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SUMS OF FINITE PRODUCTS OF ORDERED BELL FUNCTIONS

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Abstract. We study three different types of sums of finite products of ordered Bell functions. We derive Fourier series expansion for them and express each of them in terms of Bernoulli functions.

1. Introduction

The *ordered Bell numbers* (also called *Fubini numbers*) b_m arise from number theory and various counting problems in enumerative combinatorics (see [2, 5, 7]). The ordered Bell numbers b_m appeared already in 1859 work of Cayley [1], who used them to count certain plane trees with $m+1$ totally ordered leaves. While the (unordered) *Bell numbers* $\mathrm{Bel_m}$ (see [2,7]) count partitions of $[m] = \{1, 2, \ldots, m\}$ into nonempty disjoint sets, the ordered Bell numbers b_m count either the number of weak orderings on a set of m elements or the mappings from $[m]$ to itself whose image is $[\ell]$, $1 \le \ell \le m$. The generating functions of the Bell numbers $\mathrm{Bel_m}$ and of the ordered Bell numbers b_m are respectively given by (see [2, 5, 7])

$$e^{e^t-1} = \sum_{m\ge0} \mathrm{Bel_m}\frac{\mathrm{t^m}}{\mathrm{m!}}, \quad \frac{1}{2-\mathrm{e^t}} = \sum_{\mathrm{m}\ge0} \mathrm{b_m}\frac{\mathrm{t^m}}{\mathrm{m!}}.$$

As a natural companion to the ordered Bell numbers b_m, the *ordered Bell polynomials* $b_m(x)$ was introduced in [6] by defining the generating function of them as

$$\frac{1}{2-e^t}e^{xt} = \sum_{m\ge0} b_m(x)\frac{t^m}{m!}.$$

Clearly, $b_m = b_m(0)$ for all $m \ge 0$. For instance, the first six ordered Bell polynomials are given by $b_0(x) = 1$, $b_1(x) = x+1$, $b_2(x) = x^2+2x+3$, $b_3(x) = x^3+3x^2+9x+13$, $b_4(x) = x^4+4x^3+18x^2+52x+75$ and $b_5(x) = x^5+5x^4+30x^3+130x^2+375x+541$. Clearly,

$$\frac{d}{dx}b_m(x) = mb_{m-1}(x) \tag{1}$$

and

$$-b_m(x+1)+2b_m(x) = x^m. \tag{2}$$

In turn, from these we immediately (see [6]) obtain $-b_m(1)+2b_m = \delta_{m,0}$ and

$$\int_0^1 b_m(x)dx = \frac{1}{m+1}(b_{m+1}(1)-b_{m+1}(0)) = \frac{b_{m+1}}{m+1}.$$

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We note that the ordered Bell numbers b_m are positive integers for all $m \geq 0$, as it can be seen from (see [2,7]) $b_m = \sum_{n\geq 0} = \frac{n^m}{2^{n+1}} = \sum_{n=0}^{m} n!S_2(m,n)$, where $S_2(m,n)$ is the Stirling number of the second kind.

As is well known, the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m\geq 0} B_m(x)\frac{x^m}{m!}.$$

For any real number x, we let $\langle x\rangle = x - \lfloor x\rfloor \in [0,1)$ denote the fractional part of x. Later, we will need the following facts about Bernoulli functions $\tilde{B}_m(x) = B_m(\langle x\rangle)$:

$$\tilde{B}_m(x) = -m!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^m}, \quad \text{for all } m\geq 2, \tag{3}$$

$$-\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{2\pi in} = \begin{cases} \tilde{B}_1(x), & x\notin\mathbb{Z},\\ 0, & x\in\mathbb{Z}.\end{cases} \tag{4}$$

where $\mathbb{Z}' = \mathbb{Z}\backslash\{0\}$ denotes the set of all the integers except 0.

The *Fourier series* of a periodic function $f(x)$ with period 1 is given by $\sum_{n=-\infty}^{\infty} f_n e^{2\pi inx}$, where the coefficients f_n are given by $f_n = \int_0^1 f(x)e^{-2\pi inx}dx$ (for example, see [3,4,10–14]), where $i^2=-1$.

