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ON GRADED 2-ABSORBING COMPACTLY PACKED MODULES

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded 2-absorbing compactly packed modules and give a number of its properties.

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1. Introduction

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, in particular, we are dealing with graded 2-absorbing compactly packed modules.

The concept of graded compactly packed modules was introduced by F. Farzalipour and P. Ghiasvand in [8]. Here, we generalize this concept to the concept of graded 2-absorbing compactly packed modules and give a number of its properties.

Our article is organized as follows.

In Section 2 we recall important notions which will be used throughout the paper. In Section 3 we introduce the concept of the graded 2-absorbing radical of a graded submodule and give a number of its properties. In Section 4 we give a definition and a characterization of graded 2-absorbing compactly packed modules. We also study the behaviour of graded 2-absorbing compactly packed modules under graded homomorphisms.

2. Preliminaries

Convention. Throughout this paper all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [9], [11], [12] and [13] for these basic properties and more information on graded rings and modules. Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G} x_g$, where x_g is called homogeneous component of x in R_g. Let $R = \underset{g\in G}{\oplus} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \oplus_{g\in G}(I \cap R_g) := \oplus_{g\in G} I_g$. A right R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $M_g R_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called homogeneous. Let $M = \underset{g\in G}{\oplus} M_g$ be a graded R-module. A submodule N of M is said to be a graded submodule if $N = \oplus_{g\in G}(N \cap M_g) := \oplus_{g\in G} N_g$. In this case, N_g is called the g-component of N. A graded submodule N of a graded R-module M is said to be *a graded maximal* if $N \neq M$ and if there is a graded submodule K of M such that $N \subseteq K \subseteq M$, then $N = K$ or $K = M$. A graded R-module M is said to be *a graded finitely generated* if there exist $x_{g_1}, x_{g_2}, ..., x_{g_n} \in h(M)$ such that $M = Rx_{g_1} + \cdots + Rx_{g_n}$. A graded R-module M is said to be *a graded cyclic* (gr-cyclic) if $M = Rm_g$ where $m_g \in h(M)$.

Graded prime submodules of graded modules over a graded commutative rings have been introduced and studied in [1, 3-8, 14]. A proper graded submodule N of a graded module M over a G-graded ring R is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N :_R M)$ or $m \in N$.

The concept of the graded radical of a graded submodule of a graded module over a graded commutative ring has been introduced and studied by various authors,(see, for example [6, 14, 15]). The *graded radical of a graded submodule N of a graded R-module M*, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M then $Gr_M(N) = M$.

Graded 2-absorbing submodules of graded modules over graded commutative rings have been introduced and studied in [10, 2]. A proper graded submodule N of a graded module M over a G-graded ring R is said to be *a graded 2-absorbing submodule* if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$.

3. The graded 2-absorbing Radical of a graded Submodule

In this section we introduce the concept of the graded 2-absorbing radical of a graded submodule and give a number of its properties.

The following lemma is known (see [4] and [14]), but we write it here for the sake of references.

Lemma 3.1. *Let R be a G-graded ring and M a graded R-module. Then the following hold:*

(i) *If N is a graded submodule of M, $r \in h(R)$, $m \in h(M)$ and I is a graded ideal of R, then Rm, IN and rN are graded submodules of M.*
(ii) *If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are also graded submodules of M and $(N :_R M)$ is a graded ideal of R.*
(iii) *Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodules of M.*

Definition 3.2. Let R be a G-graded ring, M a graded R-module and N a graded submodule of M.

(i) The graded 2-absorbing radical of N in M denoted by $2abs\text{-}Gr_M(N)$ and is defined to be the intersection of all graded 2-absorbing submodules of M containing N. Should there be no graded 2-absorbing submodule of M containing N, then we put $2abs\text{-}Gr_M(N) = M$. By Lemma 3.1, it is easy to see that $2abs\text{-}Gr_M(N)$ is a graded submodule of M containing N.

