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A NEW LOOK AT THE M/G/1 RETRIAL QUEUE: A MARTINGALE APPROACH

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ABSTRACT. In this paper we provide a new approach which uses martingale method for studying the $M/G/1$ retrial queue. Using recursive equation for the process embedded at departure epochs, we construct a discrete-time martingale stopped at the first passage time when the system becomes empty. We derive the stability condition and study the busy period of this system.

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1. INTRODUCTION

Queueing Theory aims to provide quantitative performance evaluation methodology in connection with some practical questions arising in Communication Systems and Networks (throughput, load, response time ...) and also qualitative evaluation (stability, ergodicity, comparability ...), amongst others. Retrial queueing models are characterized by the feature that arrivals who find the server busy upon arrival may join a virtual group of blocked customers called orbit and repeat their requests for service after a random amount of time.
For a comprehensive survey of retrial queues, see the survey of Yang & Templeton [12], Falin [5], Aissani [1], Artalejo [2] and Kim [7]. We refer also to the books of Falin & Templeton [6] and Artalejo & Gomez-Corral [3]. The $M/G/1$ retrial queue is one of fundamental models of Retrial Queueing Theory which have been widely used to model many practical situations in telephone systems and telecommunication networks. It has been studied by different mathematical approaches: supplementary variable and semi-Markov methods, embedded Markov chain, regenerative approache, QBD method and so on ...
In this paper we develop an alternative method based on Martingale Theory to study stability properties of the $M/G/1$ retrial queue. The theory, originally applied by Baccelli and Makowski [4] to classical FIFO $M/G/1$ queueing model. It derives its key result from Doob's Optional Sampling Theorem [8]. Martingale is a powerful tool of probability theory which has been popularized in financial modeling (game strategy) and next in several other areas including queueing theory. Indeed, one of the most important question in Probability Theory is the study of dependency between random variates or more abstract elements [10]. There is several

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ways to investigate this dependency. In Stationary (in a large sense) Random Processes Theory (K-Theory), the basic indicators are the properties of the covariance functions. In the Markov Processes Theory, the basic elements are the transition functions (defined by a Markov dependency) which entirely define the evolution of the process or the queueing system under study. Martingales are a quiet general class of stochastic processes for which the properties are based on those of the conditional mathematical expectation. The interpretation of this stochastic process is somewhat interesting in connection with Game Theory. Indeed a martingale's value can change; however, its expectation remains constant in time. More important, the expectation of a martingale is unaffected by optional sampling.
In some recent papers, M. Rhoughan [9], (2002) develop an alternative method, which uses martingale and renewal methods to find solution for a general class of $M/G/1$ type-queue. The analysis uses the process embedded at departures to create a martingale, which makes possible the calculation of the probability generating function of the stationary occupancy distribution. The advantage of the martingale, is that it provides a deepen analysis of the system helping to study a more wide its extension than traditional methods.

The paper is organised as follows. The model is described in detail in section 2, together with notations and assumptions used throughout the paper. Section 3 defines the martingale from which we derive our results. The main results of the present paper are presented in section 4, in particular theorem 2 gives the stability conditions for the $M/G/1$ retrial queue and theorem 4 gives an explicit expression of the mean busy period.

2. The Mathematical Model

We consider a single server retrial queue in which customers arrive according to a Poisson process with rate λ, these customers are identified as primary calls. If the server is free at the instant of a primary call, the arriving customer begins service immediately and leaves the system after service completion. Otherwise, if at the arrival time, the customer finds the server busy, then it enters orbit and becomes a source of repeated calls (secondary call). The service times are independent and identically distributed with probability distribution fonction $B(x)$ and Laplace -Stieljes transform $\beta(\theta) = \int_0^\infty e^{-\theta x} dB(x)$, $\beta_k = (-1)^k \beta^{(k)}(0)$ is the k^{th} moment of service time. The retrial time (the time interval between two consecutive repeated attempts) is exponentially distributed with rate ν. We also assume that interarrival times, retrial times, and service times are mutually independent. Let $X(t)$ be the number of customers in the system at time t. Note that this process is not a Markov chain since it depends on the history of the process and not just upon the current state. So, we consider the discrete time embedded process $\{X_n,\ n \geq 1\}$, where X_n is the number of customers in the system seen by the n^{th} departing customer. Formally, if the departure epochs are $t_1, t_2, ...$ then

