hermite polynomials defined in [2, 10] and the notation in order to make the paper self-consistent, we recall here the following definitions as follows.

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [2, 4] are defined by the series:

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

The special cases of (1.1) are given as follows:

$$H_n(2x,-1) = H_n(x)$$

and

$$H_n(x,-\frac{1}{2}) = He_n(x),$$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials. Also

$$H_n(x,0) = x^n.$$

The generating function for the 2VHKdFP Hermite polynomial H_n(x,y) are given by [7, 10]:

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

Carlitz [3] introduced the degenerate Bernoulli polynomials given by the generating function:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x;\lambda)\frac{t^n}{n!},(\lambda\in\mathbb{C}) \tag{1.3}$$

so that

$$\beta_n(x;\lambda)=\sum_{n=0}^{m}\left(\begin{array}{c}n\\m\end{array}\right)\beta_m(\lambda)(\frac{x}{\lambda})_{n-m}. \tag{1.4}$$

When $x=0$, $\beta_n(\lambda)=\beta_n(0;\lambda)$ are called the degenerate Bernoulli numbers.

From (1.3), we note that

$$\sum_{n=0}^{\infty}\lim_{\lambda\longrightarrow 0}\beta_n(x;\lambda)\frac{t^n}{n!}=\lim_{\lambda\longrightarrow 0}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.5}$$

where $B_n(x)$ are called the Bernoulli polynomials (see [1-16]).

The classical polylogarithmic function $\mathrm{Li}_k(z)$ is defined by (see [5], [6], [11]):

$$\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k},(k\in\mathbb{Z}). \tag{1.6}$$

The poly-Bernoulli numbers and polynomials are defined by following generating functions (see [5], [6], [8], [11], [12]):

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}, \tag{1.7}$$

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.8}$$

In the case $k=1$ in (1.7) and (1.8), we have

$$B_n^{(1)}=B_n, B_n^{(1)}(x)=B_n(x).$$

In 2016, Khan [9] introduced the degenerate Hermite-poly-Bernoulli polynomials of two variables ${}_H\beta_n^{(k)}(x,y;\lambda)$ defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}. \tag{1.9}$$

On setting $k=1$, and $\lim_{\lambda\longrightarrow 0}$ in (1.9), we get

$$\lim_{\lambda\longrightarrow 0}\frac{\mathrm{Li}_1(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\frac{t}{e^t-1}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!}, \tag{1.10}$$

where ${}_HB_n(x,y)$ is called the Hermite-Bernoulli polynomials (see [13]).

The Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, (n \geq 0) \tag{1.11}$$

and the Stirling number of the second kind is defined by generating function:

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}. \tag{1.12}$$

A generalized falling factorial sum $\sigma_k(n;\lambda)$ can be defined by the generating function (see [16]):

$$\sum_{k=0}^{\infty} \sigma_k(n;\lambda)\frac{t^k}{k!} = \frac{(1+\lambda t)^{\frac{(n+1)}{\lambda}} - 1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}, \tag{1.13}$$

where $\lim_{\lambda \longrightarrow 0} \sigma_k(n;\lambda) = S_k(n)$.

The goal of this paper as follows, in section 2, we introduce partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ and investigate some properties of these polynomials. In section 2, we derive implicit summation formulae for degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$. In section 4, we establish general symmetry identities for degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ by applying the generating functions. These results establish a link between these families of polynomials (namely Hermite and partially degenerate poly-Bernoulli polynomials).

2. Partially degenerate Hermite poly-Bernoulli numbers and polynomials

For $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$, we consider the partially degenerate Hermite poly-Bernoulli polynomials given by the generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y|\lambda)\frac{t^n}{n!}. \tag{2.1}$$

When $x = y = 0$ in (2.1), ${}_H\beta_n^{(k)}(0,0|\lambda) = \beta_n^{(k)}(\lambda)$ are called the partially degenerate poly-Bernoulli numbers.

For $\lim_{\lambda \longrightarrow 0}$ in (2.1), we have

$$\lim_{\lambda \longrightarrow 0} \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} e^{xt+yt^2} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(k)}(x,y)\frac{t^n}{n!}, \tag{2.2}$$

where ${}_H B_n^{(k)}(x,y)$ is Hermite-poly-Bernoulli polynomials, which is defined by Khan et al. [12].

Theorem 2.1. For $n \geq 0$, we have

$$ {}_H\beta_n^{(k)}(x,y|\lambda) = \sum_{m=0}^{n} \binom{n}{m} B_m^{(k)}(\lambda) H_{n-m}(x,y). \tag{2.3}$$

Proof. From (1.2) and (2.1), we can easily obtained (2.3).