(ii) We say N is a graded 2-absorbing radical submodule if $2abs\text{-}Gr_M(N) = N$.

Since every proper graded submodule of a graded finitely generated module is contained in a graded prime submodule, see [15, Corollary 2.11] and every graded prime submodule is a graded 2-absorbing, we can conclude the following Lemma.

Lemma 3.3. *Let R be a G-graded ring and M a graded finitely generated R-module. Then every proper graded submodule of M is contained in a graded 2-absorbing submodule of M.*

Theorem 3.4. *Let R be a G-graded ring, M a graded R-module and N, L graded submodules of M. Then the following hold:*

(i) $N \subseteq 2abs\text{-}Gr_M(N)$.

(ii) *If* $N \subseteq L$, *then* $2abs\text{-}Gr_M(N) \subseteq 2abs\text{-}Gr_M(L)$.

(iii) $2abs\text{-}Gr_M(N \cap L) \subseteq 2abs\text{-}Gr_M(N) \cap 2abs\text{-}Gr_M(L)$.

(iv) $2abs\text{-}Gr_M(2abs\text{-}Gr_M(N)) = 2abs\text{-}Gr_M(N)$.

(v) $2abs\text{-}Gr_M(N + L) = 2abs\text{-}Gr_M(2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L))$.

(vi) *If* $N = M$, *then* $2abs\text{-}Gr_M(N) = M$. *Moreover, if M is a graded finitely generated, then* $2abs\text{-}Gr_M(N) = M$ *if and only if* $N = M$.

Proof. (i) It is clear.

(ii) Suppose that $N \subseteq L$. If there is no graded 2-absorbing submodule of M containing L, then $2abs\text{-}Gr_M(L) = M$. Since $2abs\text{-}Gr_M(N)$ is a graded submodule of M, we have $2abs\text{-}Gr_M(N) \subseteq 2abs\text{-}Gr_M(L)$. Now, let A be a graded 2-absorbing submodule of M with $L \subseteq A$, it follows that $N \subseteq A$. Hence $2abs\text{-}Gr_M(N) \subseteq 2abs\text{-}Gr_M(L)$.

(iii) By (ii), we have $2abs\text{-}Gr_M(N \cap L) \subseteq 2abs\text{-}Gr_M(N)$ and $2abs\text{-}Gr_M(N \cap L) \subseteq 2abs\text{-}Gr_M(L)$. Thus $2abs\text{-}Gr_M(N \cap L) \subseteq 2abs\text{-}Gr_M(N) \cap 2abs\text{-}Gr_M(L)$.

(iv) By (i) and (ii), we have $2abs\text{-}Gr_M(N) \subseteq 2abs\text{-}Gr_M(2abs\text{-}Gr_M(N))$. we will show that $2abs\text{-}Gr_M(2abs\text{-}Gr_M(N)) \subseteq 2abs\text{-}Gr_M(N)$. If there is no graded 2-absorbing submodule of M containing N, then $2abs\text{-}Gr_M(N) = M$. Thus $2abs\text{-}Gr_M(2abs\text{-}Gr_M(N)) \subseteq 2abs\text{-}Gr_M(N)$. Now, let A be a graded 2-absorbing submodule of M such that $N \subseteq A$. Then by definition of $2abs\text{-}Gr_M(N)$, we have $2abs\text{-}Gr_M(N) \subseteq A$. So $2abs\text{-}Gr_M(2abs\text{-}Gr_M(N)) \subseteq 2abs\text{-}Gr_M(N)$. Thus $2abs\text{-}Gr_M(2abs\text{-}Gr_M(N)) = 2abs\text{-}Gr_M(N)$.