$$\{X_n = X(t_n);\ n = 1, 2, ...\},$$

is now an embedded Markov chain.
With these definitions, the queue size sequence $\{X_n,\ n = 0, 1, ...\}$ is readily seen to

satisfy the recursion

$$X_{n+1} = X_n + A_{n+1} - \delta_{X_n},\ n \geq 0, \tag{1}$$

where A_n is the number of primary calls which arrive in the system during the service time of the n^{th} call and δ_{X_n} is the number of sources which enter service at time t_n, $\delta_{X_n} = 1$ if the $(n+1)^{th}$ customer comes from orbit and $\delta_{X_n} = 0$ if the customer is a primary call. The Bernoulli random variable δ_{X_n} depends on the history of the system and its conditional distribution is given by

$$P\{\delta_{X_n} = 0/X_n = k\} = \frac{\lambda}{\lambda + k\nu}, \tag{2}$$

$$P\{\delta_{X_n} = 1/X_n = k\} = \frac{k\nu}{\lambda + k\nu}. \tag{3}$$

The random variables A_n are independent with common probability fonction

$$P(A_n = k) = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dB(t),\ k = 0, 1, 2, ...;\ n \geq 1, \tag{4}$$

and probability generating function

$$a(z) = \sum_{k=0}^{\infty} z^k P(A_n = k) = B^*(\lambda - \lambda z),\ 0 \leq z \leq 1. \tag{5}$$

We take $\rho = a^{'}(1)$ which is the mean number of arrivals during a single service and represents the traffic intensity. Note that $\rho = \lambda\beta_1$

2.1. **Probabilistic elements.** All random variables (rvs) and stochastic elements occuring in this paper are defined on some probability space $(\Omega, \Im, P)$. The rvs $\{\delta_{X_n},\ n = 0, 1, ...\}$ can be represented as

$$\delta_{X_n} = 1\left[U_{n+1} \leq \frac{X_n\nu}{\lambda + X_n\nu}\right],\ n = 0, 1, ... \tag{6}$$

where the rvs $\{U_{n+1},\ n = 0, 1, ...\}$ are independent and identically distributed rvs, each uniformly distributed on the unit interval $(0, 1)$.
Consider a basic filtration $\{\Im_n,\ n = 0, 1, ...\}$ where $\Im_n = \sigma(A_0, A_1, ..., A_n;\ U_1, ..., U_{n+1})$. Consequently the rvs X_n are $\Im_n$-measurable and the rvs $\{A_{n+1},\ n = 0, 1, ...\}$ and $\{U_{n+1},\ n = 0, 1, ...\}$ are mutually independent. We take $\Im = \bigcup_1^\infty \Im_n$.
With the above notations and using the properties of the conditional expectation, we get

$$E(z^{X_{n+1}}/\Im_n) = z^{X_n - \delta_{X_n}} a(z) \qquad a.s. \tag{7}$$

2.2. **Stopping times.** We consider σ as an arbitrary stopping time for $\Im_n$ and we define the random variable $\nu(\sigma)$ as the first instant after the time σ, when the system comes back to its empty state. That is

$$\nu(\sigma) = \begin{cases} inf\{n \geq 1 : X_{\sigma+n} = 0\} & if\ \sigma < \infty, \\ \infty & otherwise \end{cases} \tag{8}$$

with the convention that $inf\{\emptyset\} = +\infty$.

3. The Martingale

We can now define a martingale $M_n(z)$ with filtration $(\Im_n)$ which will help to obtain the majority of the results.

Theorem 3.1. *For each $0 < z \leq 1$, define*

$$M_n(z) = \begin{cases} z^{X_0}, & if\ n = 0, \\ z^{X_n} \dfrac{z^{\sum_{k=0}^{n-1} \delta_{X_k}}}{a(z)^n}, & for\ n = 1, 2, ... \end{cases} \tag{9}$$

The stochastic sequence $\{M_n(z), n \in N\}$ is a positive integrable martingale with respect to $(\Im_n)$.