Theorem 2.2. For $n \geq 0$, we have

$$ {}_H\beta_n^{(2)}(x,y;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m+1} {}_H\beta_{n-m}(x,y;\lambda). \tag{2.4}$$

Proof. Applying definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1} e^{xt+yt^2}$$

$$= \frac{e^{xt+yt^2}}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \underbrace{\int_0^t \frac{1}{e^z-1}\int_0^t \frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t \frac{z}{e^z-1}}_{(k-2)-\text{times}} dz\cdots dz. \tag{2.5}$$

For $k=2$ in (2.5), we have

$$\sum_{n=0}^{\infty} {}_H\beta_n^{(2)}(x,y|\lambda)\frac{t^n}{n!} = \frac{e^{xt+yt^2}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t \frac{z}{e^z-1}dz$$

$$= \left(\sum_{m=0}^{\infty} \frac{B_m m!}{m+1}\frac{t^m}{m!}\right)\frac{te^{xt+yt^2}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$$

$$= \left(\sum_{m=0}^{\infty} \frac{B_m m!}{m+1}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_H\beta_n(x,y;\lambda)\frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} \frac{B_m m!}{m+1} {}_H\beta_{n-m}(x,y|\lambda)\frac{t^n}{n!}.$$

On equating the coefficients of the like powers of t in both sides, we get (2.4).

Theorem 2.3. For $n \geq 0$, we have

$$ {}_H\beta_n^{(k)}(x,y|\lambda) = \sum_{p=0}^{n} \binom{n}{p} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k(p+1)}\right) {}_H\beta_{n-p}(x,y|\lambda). \tag{2.6}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y|\lambda)\frac{t^n}{n!} = \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{te^{xt+yt^2}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right). \tag{2.7}$$

Now

$$\frac{1}{t}\mathrm{Li}_k(1-e^{-t}) = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{l^k} = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}(1-e^{-t})^l$$

$$= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k} l! \sum_{p=l}^{\infty}(-1)^p S_2(p,l)\frac{t^p}{p!}$$

$$= \frac{1}{t}\sum_{p=1}^{\infty}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k} l! S_2(p,l)\frac{t^p}{p!}$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}. \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$\sum_{n=0}^{\infty} {}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\left(\sum_{n=0}^{\infty}{}_H\beta_n(x,y|\lambda)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the r.h.s. of above equation and comparing the coefficients of t^n in both sides, we obtain (2.6).

Theorem 2.4. For $n \geq 0$, $d \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$${}_H\beta_n^{(k)}(x,y;\lambda) = \sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\binom{n}{l} d^{n-l-1}\frac{(-1)^{l+p+1}p!S_2(l+1,p)}{p^k l+1}{}_H\beta_{n-l}\left(\frac{a+x}{d},y|\frac{\lambda}{d}\right). \tag{2.9}$$

Proof. Using definition (2.1), we can be written as

$$\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y|\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt+yt^2} \tag{2.10}$$

$$= \frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}e^{(a+x)t+yt^2}$$

$$= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{t}\right)\left(\frac{1}{d}\sum_{a=0}^{d-1}\frac{dt}{(1+\lambda t)^{\frac{d}{\lambda}}-1}e^{(a+x)t+yt^2}\right)$$

$$= \left(\sum_{l=0}^{\infty}\left(\sum_{p=1}^{l+1}\frac{(-1)^{l+p+1}}{p^k}p!\frac{S_2(l+1,p)}{l+1}\right)\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}d^{n-1}\sum_{a=0}^{d-1}{}_H\beta_n\left(\frac{a+x}{d},y|\frac{\lambda}{d}\right)\frac{t^n}{n!}\right).$$

Replacing n by $n-l$ in above equation and comparing the coefficient of t^n in both sides, we get (2.9).

3. Summation formulae for partially degenerate Hermite poly-Bernoulli polynomials

In this section, we prove the following result involving partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ by using series rearrangement techniques and considered its special case.

