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MODIFIED DEGENERATE DAEHEE NUMBERS AND POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS

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Abstract. In the paper, the author considers differential equations arising from modified degenerate Daehee numbers and polynomials. Along the idea of Kwon *et al.*, we derive some explicit identities for the modified degenerate Daehee numbers and polynomials which are derived from differential equations.

1. Introduction

It is common knowledge that the Bernoulli polynomials $\mathrm{B}_n(x)$ for $n \geq 0$ can be generated by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}\mathrm{B}_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x=0$, $\mathrm{B}_n=\mathrm{B}_n(0)$ are called Bernoulli numbers, (see [1-21]).

L. Caritz considered the degenerate Bernoulli numbers which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}=\sum_{n=0}^{\infty}\beta_{n,\lambda}\frac{t^n}{n!}, \tag{2}$$

(see [1, 2, 16, 18]).

Recently, Dolgy *et al.* studied the modified degenerate Bernoulli numbers, which are slightly different from Carlitz's degenerate Bernoulli numbers. These are defined by the generating function to be

$$\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}=\sum_{n=0}^{\infty}\widetilde{\beta}_{n,\lambda}\frac{t^n}{n!}, \tag{3}$$

(see [3]).

Kim considered the Daehee polynomials $\mathrm{D}_n(x)$ for $n \geq 0$ which are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}\mathrm{D}_n(x)\frac{t^n}{n!}. \tag{4}$$

When $x=0$, $\mathrm{D}_n=\mathrm{D}_n(0)$ are called Daehee numbers, (see [5]). There have been many works related to Daehee numbers and polynomials, (see [4, 6, 12, 19, 20, 21]).

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As a degenerate version of Daehee numbers D_n, the degenerate Daehee numbers $\mathrm{D}_{n,\lambda}$ are defined by Kim and Kim

$$\frac{\lambda\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)}{\log\left(1+\lambda t\right)}=\sum_{n=0}^{\infty}\mathrm{D}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{5}$$

Note that $\mathrm{D}_{n,\lambda}\to\mathrm{D}_n$ as $\lambda\to 0$, (see [11]).

Motivated from the concept of modified Bernoulli and Euler number (see [13, 3]), we introduce the modifed degenerate Daehee numbers $\mathfrak{D}_{n,\lambda}$, which are defined by the generating function to be

$$\frac{\lambda\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)}{t\log(1+\lambda)}=\sum_{n=0}^{\infty}\mathfrak{D}_{n,\lambda}\frac{t^n}{n!}. \tag{6}$$

Note that $\mathfrak{D}_{n,\lambda}\to\mathrm{D}_n$ as $\lambda\to 0$.

And for $r\in\mathbb{N}$, the higher-order modifed degenerate Daehee numbers are defined by the generating function to be

$$\left(\frac{\lambda\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)}{t\log(1+\lambda)}\right)^r=\sum_{n=0}^{\infty}\mathfrak{D}_{n,\lambda}^{(r)}\frac{t^n}{n!}. \tag{7}$$

We note that many authors have studied some properties on modified degenerate numbers and polynomials, (see [3, 13, 14, 15]).

In combinatorics, the following are well known for the Stirling numbers of the first kind $S_1(n,l)$ for $n\geq 0$ are defined by

$$\begin{aligned}(x)_n&=x(x-1)\cdots(x-n+1)=\prod_{l=0}^{n-1}(x-l)\\&=\sum_{l=0}^{n}S_1(n,l)x^l\end{aligned} \tag{8}$$

and the Stirling numbers of the second kind $S_2(n,l)$ for $n\geq 0$ are given by

$$x^n=\sum_{l=0}^{\infty}S_2(n,l)(x)_l \tag{9}$$

From (8) and (9), the Stirling numbers may be generated by

$$\frac{(\log\left(1+t\right))^n}{n!}=\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!}$$

and

$$\frac{(e^t-1)^n}{n!}=\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!},$$

respectively (see [5, 6, 7, 8, 9, 10, 11, 12]).