(v) By (i), $N \subseteq 2abs\text{-}Gr_M(N)$ and $L \subseteq 2abs\text{-}Gr_M(L)$. Hence $N + L \subseteq 2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L)$. By (ii), we conclude that $2abs\text{-}Gr_M(N + L) \subseteq 2abs\text{-}Gr_M(2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L))$. Since N, $L \subseteq N +$

L, we have $2abs\text{-}Gr_M(N)$, $2abs\text{-}Gr_M(L) \subseteq 2abs\text{-}Gr_M(N+L)$. Hence $2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L) \subseteq 2abs\text{-}Gr_M(N+L)$. By (*ii*) and (*iv*), we have $2abs\text{-}Gr_M(2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L)) \subseteq 2abs\text{-}Gr_M(N+L)$. Therefore $2abs\text{-}Gr_M(N+L) = 2abs\text{-}Gr_M(2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L))$.

(vi) Suppose that $N = M$, hence $2abs\text{-}Gr_M(N) = 2abs\text{-}Gr_M(M) = M$. Now, let M be a graded finitely generated and $2abs\text{-}Gr_M(N) = M$. Suppose to the contrary that $N \neq M$. By Lemma 3.3, $N \subseteq A$ for some graded 2-absorbing submodule A of M. Thus $2abs\text{-}Gr_M(N) \neq M$, which is a contradiction.

□

Theorem 3.5. *Let R be a G-graded ring, M a graded finitely generated R-module and N, L graded submodules of M. Then $2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L) = M$ if and only if $N + L = M$.*

Proof. ($\Rightarrow$) Assume that $2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L) = M$ and $N+L \neq M$. By Lemma 3.3, there exists a graded 2-absorbing submodule A of M such that $N + L \subseteq A$. Since N, $L \subseteq N + L \subseteq A$, we have $2abs\text{-}Gr_M(N)$, $2abs\text{-}Gr_M(L) \subseteq A$. Thus $2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L) \subseteq A$, which is a contradiction.
($\Leftarrow$) Since $N \subseteq 2abs\text{-}Gr_M(N)$ and $L \subseteq 2abs\text{-}Gr_M(L)$, it is clear that $N+L = M$ implies $2abs\text{-}Gr_M(N) + 2abs\text{-}Gr_M(L) = M$. □

4. GRADED 2-ABSORBING COMPACTLY PACKED MODULES

In [8], the concept of graded compactly packed modules was introduced. A proper graded submodule N of a graded R-module M is said to be *a graded compactly packed by graded prime submodules* if for each family $\{N_\alpha\}_{\alpha\in\triangle}$ of graded prime submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, we have that $N \subseteq N_\beta$ for some $\beta \in \triangle$. A graded module M over a G-graded ring R is called a *graded compactly packed by graded prime submodules* if every graded submodule of M is a graded compactly packed by graded prime submodules. In what follows we will generalize this concept to the concept of graded 2-absorbing compactly packed modules and give a number of its properties.

Definition 4.1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is called *a graded 2-absorbing compactly packed* if for each family $\{A_\alpha\}_{\alpha\in\Delta}$ of graded 2-absorbing submodules of M with $N \subseteq \bigcup_{\alpha\in\triangle} A_\alpha$, $N \subseteq A_\beta$ for some $\beta \in \Delta$. A graded module M is called *a graded 2-absorbing compactly packed module* if every proper graded submodule of M is a graded 2-absorbing compactly packed.

Since every graded prime submodule is a graded 2-absorbing submodule, the following theorem holds.

Theorem 4.2. *Every graded 2-absorbing compactly packed module is a graded compactly packed by graded prime submodules.*

The next theorem gives a characterization of being a graded 2-absorbing compactly packed module in terms of graded 2-absorbing radical submodules.