Proof. To demonstrate that it is a martingale, we use (1) and (7). We have

$$\begin{aligned} E(M_{n+1}(z)/\Im_n) &= E\left(z^{X_{n+1}} \frac{z^{\sum_{k=0}^{n} \delta_{X_k}}}{a(z)^{n+1}} /\Im_n \right) \\ &= \frac{z^{\sum_{k=0}^{n} \delta_{X_k}}}{a(z)^{n+1}} E\big(z^{X_{n+1}}/\Im_n\big) = M_n(z)\ a.s. \end{aligned} \tag{10}$$

It is trivial to show that $M_n(z)$ is positive, therefore

$$E\big(|M_{n+1}(z)|\big) = E\big(M_n(z)\big) < \infty, \tag{11}$$

and hence the martingale is integrable. □

4. Stability results

In this section we derive a stability result for the $M/G/1$ retrial queue by means of the martingale introduced in section 3. The key of this results is the Doob's Optional Stopping Theorem [8].

Theorem 4.1. *If $\rho \leq 1$, then for $0 < z \leq 1$, the relationship below holds:*

$$E\left(1_{[\sigma\, <\, \infty,\, \nu(\sigma)\, <\, \infty]} \frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1} \delta_{X_k}}}{a(z)^{\nu(\sigma)}} /\Im_\sigma \right) = 1_{[\sigma\, <\, \infty]} z^{X_\sigma} \qquad a.s. \tag{12}$$

Proof. We consider an arbitrary $\Im_n$−stopping time σ and set $\tau(\sigma) = \sigma + \nu(\sigma)$. The rv $\tau(\sigma)$ is also $\Im_n$-stopping time.

For every $n \geq 0$, $\tau(\sigma) \wedge n$ and $\sigma \wedge n$ will still be stopping times. The stopping times σ and $\tau(\sigma)$ satisfy $\sigma \wedge n \leq \tau(\sigma) \wedge n$. Hence we may apply Doob's Optional Sampling Theorem ([8], Corollary IV-2-6, page 67) to obtain

$$E\big[M_{\tau(\sigma)\wedge n}(z)/\Im_{\sigma\wedge n}\big] = M_{\sigma\wedge n}(z) \qquad a.s. \tag{13}$$

We can rewrite(13) under the following form

$$(14)\qquad E\left[z^{X_{\tau(\sigma)\wedge n}}\frac{z^{\sum_{k=0}^{\tau(\sigma)\wedge n-1}\delta_{X_k}}}{a(z)^{\tau(\sigma)\wedge n}}/\Im_{\sigma\wedge n}\right]=z^{X_{\sigma\wedge n}}\frac{z^{\sum_{k=0}^{\sigma\wedge n-1}\delta_{X_k}}}{a(z)^{\sigma\wedge n}}\qquad a.s.$$

It is that clear that the relation

$$(15)\qquad E\big[M_{\tau(\sigma)\wedge n}(z)-M_{\sigma\wedge n}(z)/\Im_{\sigma\wedge n}\big]=0$$

hold.
Therefore

$$(16)\qquad E\big[1_{[\sigma<n,\,\nu(\sigma)<n]}M_{\tau(\sigma)}(z)/\Im_{\sigma}\big]=1_{[\sigma<n]}M_{\sigma}(z).$$

Next, from the Monotone Convergence Theorem, we get

$$(17)\qquad \lim_{n\to\infty}E\left[1_{[\sigma<n,\,\nu(\sigma)<n]}z^{X_{\tau(\sigma)}}\frac{z^{\sum_{k=0}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\tau(\sigma)}}/\Im_{\sigma}\right]$$

$$=E\left[1_{[\sigma<\infty,\,\nu(\sigma)<\infty]}z^{X_{\tau(\sigma)}}\frac{z^{\sum_{k=0}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\tau(\sigma)}}/\Im_{\sigma}\right]$$

$$=1_{[\sigma<\infty]}z^{X_{\sigma}}\frac{z^{\sum_{k=0}^{\sigma-1}\delta_{X_k}}}{a(z)^{\sigma}}\qquad a.s.$$

On $[\sigma<\infty,\nu(\sigma)<\infty]$, we have $[\tau(\sigma)<\infty]$ and therefore $X_{\tau(\sigma)}=0$.
So, we obtain

$$(18)\qquad E\left[1_{[\sigma<\infty,\,\nu(\sigma)<\infty]}\frac{z^{\sum_{k=0}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\tau(\sigma)}}/\Im_{\sigma}\right]$$

$$=1_{[\sigma<\infty]}z^{X_{\sigma}}\frac{z^{\sum_{k=0}^{\sigma-1}\delta_{X_k}}}{a(z)^{\sigma}}\qquad a.s.$$

Taking into account that the stopping time σ and $\sum_{k=0}^{\sigma-1}\delta_{X_k}$ are $\Im_{\sigma}$-measurable, we obtain (12) which proves theorem 2. □

Now, we have the following corollary of theorem 2.