In this paper, we study differential equations arising from modified degenerate Daehee numbers and polynomials. Along the idea of Kwon *et al.* in [12], we derive some identities for the modified degenerate Daehee numbers and polynomials which are derived from differential equations. We note that there has been much works on special numbers and polynomials related to certain differential equations, (see [8, 9, 10, 12, 17]).

2. Differential equations arising from the generating function of modified Daehee numbers

Let

$$\mathcal{F} = \mathcal{F}(t) = \log\left(1 + \frac{t}{\lambda}\log(1+\lambda)\right). \tag{10}$$

By taking the derivative with respect to t of (10), we can derive

$$\begin{aligned}\mathcal{F}^{(1)} &= \frac{d}{dt}\mathcal{F}(t) = \left(1 + \frac{t}{\lambda}\log(1+\lambda)\right)^{-1}\frac{\log(1+\lambda)}{\lambda}\\ &= \frac{\log(1+\lambda)}{\lambda}e^{-\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)}\\ &= \frac{\log(1+\lambda)}{\lambda}e^{-\mathcal{F}}\end{aligned} \tag{11}$$

and

$$\mathcal{F}^{(2)} = \frac{d}{dt}\mathcal{F}^{(1)} = -\frac{\log(1+\lambda)}{\lambda}e^{-\mathcal{F}}\mathcal{F}^{(1)} = -\left(\frac{\log(1+\lambda)}{\lambda}\right)^2 e^{-2\mathcal{F}}. \tag{12}$$

Similarly, we get

$$\mathcal{F}^{(3)} = \frac{d}{dt}\mathcal{F}^{(2)} = 2\left(\frac{\log(1+\lambda)}{\lambda}\right)^2 e^{-2\mathcal{F}}\mathcal{F}^{(1)} = 2\left(\frac{\log(1+\lambda)}{\lambda}\right)^3 e^{-3\mathcal{F}}. \tag{13}$$

Continuing this process, we are led to put

$$\mathcal{F}^{(N)} = \frac{d^N}{dt^N}\mathcal{F}(t) = (-1)^{N-1}(N-1)!\left(\frac{\log(1+\lambda)}{\lambda}\right)^N e^{-N\mathcal{F}}, \tag{14}$$

where $N = 0, 1, 2, \cdots$.

In addition,

$$\begin{aligned}e^{-N\mathcal{F}} &= \sum_{n=0}^{\infty}(-1)^n N^n \frac{\mathcal{F}^n}{n!}\\ &= \sum_{n=0}^{\infty}(-1)^n N^n \frac{\left[\log\left\{1 + \frac{t}{\lambda}\log(1+\lambda)\right\}\right]^n}{n!}\\ &= \sum_{n=0}^{\infty}(-1)^n N^n \sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m\\ &= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}(-1)^n N^n S_1(m,n)\left(\frac{\log(1+\lambda)}{\lambda}\right)^m\right]\frac{t^m}{m!}.\end{aligned} \tag{15}$$

Applying (15) to (14), we have

$$\mathcal{F}^{(N)} = (-1)^{N-1}(N-1)!\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}(-1)^n N^n S_1(m,n)\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m}\right]\frac{t^m}{m!}. \tag{16}$$

Also, we can present $\mathcal{F}^{(N)}$ by using generating function of modified degenerate Daehee numbers

$$\begin{aligned}\mathcal{F}^{(N)} &= \frac{d^N}{dt^N}\left(\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)\right)\\ &= \frac{d^N}{dt^N}\left(\frac{\lambda\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)}{t\log(1+\lambda)}\frac{t\log(1+\lambda)}{\lambda}\right)\\ &= \frac{d^N}{dt^N}\left(\sum_{m=0}^{\infty}\frac{\log(1+\lambda)}{\lambda}\mathfrak{D}_{m,\lambda}\frac{t^{m+1}}{m!}\right)\\ &= \frac{d^N}{dt^N}\left(\sum_{m=0}^{\infty}\frac{m\log(1+\lambda)}{\lambda}\mathfrak{D}_{m-1,\lambda}\frac{t^{m}}{m!}\right)\\ &= \sum_{m=0}^{\infty}\frac{(m+N)\log(1+\lambda)}{\lambda}\mathfrak{D}_{m+N-1,\lambda}\frac{t^{m}}{m!}.\end{aligned} \tag{17}$$