Theorem 4.3. *Let R be a G-graded ring and M a graded R-module. The following statements are equivalent :*

(i) *M is a graded 2-absorbing compactly packed module.*
(ii) *For each proper graded submodule N of M there exists $t_g \in N \cap h(M)$ such that $2abs\text{-}Gr_M(N) = 2abs\text{-}Gr_M(Rt_g)$.*
(iii) *For each proper graded submodule N of M, if $\{N_\alpha\}_{\alpha\in\Delta}$ is a family of graded submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$, then $N \subseteq 2abs\text{-}Gr_M(N_\beta)$ for some $\beta \in \Delta$.*
(iv) *For each proper graded submodule N of M, if $\{A_\alpha\}_{\alpha\in\Delta}$ is a family of graded 2-absorbing radical submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} A_\alpha$, then $N \subseteq A_\beta$ for some $\beta \in \Delta$.*

Proof. ($i \Rightarrow ii$) Let N be a proper graded submodule of M. By Theorem 3.4, $2abs\text{-}Gr_M(Rt_g) \subseteq 2abs\text{-}Gr_M(N)$ for each $t_g \in N \cap h(M)$. Now, suppose that $2abs\text{-}Gr_M(N) \nsubseteq 2abs\text{-}Gr_M(Rt_g)$ for each $t_g \in N \cap h(M)$. Then for each $t_g \in N \cap h(M)$ there exists a graded 2-absorbing submodule A_{t_g}, which contains Rt_g and $N \nsubseteq A_{t_g}$. But $N = \bigcup_{t_g\in N} Rt_g \subseteq \bigcup_{t_g\in N} A_{t_g}$, that is M is not graded 2-absorbing compactly packed which contradicts (i).

($ii \Rightarrow iii$) Let N be a proper graded submodule of M and let $\{N_\alpha\}_{\alpha\in\Delta}$ be a family of graded submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} N_\alpha$. By (ii) there exists $t_g \in N \cap h(M)$ such that $2abs\text{-}Gr_M(N) = 2abs\text{-}Gr_M(Rt_g)$. Then $t_g \in \bigcup_{\alpha\in\triangle} N_\alpha$ and hence $t_g \in N_\beta$ for some $\beta \in \Delta$, so that $Rt_g \subseteq N_\beta$ and by Theorem 3.4 we conclude that $N \subseteq 2abs\text{-}Gr_M(N) = 2abs\text{-}Gr_M(Rt_g) \subseteq 2abs\text{-}Gr_M(N_\beta)$.

($iii \Rightarrow iv$) Let N be a proper graded submodule of M. Let $\{A_\alpha\}_{\alpha\in\triangle}$ be a family of graded 2-absorbing radical submodules of M and $N \subseteq \bigcup_{\alpha\in\triangle} A_\alpha$. By (iii) there exists $\beta \in \triangle$ such that $N \subseteq 2abs\text{-}Gr_M(A_\beta)$. Since A_β is a graded 2-absorbing radical submodule $A_\beta = 2abs\text{-}Gr_M(A_\beta)$. Thus $N \subseteq A_\beta$.

($iv \Rightarrow i$) Let N be a proper graded submodule of M and let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of graded 2-absorbing submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} A_\alpha$. Since A_α is a graded 2-absorbing for each $\alpha \in \Delta$, we get $A_\alpha = 2abs\text{-}Gr_M(A_\alpha)$. Thus $N \subseteq \bigcup_{\alpha\in\triangle} A_\alpha = \bigcup_{\alpha\in\triangle} 2abs\text{-}Gr_M(A_\alpha)$. By (iv), there exists $\beta \in \triangle$ such that $N \subseteq 2abs\text{-}Gr_M(A_\beta) = A_\beta$. Thus M is a graded 2-absorbing compactly packed module. □

Theorem 4.4. *Let R be a G-graded ring and M a graded R-module such that $2abs\text{-}Gr_M(N) = N$ for all graded submodules N of M. Then M is a graded 2-absorbing compactly packed module if and only if every proper graded submodule of M is gr-cyclic.*

Proof. ($\Rightarrow$) Let N be a proper graded submodule of M. Since M is a graded 2-absorbing compactly packed module, by Theorem 4.3, there exists $t_g \in N \cap h(M)$ such that $2abs\text{-}Gr_M(N) = 2abs\text{-}Gr_M(Rt_g)$. By our assumption $N = Rt_g$. Therefore N is gr-cyclic.