Theorem 3.1. The following summation formula for partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ holds true:

$${}_H\beta_{q+l}^{(k)}(x,w|\lambda) = \sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}(w-y)^{n+p}{}_H\beta_{q+l-n-p}^{(k)}(x,y|\lambda). \tag{3.1}$$

Proof. Replacing t by $t+u$ in (2.1) and then using the formula [15,p.52(2)]:

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.2}$$

in the resultant equation, we find the following generating function for the partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$:

$$\left(\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)e^{yt^2} = e^{-x(t+u)}\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,y|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.3}$$

Replacing x by w in the above equation and equating the resultant equation to the above equation, we find

$$\exp((w-x)(t+u))\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,y|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$=\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(w,y|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.4}$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty}\frac{[(w-x)(t+u)]^N}{N!}\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,y|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$=\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,w|\lambda)\frac{t^q}{q!}\frac{u^l}{l!} \tag{3.5}$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(w-x)^{n+p}t^nu^p}{n!p!}\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,y|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$=\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,w|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.6}$$

Now replacing q by $q-n$, l by $l-p$ and using the lemma ([15, p.100(1)]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A(n,k)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}A(n,k-n), \tag{3.7}$$

in the l.h.s. of (3.6), we find

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l}\frac{(w-y)^{n+p}}{n!p!}{}_H\beta_{q+l-n-p}^{(k)}(x,y|\lambda)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!}$$
$$=\sum_{q,l=0}^{\infty}{}_H\beta_{q+l}^{(k)}(x,w|\lambda)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.8}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem (3.1).

Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for partially degenerate Hermite-poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ holds true:

$${}_H\beta_q^{(k)}(x,w|\lambda) = \sum_{n=0}^{q}\binom{q}{n}(w-x)^n{}_H\beta_{q-n}^{(k)}(x,y|\lambda). \tag{3.9}$$

Remark 3.2. Replacing w by $w+x$ in (3.8), we obtain

$${}_H\beta_q^{(k)}(x+w,w|\lambda) = \sum_{n=0}^{q}\binom{q}{n}w^n{}_H\beta_{q-n}^{(k)}(x,y|\lambda). \tag{3.10}$$

Theorem 3.2. The following summation formula for partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ holds true:

$$ {}_H\beta_n^{(k)}(x+u,y+w|\lambda)=\sum_{m=0}^{n}\binom{n}{m}{}_H\beta_{n-m}^{(k)}(x,y|\lambda)H_m(u,w). \tag{3.11}$$

Proof. From (2.1) and (1.2), we have

$$\frac{\mathrm{Li}_k(1-(e)^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}-1}}e^{(x+u)t+(y+w)t^2}=\left(\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y|\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(u,w)\frac{t^m}{m!}\right).$$

Now replacing n by $n-m$ and comparing the coefficients of t^n in both sides, we get (3.11).

Theorem 3.3. The following summation formula for partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ holds true:

$$ {}_H\beta_n^{(k)}(x,y|\lambda)=\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}\beta_m^{(k)}(\lambda)x^{n-m-2j}y^j\frac{1}{(n-2j)!j!}. \tag{3.12}$$

Proof. Applying the definition (2.1) to the term $\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}-1}}$ and expanding the function e^{xt+yt^2} at $t=0$ yields

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}-1}}e^{xt+yt^2}=\left(\sum_{m=0}^{\infty}\beta_m^{(k)}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\beta_m^{(k)}(\lambda)x^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}\beta_m^{(k)}(\lambda)x^{n-m-2j}y^j\right)\frac{t^n}{(n-2j)!j!}. \tag{3.13}$$

Equating their coefficients of t^n, we get the result (3.12).

Theorem 3.4. For $x,y\in\mathbb{R}$ and $n\geq 0$. Then

$$ {}_H\beta_n^{(k)}(x,y;\lambda)=\sum_{m=0}^{n}\binom{n}{m}z^{n-m}{}_H\beta_m^{(k)}(x-z,y;\lambda). \tag{3.14}$$

Proof. By exploiting the generating function (2.1), we can write the equation

$$\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-e^{-t})}{e}^{(x-z)t+yt^2}e^{zt}. \tag{3.15}$$

$$=\left(\sum_{m=0}^{\infty}{}_H\beta_m^{(k)}(x-z,y;\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}z^n\frac{t^n}{n!}\right)$$

Replacing n by $n-m$ in above equation and equating their coefficients of t^n leads to formula (3.14).

Theorem 3.5. The following implicit summation formula involving partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ holds true:

$$ {}_H\beta_n^{(k)}(x+1,y;\lambda)=\sum_{r=0}^{n}\binom{n}{r}{}_H\beta_{n-r}^{(k)}(x,y;\lambda). \tag{3.16}$$

Proof. By the definition of partially degenerate Hermite poly-Bernoulli polynomials, we have

$$\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x+1,y;\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}=\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt+yt^2}(e^t+1)$$

$$=\left(\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}\right)+\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_H\beta_{n-r}^{(k)}(x,y;\lambda)\frac{t^n}{(n-r)!r!}+\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.16).