Corollary 4.1. *Under the condition* $\rho\le 1$ *and for* $0<z<1$ *we have*

$$(19)\qquad P\big[\sigma<\infty,\nu(\sigma)<\infty/\Im_{\sigma}\big]=1_{[\sigma<\infty]}\qquad a.s.$$

In particular, if $\sigma<\infty$ *a.s., then* $\nu(\sigma)<\infty$ *a.s.*

Proof. Letting $z\to 1$ in (12) the corollary(4.1) is an immediate consequence of the Bounded Convergence Theorem.

□

5. Instability condition

Now, we discuss the system instability.

Theorem 5.1. *If $\rho > 1$, then*

$$lim_{n\to\infty} X_n = \infty \ a.s. \tag{20}$$

Proof. For all $0 < z < 1$ and every $n \in N$, the relation (7) takes the form

$$E\big[z^{X_{n+1}}/\Im_n\big] = z^{X_n - \delta_{X_n}} a(z) \leq z^{X_n}\big(\frac{a(z)}{z}\big) \qquad a.s. \tag{21}$$

Assume $\rho > 1$, since the function $a(.)$ is convex, and according to Takács lemma [11], there exists z_0 in the interval $(0, 1)$ such that $a(z_0) < z_0$. Let c_0 denote the constant defined by $c_0 = \frac{a(z_0)}{z_0} < 1$.

Consequently

$$E(z_0^{X_{n+1}}/\Im_n) \leq c_0 z_0^{X_n} \leq z_0^{X_n} a.s. \tag{22}$$

which proves that the rvs $\{z_0^{X_n},\ n \in N\}$ form a bounded positive $\Im_n-$submartingale and therefore converge a.s. and in the mean [8] (Th II-2-9, p.25).

Moreover, upon iterating (22), it is plain that

$$E(z_0^{X_n}) \leq c_0^n E(z_0^{X_0}) \leq c_0^n \tag{23}$$

on the other hand, by using the Dominated Convergence Theorem, we have

$$lim_n E(z_0^{X_n}) = E\big(lim_n z_0^{X_n}\big)$$

So that

$$lim_n E(z_0^{X_n}) = E(lim_n z_0^{X_n}) = 0 \tag{24}$$

Consequently, $lim_n z_0^{X_n} = 0$ a.s. for $0 < z_0 < 1$, and the conclusion (20) follows immediately.

□

The following part looks into the case where σ is the $\Im_n$-stopping time such that $X_\sigma = 0$ on the event $[\sigma < \infty]$. In this case, the random variable $\nu(\sigma)$ represents the number of customers served during a busy period.

5.1. **Busy period.**

A busy period is defined as the period that starts at an epoch when an arriving customer finds an empty system and ends at the next departure epoch at which the system is empty. The busy period of a retrial queue consists of alternating service periods and periods in which the server is free and there are customers in orbit.

Theorem 5.2. *We consider σ as a stopping time for $\Im_n$ such that $X_\sigma = 0$ on $[\sigma < \infty]$. Under the assumption $\rho \leq 1$, the mean number of customers served during a busy period is given by*

$$E[\nu(\sigma)] = \begin{cases} \frac{1}{1-\rho}\psi(1) & if \rho < 1, \\ \infty & if \rho = 1, \end{cases} \tag{25}$$

where

$$\Psi(1) = exp\left(\frac{\lambda}{\nu}\int_0^1 \frac{1-a(y)}{a(y)-y}dy\right). \tag{26}$$