In this paper, we will consider three types of sums of finite products of ordered Bell functions $\tilde{b}_m(x)$ and derive Fourier series expansion for them (see the next three sections). In addition, we will express each of them in terms of Bernoulli functions $\tilde{B}_m(x)$.

2. Sum $\tilde{\alpha}_m(x)$ of finite products of ordered Bell functions

For all $m\geq 1$, let $\alpha_m(x) = \sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r} b_{i_j}(x)$ and $\tilde{\alpha}_m(x) = \alpha_m(\langle x\rangle)$, where the sum is over all nonnegative integers $i_1,\ldots,i_r$ with $i_1+\cdots+i_r=m$ and $r\geq 1$. The aim of this section is to study the function $\tilde{\alpha}_m(x)$ defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\tilde{\alpha}_m$ is

$$\sum_{n\in\mathbb{Z}} A_n^{(m)} e^{2\pi inx},$$

where $A_n^{(m)} = \int_0^1 \tilde{\alpha}_m(x)e^{-2\pi inx}dx = \int_0^1 \alpha_m(x)e^{-2\pi inx}dx$. To proceed further, we note the following lemma.

Lemma 1. *For all $m\geq 0$, $\frac{d}{dx}\alpha_{m+1}(x) = (m+r)\alpha_m(x)$.*

Proof. By the definitions and (1), we obtain

$$\begin{aligned}\frac{d}{dx}\alpha_{m+1}(x) &= \sum_{j=1}^{r}\sum_{i_1+\cdots+i_r=m+1} i_j\frac{b_{i_j-1}(x)}{b_{i_j}(x)}\prod_{s=1}^{r} b_{i_s}(x) = \sum_{j=1}^{r}\sum_{i_1+\cdots+i_r=m}(i_j+1)\prod_{s=1}^{r} b_{i_s}(x)\\ &= \sum_{i_1+\cdots+i_r=m}(m+r)\prod_{s=1}^{r} b_{i_s}(x) = (m+r)\alpha_m(x),\end{aligned}$$

as claimed. □

By Lemma 1, we have that $\int_0^1 \alpha_m(x)dx = \frac{\alpha_{m+1}(x)}{m+r}\Big|_{x=0}^{x=1} = \frac{\alpha_{m+1}(1)-\alpha_{m+1}(0)}{m+r}$. Define $\Delta_m = \alpha_m(1) - \alpha_m(0)$. Then, by (2),

$$\begin{aligned}\Delta_m &= \sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r} b_{i_j}(1) - \sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r} b_{i_j}\\ &= \sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r}(2b_{i_j}-\delta_{i_j,0}) - \sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r} b_{i_j}\\ &= \sum_{a=1}^{r}\binom{r}{a}2^a(-1)^{r-a}\sum_{i_1+\cdots+i_a=m}\prod_{j=1}^{a} b_{i_j} - \sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r} b_{i_j}.\end{aligned}$$

Observe here that the sum over all $i_1+\cdots+i_r = m$ of any term with a of $2b_{i_k}$ and b of $-\delta_{i_\ell,0}$ $(1 \le k, \ell \le r$ and $k+\ell = r)$ all give the same sum

$$\sum_{i_1+\cdots+i_r=m}(2b_{i_1})\cdots(2b_{i_a})(-\delta_{i_{a+1},0})\cdots(-\delta_{i_{a+b},0}) = 2^a(-1)^{r-a}\sum_{i_1+\cdots+i_a=m} b_{i_1}\cdots b_{i_a}.$$

Note here that, for $a = 0$, there is only one term which is $(-1)^r\delta_{m,0} = 0$. Thus, $\alpha_m(1) = \alpha_m(0)$ if and only if $\Delta_m = 0$, and

$$\int_0^1 \alpha_m(x)dx = \frac{\Delta_{m+1}}{m+r}. \tag{5}$$

Now, we are ready to determine the Fourier coefficients $A_n^{(m)}$. First, let us consider the case $n \neq 0$. By Lemma 1, we obtain

$$\begin{aligned}A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx = \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}\alpha_m(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}\,\alpha_m(x)e^{-2\pi inx}\Big|_{x=0}^{x=1}\\ &= \frac{m+r-1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx - \frac{\Delta_m}{2\pi in}\\ &= \frac{m+r-1}{2\pi in}A_n^{(m-1)} - \frac{\Delta_m}{2\pi in}.\end{aligned}$$