($\Leftarrow$) Let N be a proper graded submodule of M, and let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of graded 2-absorbing submodules of M such that $N \subseteq \bigcup_{\alpha\in\triangle} A_\alpha$. Since N is a gr-cyclic, $N = Rt_g$ for some $t_g \in N \cap h(M)$. Since $t_g \in N = Rt_g \subseteq \bigcup_{\alpha\in\triangle} A_\alpha$, $t_g \in A_\beta$ for some $\beta \in \Delta$. Thus $N = Rt_g \subseteq A_\beta$. Therefore M is a graded 2-absorbing compactly packed module. □

Theorem 4.5. *Let R be a G-graded ring and M a graded R-module. If M is a graded 2-absorbing compactly packed which has at least one graded maximal submodule, then M satisfies the ascending chain condition on graded 2-absorbing radical submodules.*

Proof. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be an ascending chain of graded 2-absorbing radical submodules of M. If $A_k = M$ for some k, then the result follows immediately, so assume that none of A_k's is M and let $A = \cup_{i=1}^{\infty} A_i$. We claim that A is a proper graded submodule of M. Assume on contrary that $A = M$ and let L be a graded maximal submodule of M. Then $L \subseteq \cup_{i=1}^{\infty} A_i$. Since M is a graded 2-absorbing compactly packed, by Theorem 4.3, $L \subseteq A_k$ for some k. Hence $L = A_k$ and thus A_k is a graded maximal. Hence $A_k = A_i$ for all $i \geq k$ it follows that $A_k = \cup_{i=1}^{\infty} A_i = M$, which is impossible. Thus A is a proper graded submodule of M. Since M is a graded 2-absorbing compactly packed, by Theorem 4.3, we have $A \subseteq A_s$ for some s and hence $A_s = A_i$ for all $i \geq s$. Therefore the ascending chain condition is satisfied on graded 2-absorbing radical submodules. □

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$, (see [11].) A graded homomorphism that is an injective function will be referred to simply as a monomorphism and a graded homomorphism that is a surjective function will be called an epimorphism. If $\varphi : M \to M'$ is an epimorphism, then M' is said to be a graded homomorphic image of M.

Lemma 4.6. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded homomorphism.*
(i) *If A' is a graded 2-absorbing submodule of M', then $\varphi^{-1}(A')$ is a graded 2-absorbing submodule of M.*
(ii) *If φ is an epimorphism and A is a graded 2-absorbing submodule of M containing $ker(\varphi)$, then $\varphi(A)$ is a graded 2-absorbing submodule of M'.*

Proof. (i) Assume that A' is a graded 2-absorbing submodule of M' and let $r, s \in h(R)$ and $m \in h(M)$ such that $rsm \in \varphi^{-1}(N')$. Then $rs\varphi(m) \in A'$. Since A' is a graded 2-absorbing submodule of M', we have either $rs \in (A' :_R M')$ or $r\varphi(m) \in A'$ or $s\varphi(m) \in A'$ and thus $rs \in (\varphi^{-1}(A') :_R M)$ or $rm \in \varphi^{-1}(A')$ or $sm \in \varphi^{-1}(A')$. Therefore $\varphi^{-1}(A')$ is a graded 2-absorbing submodule of M.
(ii) Assume that A is a graded 2-absorbing submodule of M. Let $r, s \in h(R)$, $m' \in h(M')$ and $rsm' \in \varphi(A)$. Since $rsm' \in \varphi(A)$, there exists $a \in A \cap h(M)$ such that $\varphi(a) = rsm'$. Since $m' \in h(M')$ and φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$ and so