Comparing the coefficients of (16) and (17) gives

$$\frac{(m+N)\log(1+\lambda)}{\lambda}\mathfrak{D}_{m+N-1,\lambda} = (-1)^{N-1}(N-1)!\sum_{n=0}^{m}(-1)^n N^n S_1(m,n)\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m}. \tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 2.1. *For $N\in\mathbb{N}$ and $m\geq 0$, we have*

$$\mathfrak{D}_{m+N-1,\lambda} = \frac{(-1)^{N-1}(N-1)!}{m+N}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m-1}\sum_{n=0}^{m}(-1)^n N^n S_1(m,n).$$

On the other hand, we note that

$$\begin{aligned}\mathcal{F}^{n} &= \left(\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)\right)^n\\ &= \left(\frac{\lambda\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)}{t\log(1+\lambda)}\right)^n\left(\frac{t\log(1+\lambda)}{\lambda}\right)^n\\ &= \left(\sum_{l=0}^{\infty}\mathfrak{D}_{l,\lambda}^{(n)}\frac{t^l}{l!}\right)\left(\frac{\log(1+\lambda)}{\lambda}\right)^n t^n\\ &= \sum_{l=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n\mathfrak{D}_{l,\lambda}^{(n)}(l+n)_n\frac{t^{l+n}}{(l+n)!}.\end{aligned} \tag{19}$$

Thus by (19), it follows that

$$\begin{aligned} e^{-N\mathcal{F}} &= \sum_{n=0}^{\infty}(-1)^n N^n \frac{1}{n!}\mathcal{F}^n \\ &= \sum_{n=0}^{\infty}(-1)^n N^n \frac{1}{n!}\sum_{l=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n \mathfrak{D}_{l,\lambda}^{(n)}(l+n)_n \frac{t^{l+n}}{(l+n)!} \\ &= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}(-1)^n N^n \binom{m}{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n \mathfrak{D}_{m-n,\lambda}^{(n)}\right]\frac{t^m}{m!}. \end{aligned} \tag{20}$$

From (14) and (20), we derive

$$\mathcal{F}^{(N)} = (-1)^{N-1}(N-1)!\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}(-1)^n N^n \binom{m}{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n \mathfrak{D}_{m-n,\lambda}^{(n)}\right]\frac{t^m}{m!}. \tag{21}$$

Therefore, by(17) and (21), we observe the following theorem.

Theorem 2.2. *For $N \in \mathbb{N}$ and $m \geq 0$, we have*

$$\mathfrak{D}_{m+N-1,\lambda} = \frac{(-1)^N(N-1)!}{m+N}\sum_{n=0}^{m}\binom{m}{n}(-1)^n N^n \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+n-1}\mathfrak{D}_{m-n,\lambda}^{(n)}.$$

3. Modified degenerate Daehee polynomials associated to differential equations

In this section, we consider the modified degenerate Daehee polynomials $\mathfrak{D}_{n,\lambda}(x)$ by the generating function

$$\frac{\lambda\log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)}{t\log(1+\lambda)}\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)^x = \sum_{n=0}^{\infty}\mathfrak{D}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{22}$$

When $x = 0$, we have the modified degenerate Daehee numbers i.e, $\mathfrak{D}_{n,\lambda} = \mathfrak{D}_{n,\lambda}(0)$.