Hence, by induction on m, we obtain

$$A_n^{(m)} = \frac{(m+r-1)_m}{(2\pi in)^m}A_n^{(0)} - \sum_{j=1}^{m}\frac{(m+r-1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1} = -\frac{1}{m+r}\sum_{j=1}^{m}\frac{(m+r)_j}{(2\pi in)^j}\Delta_{m-j+1}. \tag{6}$$

where $(x)_j = x(x-1)\cdots(x-j+1)$ with $(x)_0 = 1$.

The case $n = 0$ follows immediately from (5):

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{\Delta_{m+1}}{m+r}. \tag{7}$$

Note that the function $\tilde{\alpha}_m$, $m \ge 1$, is piecewise C^∞. Moreover, the function $\tilde{\alpha}_m$ is continuous for those positive integers m with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m \neq 0$.

2.1. **Case $\Delta_m = 0$.** Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. So, the function $\tilde{\alpha}_m$ is piecewise C^∞, and continuous. Thus, the Fourier series of $\tilde{\alpha}_m$ converges uniformly to $\tilde{\alpha}_m$. So, by (6) and (7), we have

$$\tilde{\alpha}_m(x) = \frac{\Delta_{m+1}}{m+r} + \sum_{n\in\mathbb{Z}'} \left(-\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx},$$

which, by (3) and (4), implies

$$\begin{aligned}\tilde{\alpha}_m(x) &= \frac{\Delta_{m+1}}{m+r} + \frac{1}{m+r} \sum_{j=1}^{m} \binom{m+r}{j} \Delta_{m-j+1} \left(-j! \sum_{n\in\mathbb{Z}'} \frac{e^{2\pi inx}}{(2\pi in)^j} \right) \\ &= \frac{\Delta_{m+1}}{m+r} + \frac{1}{m+r} \sum_{j=2}^{m} \binom{m+r}{j} \Delta_{m-j+1} \tilde{B}_j(x) + \Delta_m \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}\end{aligned}$$

Thus, we can state the following result.

Theorem 2. *Let m be a positive integer with $\Delta_m = 0$. Then the function $\tilde{\alpha}_m(x)$ has the Fourier series expansion*

$$\tilde{\alpha}_m(x) = \frac{\Delta_{m+1}}{m+r} + \sum_{n\in\mathbb{Z}'} \left(-\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\alpha}_m(x) = \frac{\Delta_{m+1}}{m+r} + \frac{1}{m+r} \sum_{j=2}^{m} \binom{m+r}{j} \Delta_{m-j+1} \tilde{B}_j(x),$$

for all $x \in \mathbb{R}$.

2.2. **Case $\Delta_m \neq 0$.** Assume next that m is a positive integer with $\Delta_m \neq 0$. Then $\alpha_m(1) \neq \alpha_m(0)$. So, the function $\tilde{\alpha}_m$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\alpha}_m(x)$ converges piecewise to $\tilde{\alpha}_m(x)$ for all $x \notin \mathbb{Z}$, and converges to

$$\frac{\alpha_m(1) + \alpha_m(0)}{2} = \alpha_m(0) + \frac{\Delta_m}{2}, \quad \text{for all } x \in \mathbb{Z}.$$

Then, by Theorem 2, we obtain the following result.