Proof. As $\rho < 1$, for all z in $(0,1)$ we have $z < a(z) \leq 1$, pick a finit $\Im_n$-stopping time σ, it is plain that

$$\begin{aligned} &E\left[1_{[\nu(\sigma)\,<\,\infty]}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}}\right] - E\left[1_{[\nu(\sigma)\,<\,\infty]}z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}\right] \\ &= E\left[1_{[\nu(\sigma)\,<\,\infty]}z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}\left(\frac{1-a(z)^{\nu(\sigma)}}{a(z)^{\nu(\sigma)}}\right)\right] \\ &= \big(1-a(z)\big)E\left[1_{[\nu(\sigma)\,<\,\infty]}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}}\left(\frac{1-a(z)^{\nu(\sigma)}}{1-a(z)}\right)\right] \\ &= \big(1-a(z)\big)E\left[1_{[\nu(\sigma)\,<\,\infty]}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}}\sum_{k=0}^{\nu(\sigma)-1}a(z)^k\right]. \end{aligned} \tag{27}$$

Besides,

$$\lim_{z\to 1}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}} = 1.$$

When $z \to 1$, hence from (27) and the Monotone Convergence Theorem, now yields

$$\lim_{z\to 1}E\left[1_{[\nu(\sigma)\,<\,\infty]}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}}\sum_{k=0}^{\nu(\sigma)-1}a(z)^k\right] = E\big[1_{[\nu(\sigma)\,<\,\infty]}\nu(\sigma)\big].$$

It follows that

$$\begin{aligned} &\lim_{z\to 1}\big(1-a(z)\big)^{-1}\left(E\left[1_{[\nu(\sigma)\,<\,\infty]}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}}\right] - E\left[1_{[\nu(\sigma)\,<\,\infty]}z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}\right]\right) \\ &= E\big[1_{[\nu(\sigma)\,<\,\infty]}\nu(\sigma)\big]. \end{aligned} \tag{28}$$

We use theorem (4.1) to get

$$E\left[1_{[\nu(\sigma)\,<\,\infty]}\frac{z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}}{a(z)^{\nu(\sigma)}}\right]=E\left[z^{X_\sigma}\right], \tag{29}$$

now (28) can be rewritten as

$$E\left[1_{[\nu(\sigma)\,<\,\infty]}\nu(\sigma)\right] \tag{30}$$

$$=\lim_{z\to 1}\left(1-a(z)\right)^{-1}\left(E\left[z^{X_\sigma}\right]-E\left[1_{[\nu(\sigma)\,<\,\infty]}z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}\right]\right).$$

Applying the equality

$$E\left[1_{[\nu(\sigma)\,<\,\infty]}z^{\sum_{k=\sigma}^{\tau(\sigma)-1}\delta_{X_k}}\right]=E\left[1_{[\nu(\sigma)\,<\,\infty]}z^{\sum_{i=1}^{\nu(\sigma)}\delta_{X_i}}\right]=\varphi_{\nu(\sigma)}[\varphi_{\delta_{X_1}}(z)], \tag{31}$$

we obtain by virtue of (30) and (31)

$$E\left[1_{[\nu(\sigma)\,<\,\infty]}\nu(\sigma)\right]=\lim_{z\to 1}(a(z)-1)^{-1}\left[\varphi_{\nu(\sigma)}(\varphi_{\delta_{X_1}}(z))-E[z^{X_\sigma}]\right], \tag{32}$$

where $\varphi_{\nu(\sigma)}$ is the generating function of the number of custormers served during a busy period (see[5]).

On the event $[\sigma<\infty]$, $X_\sigma=0$ and by the corollary (4.1) $[\nu(\sigma)<\infty]$, we have

$$E[\nu(\sigma)]=\lim_{z\to 1}\frac{\varphi_{\nu(\sigma)}(\varphi_{\delta_{X_1}}(z))-1}{a(z)-1}. \tag{33}$$

Applying l'Hospital's rule, when $z\to 1$, from (33) we have

$$E[\nu(\sigma)]=\lim_{z\to 1}\frac{[\varphi_{\nu(\sigma)}(\varphi_{\delta_{X_1}}(z))]'}{a'(z)}=\lim_{z\to 1}\frac{\varphi'_{\delta_{X_1}}(z)\varphi'_{\nu(\sigma)}\left(\varphi_{\delta_{X_1}}(z)\right)}{a'(z)}. \tag{34}$$