From (22), we have the following

$$\begin{aligned} \sum_{n=0}^{\infty}\mathfrak{D}_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty}\mathfrak{D}_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x)_l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}(x)_{n-m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-m}\mathfrak{D}_{m,\lambda}(x)\right]\frac{t^n}{n!}. \end{aligned} \tag{23}$$

Thus by (23), we have

$$\begin{aligned} \mathfrak{D}_{n,\lambda}(x) &= \sum_{m=0}^{n}(x)_{n-m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-m}\mathfrak{D}_{m,\lambda}(x) \\ &= \sum_{m=0}^{n}(x)_m\left(\frac{\log(1+\lambda)}{\lambda}\right)^m\mathfrak{D}_{n-m,\lambda}(x). \end{aligned} \tag{24}$$

Using the notation in (10) gives

$$e^{\mathcal{F}} - 1 = \frac{t\log(1+\lambda)}{\lambda}. \tag{25}$$

Accordingly, (22) can be rewritten as follows

$$\frac{\mathcal{F}}{e^{\mathcal{F}}-1}e^{\mathcal{F}x}=\sum_{n=0}^{\infty}\mathfrak{D}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{26}$$

Thus by using the generating function of Bernoulli polynomials in (1), we have

$$\begin{aligned}\frac{\mathcal{F}}{e^{\mathcal{F}}-1}e^{\mathcal{F}x}&=\sum_{m=0}^{\infty}\mathrm{B}_m(x)\frac{\mathcal{F}^m}{m!}\\&=\sum_{m=0}^{\infty}\mathrm{B}_m(x)\sum_{n=m}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n S_1(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\mathrm{B}_m(x)\right]\frac{t^n}{n!}.\end{aligned} \tag{27}$$

Consequently, from (26) and (27), it follows that

$$\mathfrak{D}_{n,\lambda}(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\mathrm{B}_m(x). \tag{28}$$

Also, replacing t by $\frac{\lambda}{\log(1+\lambda)}(e^t-1)$ in (22) derives

$$\mathrm{B}_n(x)=\sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_2(n,m)\mathfrak{D}_{m,\lambda}(x). \tag{29}$$

Let

$$\mathcal{G}=\mathcal{G}(t,x)=\mathcal{F}(t)e^{\mathcal{F}(t)x}. \tag{30}$$

Then by (11) and (30), we get

$$\begin{aligned}\mathcal{G}'&=\frac{d}{dt}\mathcal{G}(t,x)=\mathcal{F}'e^{\mathcal{F}x}+\mathcal{F}e^{\mathcal{F}x}\mathcal{F}'x\\&=\frac{\log(1+\lambda)}{\lambda}e^{\mathcal{F}(x-1)}(1+x\mathcal{F}),\end{aligned} \tag{31}$$

$$\mathcal{G}^{(2)}=\left(\frac{\log(1+\lambda)}{\lambda}\right)^2 e^{\mathcal{F}(x-2)}\left\{(x-1)+x+(x)_2\mathcal{F}\right\} \tag{32}$$

and

$$\mathcal{G}^{(3)}=\left(\frac{\log(1+\lambda)}{\lambda}\right)^3 e^{\mathcal{F}(x-3)}\left\{(x-2)(x-1)+(x-2)x+(x-1)x+(x)_3\mathcal{F}\right\}. \tag{33}$$

Continuing this process, we are led to put

$$\mathcal{G}^{(N)}=\left(\frac{\log(1+\lambda)}{\lambda}\right)^N e^{\mathcal{F}(x-N)}\left\{a_N(x)+(x)_N\mathcal{F}\right\}, \tag{34}$$

for $N=1,2,3,\cdots$.

From (11) and (34), we yield

$$\begin{aligned}\mathcal{G}^{(N+1)} &= \frac{d}{dt}\mathcal{G}^{(N)}\\ &= \left(\frac{\log(1+\lambda)}{\lambda}\right)^N e^{\mathcal{F}(x-N)}\mathcal{F}'\left\{a_N(x)+(x)_N\mathcal{F}\right\}\\ &\quad+\left(\frac{\log(1+\lambda)}{\lambda}\right)^N e^{\mathcal{F}(x-N)}(x)_N\mathcal{F}'\\ &= \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+1} e^{\mathcal{F}(x-(N+1))}\left\{(x-N)a_N(x)+(x)_N+(x)_{N+1}\mathcal{F}\right\}.\end{aligned} \tag{35}$$

Furthermore, by replacing N by $N+1$ in (34), we get

$$\mathcal{G}^{(N+1)} = \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+1} e^{\mathcal{F}(x-(N+1))}\left\{a_{N+1}(x)+(x)_{N+1}\mathcal{F}\right\}. \tag{36}$$

Now comparing (35) and (36) result in

$$a_{N+1}(x) = (x-N)a_N(x)+(x)_N, \tag{37}$$

where $N = 1, 2, 3, \cdots$.