Theorem 3. *Let m be a positive integer with $\Delta_m \neq 0$. Then*

$$\begin{aligned}&\frac{\Delta_{m+1}}{m+r} + \sum_{n\in\mathbb{Z}'} \left(-\frac{1}{m+r} \sum_{j=1}^{m} \frac{(m+r)_j}{(2\pi in)^j} \Delta_{m-j+1} \right) e^{2\pi inx} \\ &\qquad = \begin{cases} \tilde{\alpha}_m(x), & x \notin \mathbb{Z}, \\ \sum_{i_1+\cdots+i_r=m} \prod_{j=1}^{r} b_{i_j} + \frac{\Delta_m}{2}, & x \in \mathbb{Z}. \end{cases}\end{aligned}$$

Moreover,

$$\frac{\Delta_{m+1}}{m+r} + \frac{1}{m+r} \sum_{j=1}^{m} \binom{m+r}{j} \Delta_{m-j+1} \tilde{B}_j(x) = \tilde{\alpha}_m(x), \text{ for all } x \notin \mathbb{Z},$$

and

$$\frac{\Delta_{m+1}}{m+r}+\frac{1}{m+r}\sum_{j=2}^{m}\binom{m+r}{j}\Delta_{m-j+1}\tilde{B}_j(x)=\sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r}b_{i_j}+\frac{\Delta_m}{2},\ \textit{for all } x\in\mathbb{Z}.$$

3. Sum $\tilde{\beta}_m(x)$ of finite products of ordered Bell functions

Define, for all $m\geq 1$, $\beta_m(x)=\sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r}\frac{b_{i_j}(x)}{i_j!}$ and $\tilde{\beta}_m(x)=\beta_m(\langle x\rangle)$, where the sum is over all nonnegative integers $i_1,\dots,i_r$ with $i_1+\cdots+i_r=m$ and $r\geq 1$. In this section, we consider the function $\tilde{\beta}_m$ on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\tilde{\beta}_m(x)$ is $\sum_{n\in\mathbb{Z}}B_n^{(m)}e^{2\pi inx}$, where

$$B_n^{(m)}=\int_0^1\tilde{\beta}_m(x)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To proceed further, we note first the following lemma.

Lemma 4. *For all* $m\geq 1$, $\frac{d}{dx}\beta_m(x)=r\beta_{m-1}(x)$.

Proof. By the definitions and (1), we obtain

$$\begin{aligned}\frac{d}{dx}\beta_m(x)&=\sum_{j=1}^{r}\sum_{i_1+\cdots+i_r=m}\frac{i_jb_{i_j-1}(x)}{b_{i_j}(x)}\prod_{s=1}^{r}\frac{b_{i_s}(x)}{i_s!}\\&=\sum_{j=1}^{r}\sum_{i_1+\cdots+i_r=m-1}\prod_{s=1}^{r}\frac{b_{i_s}(x)}{i_s!}=\sum_{j=1}^{r}\beta_{m-1}(x)=r\beta_{m-1}(x),\end{aligned}$$

as claimed. □

By Lemma 4, we have that $\int_0^1\beta_m(x)dx=\frac{\beta_{m+1}(x)}{r}\Big|_{x=0}^{x=1}=\frac{\beta_{m+1}(1)-\beta_{m+1}(0)}{r}$. Define $\Omega_m=\beta_m(1)-\beta_m(0)$ for all m. Then, by (2),

$$\begin{aligned}\Omega_m&=\sum_{i_1+\cdots+i_r=m}\prod_{s=1}^{r}\frac{b_{i_s}(1)}{i_s!}-\sum_{i_1+\cdots+i_r=m}\prod_{s=1}^{r}\frac{b_{i_s}}{i_s!}\\&=\sum_{i_1+\cdots+i_r=m}\prod_{s=1}^{r}\frac{2b_{i_s}-\delta_{i_s,0}}{i_s!}-\sum_{i_1+\cdots+i_r=m}\prod_{s=1}^{r}\frac{b_{i_s}}{i_s!}\\&=\sum_{a=1}^{r}\binom{r}{a}2^a(-1)^{r-a}\sum_{i_1+\cdots+i_a=m}\prod_{s=1}^{a}\frac{b_{i_s}}{i_s!}-\sum_{i_1+\cdots+i_r=m}\prod_{s=1}^{r}\frac{b_{i_s}}{i_s!},\end{aligned}$$

where the sum over all $i_1+\cdots+i_r=m$ of any term with a of $2b_{i_\ell}$ and b of $-\delta_{i_k,0}$ $(1\leq\ell,k\leq r$ and $\ell+k=r)$ all give the same sum

$$\sum_{i_1+\cdots+i_r=m}\prod_{j=1}^{r}\frac{1}{i_j!}(2b_{i_1})\cdots(2b_{i_a})(-\delta_{i_{a+1},0})\cdots(-\delta_{i_{a+b},0})=2^a(-1)^{r-a}\sum_{i_1+\cdots+i_a=m}\prod_{s=1}^{a}\frac{b_{i_s}}{i_s!}.$$