In order to prouve this, let us calculate the first derivative at $\varphi_{\delta_{X_1}}$ with respect to z. Using the equation

$$\varphi_{\delta_{X_1}}(z)=\sum_{k=0}^{1}z^kP(\delta_{X_1}=k), \tag{35}$$

we get

$$\lim_{z\to 1}\varphi'_{\delta_{X_1}}(z)=E(\delta_{X_1})=\rho, \tag{36}$$

and

$$\varphi'_{\nu(\sigma)}\left(\varphi_{\delta_{X_1}}(z)\right) \tag{37}$$

$$=\pi'_\infty\left(0,\varphi_{\delta_{X_1}}(z)\right)exp\left\{\frac{\lambda}{\nu}\int_0^{\pi_\infty\left(0,\varphi_{\delta_{X_1}}(z)\right)}\frac{1-\pi_\infty\left(0,\varphi_{\delta_{X_1}}(z)\right)a(y)}{\pi_\infty\left(0,\varphi_{\delta_{X_1}}(z)\right)a(y)-y}dy\right\}.$$

Note that

$$\pi_\infty^{'}\big(0, \varphi_{\delta_{X_1}}(z)\big) = E^{'}(\varphi_{\delta_{X_1}}(z)^{I_\infty}) = \sum_{k=0}^{\infty} k\varphi_{\varphi_{\delta_{X_1}}}^{k-1}(z)P(I_\infty = k), \tag{38}$$

where I_∞ is the number of customers served during a busy period in the standard M/G/1 queue.

Putting (36) and (37) into (34), we get for $0 < z < 1$

$$E[\nu(\sigma)] \tag{39}$$

$$= \lim_{z\to 1} \frac{\varphi_{\delta_{X_1}}^{'}(z)}{a^{'}(z)} \pi_\infty^{'}\big(0, \varphi_{\delta_{X_1}}(z)\big) exp\Big\{\frac{\lambda}{\nu} \int_0^{\pi_\infty\left(0, \varphi_{\delta_{X_1}}(z)\right)} \frac{1 - \pi_\infty\big(0, \varphi_{\delta_{X_1}}(z)\big)a(y)}{\pi_\infty\big(0, \varphi_{\delta_{X_1}}(z)\big)a(y) - y} dy\Big\}.$$

Taking into account that $\lim_{z\to 1} a^{'}(z) = \rho$, the equation (39) yields:

$$E[\nu(\sigma)] = E(I_\infty) exp\{\frac{\lambda}{\nu} \int_0^1 \frac{1 - a(y)}{a(y) - y} dy\}. \tag{40}$$

□

Finally the following theorem gives a complete image of the M/G/1 retrial system evolution under the $\rho < 1$ hypothesis.

Theorem 5.3. *Assume that $\rho = -\lambda\beta^{'}(0) \leq 1$, and the sequence of service times forms a renewal sequence, then there exists a sequence $\{\tau_n,\ n = 1, 2, ...\}$ of a.s. finite $\Im_n$-stopping times defined by the recursion $\tau_{n+1} = \tau_n + \nu_{n+1}$ for all $n > 0$ such that $X_{\tau_n} = 0$ on $\{\tau_n < \infty\}$ and $\tau_n + 1 \leq \tau_{n+1}$, $\forall n \in N^*$. In this case the random variables $\{\nu_n\}_2^\infty$ form an i.i.d. sequence independent of τ_1 and*

$$E(\nu_{n+2}) = \begin{cases} \frac{1}{1-\rho}\Psi(1) & if\ \rho < 1, \\ \infty & if\ \rho = 1, \end{cases} \tag{41}$$

where

$$\Psi(1) = exp\Big(\frac{\lambda}{\nu} \int_0^1 \frac{1 - a(y)}{a(y) - y} dy\Big). \tag{42}$$

Proof. Define $\tau_{n+1} = \tau_n + \nu(\tau_n)$, $n = 0,\ 1,\ ...$, with $\nu(\tau_n) = \nu_{n+1}$ and $\tau_0 = 0$.

The proof of the theorem 5.3 follows the methodology of Baccelli and Makowski [4] using the above Martingale relations and the fact that for the $M/G/1$ Retrial Queue, the busy period satisfies the equation

$$E\big[\nu(\sigma)\big] = E(I_\infty) exp\Big(\frac{\lambda}{\nu} \int_0^1 \frac{1 - a(y)}{a(y) - y} dy\Big). \tag{43}$$

□

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