4. Symmetry identities for partially degenerate Hermite poly-Bernoulli polynomials

In this section, we prove general symmetry identities for partially degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y|\lambda)$ by applying the generating function(2.1) and (2.3).

Theorem 4.1. Let $a,b>0$ and $a\neq b$. For $x,y\in\mathbb{R}$ and $n\geq 0$, then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}b^m a^{n-m}{}_H\beta_{n-m}^{(k)}(bx,b^2y|\lambda){}_H\beta_m^{(k)}(ax,a^2y|\lambda)$$

$$=\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}{}_H\beta_{n-m}^{(k)}(ax,a^2y|\lambda){}_H\beta_m^{(k)}(bx,b^2y|\lambda). \tag{4.1}$$

Proof. Let us consider

$$H(t)=\left(\frac{\mathrm{Li}_k(1-e^{-at})\mathrm{Li}_k(1-e^{-bt})}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)}\right)e^{abxt+a^2b^2yt^2}. \tag{4.2}$$

Then the expression for $H(t)$ is symmetric in a and b and we can expand $H(t)$ into series in two ways to obtain

$$H(t)=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(bx,b^2y|\lambda)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_H\beta_m^{(k)}(ax,a^2y|\lambda)\frac{(bt)^m}{m!}$$

$$H(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m{}_H\beta_{n-m}^{(k)}(bx,b^2y|\lambda){}_H\beta_m^{(k)}(ax,a^2y|\lambda)\right)\frac{t^n}{n!}. \tag{4.3}$$

Similarly

$$H(t)=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(ax,a^2y|\lambda)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}{}_H\beta_m^{(k)}(bx,b^2y|\lambda)\frac{(at)^m}{m!}$$

$$H(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}{}_H\beta_{n-m}^{(k)}(ax,a^2y|\lambda){}_H\beta_m^{(k)}(bx,b^2y|\lambda)\right)\frac{t^n}{n!}. \tag{4.4}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (4.3) and (4.4), we arrive at the desired result.

Corollary 4.1. On setting $b = 1$ in Theorem 4.1, we reduce the following result

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}{}_H\beta_{n-m}^{(k)}(x,y|\lambda){}_H\beta_{m}^{(k)}(ax,a^2y|\lambda)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}{}_H\beta_{n-m}^{(k)}(ax,a^2y|\lambda){}_H\beta_{m}^{(k)}(x,y|\lambda). \tag{4.5}$$

Theorem 4.2. For all integers $a > 0, b > 0$, and $n \geq 0$, the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}{}_H\beta_{n-m}^{(k)}\left(bx,b^2z|\lambda\right)\sum_{i=0}^{m}\binom{m}{i}\sigma_i(a-1|\lambda)\beta_{m-i}^{(k)}(ay|\lambda)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}{}_H\beta_{n-m}^{(k)}\left(ax,a^2z|\lambda\right)\sum_{i=0}^{m}\binom{m}{i}\sigma_i(b-1|\lambda)\beta_{m-i}^{(k)}(by|\lambda). \quad (4.6),$$

where generalized falling factorial sum $\sigma_k(n;\lambda)$ is given by (1.13).

Proof. Let

$$G(t)=\frac{\mathrm{Li}_k(1-e^{-at})\mathrm{Li}_k(1-e^{-bt})((1+\lambda t)^{\frac{ab}{\lambda}}-1)e^{ab(x+y)t+a^2b^2yt^2}}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)^2}$$

to find that

$$G(t)=\left(\frac{\mathrm{Li}_k(1-e^{-at})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}\right)e^{abxt+a^2b^2zt^2}\left(\frac{(1+\lambda t)^{\frac{ab}{\lambda}}-1}{(1+\lambda t)^{\frac{b}{\lambda}}-1}\right)$$
$$\times\left(\frac{\mathrm{Li}_k(1-e^{-bt})}{(1+\lambda t)^{\frac{b}{\lambda}}-1}\right)e^{abyt}$$

$$G(t)=\sum_{n=0}^{\infty}{}_H\beta_{n}^{(k)}\left(bx,b^2z|\lambda\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\sigma_n(a-1|\lambda)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\beta_{n}^{(k)}(ay|\lambda)\frac{(bt)^n}{n!}. \tag{4.7}$$

Using a similar plan, we get

$$g(t)=\sum_{n=0}^{\infty}{}_H\beta_{n}^{(k)}\left(ax,a^2z|\lambda\right)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\sigma_n(b-1|\lambda)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\beta_{n}^{(k)}(by|\lambda)\frac{(at)^n}{n!}. \tag{4.8}$$

On comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result.

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