Note here that, for $a = 0$, there is only one term which is $(-1)^r\delta_{0,m} = 0$. Thus, $\beta_m(0) = \beta_m(1)$ if and only if $\Omega_m = 0$ and

$$\int_0^1 \beta_m(x)dx = \frac{\Omega_{m+1}}{r}. \tag{8}$$

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$. First, let us consider the case $n \neq 0$. By Lemma 4 and (8), we have

$$\begin{aligned} B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx = \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}\beta_m(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}\,\beta_m(x)e^{-2\pi inx}\Big|_{x=0}^{x=1} \\ &= \frac{r}{2\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx - \frac{\Omega_m}{2\pi in} = \frac{r}{2\pi in}B_n^{(m-1)} - \frac{\Omega_m}{2\pi in}. \end{aligned}$$

Hence, by induction on m, we obtain

$$B_n^{(m)} = \frac{r^m}{(2\pi in)^m}B_n^{(0)} - \sum_{j=1}^{m}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1} = -\sum_{j=1}^{m}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}.$$

The case $n = 0$ follows immediately from (8):

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{\Omega_{m+1}}{r}. \tag{9}$$

Note that $\tilde{\beta}_m(x)$, $m \geq 1$, is piecewise C^∞. Moreover, $\tilde{\beta}_m(x)$ is continuous for those positive integers m with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

3.1. **Case $\Omega_m = 0$.** Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. So, the function $\tilde{\beta}_m(x)$ is piecewise C^∞, and continuous. Thus, the Fourier series of $\tilde{\beta}_m(x)$ converges uniformly to $\tilde{\beta}_m(x)$. So, by (9), we have

$$\tilde{\beta}_m(x) = \frac{\Omega_{m+1}}{r} - \sum_{n\in\mathbb{Z}'}\left(\sum_{j=1}^{m}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},$$

which implies

$$\begin{aligned} \tilde{\beta}_m(x) &= \frac{\Omega_{m+1}}{r} + \sum_{j=1}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1}\left(-j!\sum_{n\in\mathbb{Z}'}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{\Omega_{m+1}}{r} + \sum_{j=2}^{m}\frac{r^{j-1}}{j!}\Omega_{m-j+1}\tilde{B}_j(x) + \Omega_m\cdot\begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases} \end{aligned}$$

for all $x \in \mathbb{R}$. Thus, we can state the following result.

Theorem 5. *Let m be a positive integer with $\Omega_m = 0$. Then the function $\tilde{\beta}_m(x)$ has the Fourier series expansion*

$$\tilde{\beta}_m(x) = \frac{\Omega_{m+1}}{r} - \sum_{n\in\mathbb{Z}'}\left(\sum_{j=1}^{m}\frac{r^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},$$

for all $x \in \mathbb{R}$*, where the convergence is uniform. Moreover,*

$$\tilde{\beta}_m(x) = \frac{\Omega_{m+1}}{r} + \sum_{j=2}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1}\tilde{B}_j(x),$$

for all $x \in \mathbb{R}$*.*

3.2. **Case** $\Omega_m \neq 0$**.** Assume next that m is a positive integer with $\Omega_m \neq 0$. Then $\beta_m(1) \neq \beta_m(0)$. So, the function $\tilde{\beta}_m(x)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\beta}_m(x)$ converges pointwise to $\tilde{\beta}_m(x)$ for all $x \notin \mathbb{Z}$, and converges to

$$\frac{\beta_m(1) + \beta_m(0)}{2} = \beta_m(0) + \frac{1}{2}\Omega_m, \text{ for all } x \in \mathbb{Z}.$$

Then, by Theorem 5, we obtain the following result.