From (31), (32), (33) and (34), we have

$$\begin{aligned}&a_1(x)=1, \quad a_2(x)=(x-1)+x=-1,\\ &a_3(x)=(x-2)(x-1)+(x-2)x+(x-1)x,\\ &\cdots.\end{aligned} \tag{38}$$

This implies that

$$a_N(x) = \sum_{j=0}^{N-1}\frac{(x)_N}{x-j}. \tag{39}$$

By (34) and (37), we arrive at

$$\mathcal{G}^{(N)} = \left(\frac{\log(1+\lambda)}{\lambda}\right)^N\left(\sum_{j=0}^{N-1}\frac{(x)_N}{x-j}e^{\mathcal{F}(x-N)}+(x)_N\mathcal{F}e^{\mathcal{F}(x-N)}\right). \tag{40}$$

Therefore, we obtain the following theorem.

Theorem 3.1. *For $N \in \mathbb{N}$, let we consider the following family of differential equations*

$$\mathcal{G}^{(N)} = \left(\frac{\log(1+\lambda)}{\lambda}\right)^N (x)_N\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)^{-N}\left(\frac{1}{\mathcal{F}}\sum_{j=1}^{N-1}\frac{1}{x-j}+1\right)\mathcal{G}$$

have a solution

$$\mathcal{G} = \log\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)\left(1+\frac{t}{\lambda}\log(1+\lambda)\right)^x,$$

where $\mathcal{F}$ is in (10).

Now we acquire that

$$\begin{aligned}\mathcal{G}^{(N)} &= \frac{d^N}{dt^N}\left(\frac{\mathcal{F}e^{x\mathcal{F}}}{\frac{t}{\lambda}\log(1+\lambda)}\frac{t}{\lambda}\log(1+\lambda)\right)\\ &= \frac{d^N}{dt^N}\left(\sum_{m=0}^{\infty}\mathfrak{D}_{m,\lambda}(x)\frac{t^m}{m!}\frac{t}{\lambda}\log(1+\lambda)\right)\\ &= \frac{\log(1+\lambda)}{\lambda}\frac{d^N}{dt^N}\left(\sum_{m=0}^{\infty}m\mathfrak{D}_{m-1,\lambda}(x)\frac{t^m}{m!}\right)\\ &= \frac{\log(1+\lambda)}{\lambda}\sum_{m=0}^{\infty}(m+N)\mathfrak{D}_{m+N-1,\lambda}(x)\frac{t^m}{m!}.\end{aligned} \tag{41}$$

For the right hand side of (40), we observe the followings

$$\begin{aligned}e^{\mathcal{F}(x-N)} &= \sum_{n=0}^{\infty}\frac{\mathcal{F}^n}{n!}(x-N)^n\\ &= \sum_{n=0}^{\infty}(x-N)^n\sum_{m=n}^{\infty}S_1(m,n)\frac{t^m}{m!}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m\\ &= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}(x-N)^n\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(m,n)\right]\frac{t^m}{m!}\end{aligned} \tag{42}$$

and

$$\begin{aligned}\mathcal{F}e^{\mathcal{F}(x-N)} &= \sum_{m=0}^{\infty}\mathfrak{D}_{m,\lambda}(x-N)\frac{t^{m+1}}{m!}\frac{\log(1+\lambda)}{\lambda}\\ &= \sum_{m=0}^{\infty}\frac{m\log(1+\lambda)}{\lambda}\mathfrak{D}_{m-1,\lambda}(x-N)\frac{t^m}{m!}.\end{aligned} \tag{43}$$

Applying (42) and (43) to (41) gives

$$\begin{aligned}&\frac{\log(1+\lambda)}{\lambda}(m+N)\mathfrak{D}_{m+N-1,\lambda}(x)\\ &= \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m}\sum_{j=0}^{N-1}\frac{(x)_N}{x-j}\sum_{n=0}^{m}(x-N)^n S_1(m,n)\\ &\quad + (x)_N m\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+1}\mathfrak{D}_{m-1,\lambda}(x-N).\end{aligned} \tag{44}$$

As a result of (44), we have the following theorem.