$\varphi(rsm) \in \varphi(A)$. Hence $\varphi(a) = rs\varphi(m)$ this implies $\varphi(a - rsm) = 0$. Thus $a - rsm \in Ker\varphi \subseteq A$ which implies that $rsm \in A$. Since A is a graded 2-absorbing submodule, we have either $rs \in (A :_R M)$ or $rm \in A$ or $sm \in A$. If $rs \in (A :_R M)$, then $rsM \subseteq A$ and so $\varphi(rsM) \subseteq \varphi(A)$ which implies that $rsM' \subseteq \varphi(A)$ and we get that $rs \in (\varphi(A) :_R M')$. If $rm \in A$, then $\varphi(rm) = r\varphi(m) \in \varphi(A)$ so $rm' \in \varphi(A)$. Similarly, $sm \in A$ implies that $sm' \in \varphi(A)$. Consequently $\varphi(A)$ is a graded 2-absorbing submodule of M'.

□

In what follows we show that a graded module is graded 2-absorbing compactly packed if and only if its homomorphic image is graded 2-absorbing compactly packed whenever the respective kernel is contained in any graded 2-absorbing submodule.

Theorem 4.7. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be a graded epimorphism. If M is a graded 2-absorbing compactly packed, then so is M'. The converse is true if $ker(\varphi) \subseteq 2abs\text{-}Gr_M(\{0\})$.*

Proof. Assume that M is a graded 2-absorbing compactly packed module. Let N' be a proper graded submodule of M' and let $\{A'_\alpha\}_{\alpha\in\Delta}$ be a family of graded 2-absorbing submodules of M' such that $N' \subseteq \cup_{\alpha\in\Delta} A'_\alpha$. Since φ is a graded epimorphism, $\varphi^{-1}(N') \subseteq \varphi^{-1}(\cup_{\alpha\in\Delta} A'_\alpha)$. Thus $\varphi^{-1}(N') \subseteq \cup_{\alpha\in\Delta}\varphi^{-1}(A'_\alpha)$. By Lemma 4.6(i), $\varphi^{-1}(A'_\alpha)$ is a graded 2-absorbing submodule of M for each $\alpha \in \Delta$. Since M is a graded 2-absorbing compactly packed, there exists $\beta \in \Delta$ such that $\varphi^{-1}(N') \subseteq \varphi^{-1}(A'_\beta)$. Thus $N' \subseteq A'_\beta$ for some $\beta \in \Delta$. Therefore M' is a graded 2-absorbing compactly packed. Now assume that M' is a graded 2-absorbing compactly packed module and $ker(\varphi) \subseteq 2abs\text{-}Gr_M(\{0\})$. Let N be a proper graded submodule of M and let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of graded 2-absorbing submodules of M such that $N \subseteq \cup_{\alpha\in\Delta} A_\alpha$. Then $\varphi(N) \subseteq \varphi(\cup_{\alpha\in\Delta} A_\alpha)$ and thus $\varphi(N) \subseteq \cup_{\alpha\in\Delta}\varphi(A_\alpha)$. But $Ker(\varphi) \subseteq A_\alpha$ for each $\alpha \in \Delta$. Therefore by Lemma 4.6(ii), $\varphi(A_\alpha)$ is a graded 2-absorbing submodule of M'. Since M' is a graded 2-absorbing compactly packed, we have $\varphi(N) \subseteq \varphi(A_\beta)$ for some $\beta \in \Delta$. Now, we show that $N \subseteq A_\beta$. Let $t = \sum_{g\in G} t_g \in N$, for $g \in G$, $t_g \in N$ and so $\varphi(t_g) \in \varphi(N) \subseteq \varphi(A_\beta)$. Thus there exists $a \in A_\beta \cap h(M)$ such that $\varphi(t_g) = \varphi(a)$. So $t_g - a \in Ker(\varphi) \subseteq A_\beta$, it follows that $t_g \in A_\beta$. Hence $N \subseteq A_\beta$. Therefore M is a graded 2-absorbing compactly packed. □

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