Theorem 6. *Let* m *be a positive integer with* $\Omega_m \neq 0$*. Then we have the following.*

$$\frac{\Omega_{m+1}}{r} - \sum_{n\in\mathbb{Z}'} \left(\sum_{j=1}^{m} \frac{r^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \right) e^{2\pi i n x} = \begin{cases} \tilde{\beta}_m(x), & x \notin \mathbb{Z}, \\ \beta_m(0) + \frac{1}{2}\Omega_m, & x \in \mathbb{Z}, \end{cases}$$

where the convergence is pointwise. Moreover,

$$\frac{\Omega_{m+1}}{r} + \sum_{j=1}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1}\tilde{B}_j(x) = \tilde{\beta}_m(x), \text{ for all } x \notin \mathbb{Z},$$

and

$$\frac{\Omega_{m+1}}{r} + \sum_{j=2}^{m} \frac{r^{j-1}}{j!}\Omega_{m-j+1}\tilde{B}_j(x) = \beta_m(0) + \frac{\Omega_m}{2}, \text{ for all } x \in \mathbb{Z}.$$

Note that for $r = 2$, Ω_m has the following expression

$$\Omega_m = 3\sum_{k=0}^{m} \frac{b_k b_{m-k}}{k!(m-k)!} - 4\frac{b_m}{m!} = 3\sum_{k=1}^{m-1} \frac{b_k b_{m-k}}{k!(m-k)!} + 2\frac{b_m}{m!} > 0,$$

for all $m \geq 1$.

4. Sum $\tilde{\gamma}_{r,m}(x)$ of finite products of ordered Bell functions

Define $\gamma_{r,m}(x) = \sum_{i_1+\cdots+i_r=m} \prod_{j=1}^{r} \frac{b_{i_j}(x)}{i_j}$ and $\tilde{\gamma}_{r,m}(x) = \gamma_{r,m}(\langle x \rangle)$, where the sum is over all positive integers $i_1, \dots, i_r$ with $i_1+\cdots+i_r = m$ and $m \geq r \geq 1$. In this section, we consider the function $\tilde{\gamma}_{r,m}$ on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\tilde{\gamma}_{r,m}(x)$ is $\sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi i n x}$ with

$$C_n^{(r,m)} = \int_0^1 \tilde{\gamma}_{r,m}(x)e^{-2\pi i n x}dx = \int_0^1 \gamma_{r,m}(x)e^{-2\pi i n x}dx.$$

To proceed further, we note first the following lemma.

Lemma 7. *For all* $m \geq 1$*,*

$$\frac{d}{dx}\gamma_{r,m}(x) = r\gamma_{r-1,m-1}(x) + (m-1)\gamma_{r,m-1}(x)$$

with $\gamma_{r,r-1}(x) = 0$*.*

Proof. By the definitions and (1), we obtain

$$\begin{aligned}\frac{d}{dx}\gamma_m(x) &= \sum_{j=1}^{r}\sum_{i_1+\cdots+i_r=m}\frac{i_j b_{i_j-1}(x)}{b_{i_j}(x)}\prod_{s=1}^{r}\frac{b_{i_s}(x)}{i_s}\\ &= r\sum_{i_1+\cdots+i_{r-1}=m-1}\prod_{s=1}^{r-1}\frac{b_{i_s}(x)}{i_s}+\sum_{i_1+\cdots+i_r=m-1}\frac{i_1+\cdots+i_r}{i_1\cdots i_r}b_{i_1}(x)\cdots b_{i_r}(x)\\ &= r\gamma_{r-1,m-1}(x)+(m-1)\gamma_{r,m-1}(x),\end{aligned}$$

as claimed. □

Define $\Lambda_{r,m}=\gamma_{r,m}(1)-\gamma_{r,m}(0)$. By Lemma 7, if $a_{r,m}=\int_0^1\gamma_{r,m}(x)dx$ then

$$a_{r,m}=\frac{-r}{m}a_{r-1,m}+\frac{\Lambda_{r,m+1}}{m},$$

which implies, by induction on r, that

$$\int_0^1\gamma_{r,m}(x)dx=\sum_{j=1}^{r}\frac{(-1)^{j-1}(r)_{j-1}}{m^j}\Lambda_{r-j+1,m+1},$$