Theorem 3.2. *For $N \in \mathbb{N}$ and $m \geq 0$, we have*

$$\begin{aligned}&\mathfrak{D}_{m+N-1,\lambda}(x)\\ &= \frac{1}{m+N}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m-1}\sum_{j=0}^{N-1}\frac{(x)_N}{x-j}\sum_{n=0}^{m}(x-N)^n S_1(m,n)\\ &\quad + \frac{m(x)_N}{m+N}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N}\mathfrak{D}_{m-1,\lambda}(x-N).\end{aligned}$$

Specially, if we take $x = 0$, we have just the same result as that of Theorem 2.1.

Now, we consider the higher order modified degenerate Daehee polynomials $\mathfrak{D}_{n,\lambda}^{(r)}(x)$ give by the generating function to be, for $r \in \mathbb{N}$,

$$\left(\frac{\lambda \log\left(1 + \frac{t}{\lambda}\log(1+\lambda)\right)}{t\log(1+\lambda)}\right)^r \left(1 + \frac{t}{\lambda}\log(1+\lambda)\right)^x = \sum_{n=0}^{\infty} \mathfrak{D}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{45}$$

When $x = 0$, $\mathfrak{D}_{n,\lambda}^{(r)} = \mathfrak{D}_{n,\lambda}^{(r)}(0)$ are the higher order modified degenerate Daehee numbers.

On the other hand, we observe

$$e^{(\mathcal{F}-N)x} = \sum_{n=0}^{\infty} \frac{(x-N)^n}{n!}\mathcal{F}^n. \tag{46}$$

By applying (19) to (46), we have

$$\begin{aligned} e^{(\mathcal{F}-N)x} &= \sum_{n=0}^{\infty} \frac{(x-N)^n}{n!} \sum_{l=0}^{\infty} \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \mathfrak{D}_{l,\lambda}^{(n)}(l+n)_n \frac{t^{l+n}}{(l+n)!} \\ &= \sum_{m=0}^{\infty} \left[\sum_{n=0}^{m} \binom{m}{n} \left(\frac{(x-N)\log(1+\lambda)}{\lambda}\right)^n \mathfrak{D}_{m-n,\lambda}^{(n)}\right] \frac{t^m}{m!}. \end{aligned} \tag{47}$$

Therefore, applying (43) and (47) to (40) results in

$$\begin{aligned} &\frac{\log(1+\lambda)}{\lambda}(m+N)\mathfrak{D}_{m+N-1,\lambda}(x) \\ &= \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m} \sum_{j=0}^{N-1} \frac{(x)_N}{x-j} \sum_{n=0}^{m} (x-N)^n \binom{m}{n} \mathfrak{D}_{m-N,\lambda}^{(n)} \\ &\quad + (x)_N m \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+1} \mathfrak{D}_{m-1,\lambda}^{(n)}(x-N). \end{aligned} \tag{48}$$

Finally, we have the following theorem.

Theorem 3.3. *For $N \in \mathbb{N}$ and $m \geq 0$, we have*

$$\begin{aligned} \mathfrak{D}_{m+N-1,\lambda}(x) = \frac{1}{m+N}\Bigg\{ &\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+m-1} \sum_{j=0}^{N-1} \frac{(x)_N}{x-j} \sum_{n=0}^{m} (x-N)^n \binom{m}{n} \mathfrak{D}_{m-N,\lambda}^{(n)} \\ &+ (x)_N m \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N} \mathfrak{D}_{m-1,\lambda}^{(n)}(x-N)\Bigg\}. \end{aligned}$$

Letting $x = 0$ in the above theorem gives us the same result in Theorem 2.2.

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