where we used the fact that $a_{1,m}=\frac{1}{m}\int_0^1 b_m(x)dx$ and $\Lambda_{1,m+1}=\int_0^1 b_m(x)dx$ (see (1)). Thus, by (2), we have

$$\begin{aligned}\Lambda_{r,m}=\gamma_{r,m}(1)-\gamma_{r,m}(0) &= \sum_{i_1+\cdots+i_r=m,\, i_1,\ldots,i_r\geq 1}\frac{b_{i_1}(1)\cdots b_{i_r}(1)-b_{i_1}\cdots b_{i_r}}{i_1\cdots i_r}\\ &= \sum_{i_1+\cdots+i_r=m,\, i_1,\ldots,i_r\geq 1}\frac{(2b_{i_1})\cdots(2b_{i_r})-b_{i_1}\cdots b_{i_r}}{i_1\cdots i_r}\\ &= (2^r-1)\sum_{i_1+\cdots+i_r=m,\, i_1,\ldots,i_r\geq 1}\prod_{s=1}^{r}\frac{b_{i_s}}{i_s},\end{aligned}$$

from which we see that $\Lambda_{r,m}>0$, for all $m\geq r\geq 1$.

Now, we are ready to determine the Fourier coefficients $C_n^{(r,m)}$. First, let us consider the case $n\neq 0$. By Lemma 7, we have

$$\begin{aligned}C_n^{(r,m)} &= \int_0^1\gamma_{r,m}(x)e^{-2\pi inx}dx=\frac{1}{2\pi in}\int_0^1\frac{d}{dx}\gamma_{r,m}(x)e^{-2\pi inx}dx-\frac{1}{2\pi in}\gamma_{r,m}(x)e^{-2\pi inx}\Big|_{x=0}^{x=1}\\ &= \frac{1}{2\pi in}\int_0^1(r\gamma_{r-1,m-1}(x)+(m-1)\gamma_{r,m-1}(x))e^{-2\pi inx}dx-\frac{1}{2\pi in}(\gamma_{r,m}(1)-\gamma_{r,m}(0))\\ &= \frac{m-1}{2\pi in}C_n^{(r,m-1)}+\frac{r}{2\pi in}C_n^{(r-1,m-1)}-\frac{1}{2\pi in}\Lambda_{r,m}.\end{aligned}$$

Thus, by induction on m (use only the first term of the recurrence), we see that

$$C_n^{(r,m)}=\sum_{j=1}^{m-r+1}\frac{r(m-1)_{j-1}}{(2\pi in)^j}C_n^{(r-1,m-j)}-\sum_{j=1}^{m-r+1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{r,m-j+1},$$

where

$$C_n^{(r,r)}=\int_0^1(x+1)^re^{-e\pi inx}dx=-\frac{1}{2\pi in}(2^r-1)+\frac{r}{2\pi in}C_n^{(r-1,r-1)},$$

and $\Lambda_{r,r} = 2^r - 1$. Also, we can show that

$$\begin{aligned} C_n^{(1,m)} &= \int_0^1 \gamma_{1,m}(x) e^{-2\pi inx} dx = \frac{1}{m}\int_0^1 b_m(x) e^{-2\pi inx} dx \\ &= \begin{cases} -\frac{1}{m}\sum_{j=1}^{m} \frac{(m)_{j-1}}{(2\pi in)^j} b_{m-j+1}, & n \neq 0, \\ \frac{1}{m(m+1)} b_{m+1}, & n = 0. \end{cases} \end{aligned}$$

Thus, for $n > 0$, we have

$$C_n^{(r,m)} = \sum_{j=1}^{m-r+1} \frac{r(m-1)_{j-1}}{(2\pi in)^j} C_n^{(r-1,m-j)} - \sum_{j=1}^{m-r+1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{r,m-j+1}, \tag{10}$$

$$C_n^{(1,m)} = -\frac{1}{m}\sum_{j=1}^{m} \frac{(m)_{j-1}}{(2\pi in)^j} b_{m-j+1}, \tag{11}$$

which determine all $C_n^{(r,m)}$ recursively.

When $n = 0$, we have

$$C_0^{(r,m)} = \int_0^1 \gamma_{r,m}(x) dx = \sum_{j=1}^{r} \frac{(-1)^{j-1}(r)_{j-1}}{m^j} \Lambda_{r-j+1,m+1}. \tag{12}$$

We recall here that $\Lambda_{r,m} > 0$, that is, $\gamma_{r,m}(0) \neq \gamma_{r,m}(1)$ for all $m \geq r \geq 1$. So $\tilde{\gamma}_{r,m}(x)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers for those positive integers. Thus, the Fourier series of $\tilde{\gamma}_{r,m}(x)$ converges pointwise to $\tilde{\gamma}_{r,m}(x)$ for all $x \notin \mathbb{Z}$, and converges to

$$\frac{\gamma_{r,m}(1) + \gamma_{r,m}(0)}{2} = \gamma_{r,m}(0) + \frac{1}{2}\Lambda_{r,m}.$$

Hence, we can state the following result.

Theorem 8. *Let $m \geq r \geq 1$. Then*

$$C_0^{(r,m)} + \sum_{n\in\mathbb{Z}'} C_n^{(r,m)} e^{2\pi inx} = \begin{cases} \tilde{\gamma}_{r,m}(x), & x \notin \mathbb{Z}, \\ \gamma_{r,m}(0) + \frac{1}{2}\Lambda_{r,m}, & x \in \mathbb{Z}, \end{cases}$$

where $C_n^{(m)}$ are determining by (10), (11) *and* (12).

For instance, let us consider the case $r = 2$. Theorem 8 with $r = 2$ shows

$$\begin{aligned} 2\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j} C_n^{(1,m-j)} &= -2\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}} b_{m-j-k+1} \\ &= -2\sum_{j=1}^{m-1} \frac{1}{m-j} \sum_{s=j+1}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} b_{m-s+1} \\ &= -2\sum_{s=2}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} b_{m-s+1} \sum_{j=1}^{s-1} \frac{1}{m-j} \\ &= -\frac{2}{m}\sum_{s=2}^{m} \frac{(m)_s}{(2\pi in)^s} \frac{b_{m-s+1}}{m-s+1}(H_{m-1} - H_{m-s}), \end{aligned}$$

where H_j is the j-th Harmonic number. So, for $n \neq 0$,

$$C_n^{(2,m)} = -\frac{2}{m}\sum_{s=2}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{b_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) - \frac{3}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s}\sum_{k=1}^{m-s}\frac{b_k b_{m-s-k+1}}{k(m-s-k+1)}$$

and

$$C_0^{(2,m)} = \sum_{j=1}^{2}\frac{(-1)^{j-1}(2)_{j-1}}{m^j}\Lambda_{3-j,m+1} = \frac{3}{m}\sum_{k=1}^{m}\frac{b_k b_{m+1-k}}{k(m+1-k)} - \frac{2b_{m+1}}{m^2(m+1)}.$$

By using expressions of $C_n^{(2,m)}$ and $C_0^{(2,m)}$, we obtain the following corollary.

Corollary 9. *Let $m \geq 1$. Then*

$$\frac{3}{m}\sum_{k=1}^{m}\frac{b_k b_{m+1-k}}{k(m+1-k)} - \frac{2b_{m+1}}{m^2(m+1)} - \frac{1}{m}\sum_{n\in\mathbb{Z}'}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\frac{2b_{m-s+1}}{m-s+1}(H_{m-1}-H_{m-s}) + \sum_{k=1}^{m-s}\frac{3b_k b_{m-s-k+1}}{k(m-s-k+1)}\right)e^{2\pi inx}$$
$$= \begin{cases} \sum_{k=1}^{m-1}\frac{\tilde{B}_k(x)\tilde{B}_{m-k}(x)}{k(m-k)}, & x \notin \mathbb{Z}, \\ \frac{5}{2}\sum_{k=1}^{m-1}\frac{b_k b_{m-k}}{k(m-k)}, & x \in \mathbb{Z}. \end{cases}$$

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