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GENERALIZED FRACTIONAL INTEGRATION OF k-BESSEL FUNCTION

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Abstract. In this present paper our aim is to deal with two integral transforms which involving the Gauss hypergeometric function as its kernels. We prove some compositions formulas for such a generalized fractional integrals with k-Bessel function. The results are established in terms of generalized Wright type hypergeometric function and generalized hypergeometric series. Also, the authors presented some corresponding assertions for RiemannLiouville and ErdélyiKober fractional integral transforms.

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1. Introduction and Preliminaries

The Gauss hypergeometric function is defined as:

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{1}$$

where $a,b,c\in\mathbb{C}$, $c\neq 0,-1,-2,\cdots$ and $(\lambda)_n$ is the Pochhammer symbol defined for $\lambda\in\mathbb{C}$ and $n\in\mathbb{N}$ as:

$$(\lambda)_0=1,\qquad (\lambda)_n=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1);n\in\mathbb{N}. \tag{2}$$

The series defined in (1) is absolutely convergent for $|z|<1$ and $|z|=1$ (see [2]). Saigo [9] introduced the following left and right sided generalized integral transforms defined for $x>0$ respectively as:

$$\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \times\int_0^x (x-t)^{\alpha-1}\;{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\tfrac{t}{x}\right)f(t)dx, \tag{3}$$

*corresponding author.

and

$$\left(I_{-}^{\alpha,\beta,\eta}f\right)(x) = \frac{1}{\Gamma(\alpha)}$$

$$\times \int_{x}^{\infty} (x-t)^{\alpha-1} t^{-\alpha-\beta} \quad {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\tfrac{x}{t} \right) f(t)dx, \tag{4}$$

where $\alpha, \beta, \eta \in \mathbb{C}$ and $\mathbb{R}e(\alpha) > 0$ and ${}_2F_1(a, b; c; z)$ is Gauss hypergeometric function defined in (1). When $\beta = -\alpha$, then (2) and (4) will lead to the classical Riemann-Liouville left and right-sided fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, (see [11]):

$$\left(I_{0+}^{\alpha,\beta,\eta}f \right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t)dx (x > 0), \tag{5}$$

and

$$\left(I_{0+}^{\alpha,\beta,\eta}f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (x-t)^{\alpha-1} t^{-\alpha-\beta} f(t)dx (x > 0). \tag{6}$$

If $\beta = 0$, then equations (3) and (4) will reduce to the well known Erdélyi-Kober fractional defined as:

$$\left(I_{0+}^{\alpha,0,\eta}f \right)(x) = \left(K_{\eta,\alpha}^{+}f \right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} t^{\eta} f(t)dx \tag{7}$$

and

$$\left(I_{0+}^{\alpha,0,\eta}f \right)(x) = \left(K_{\eta,\alpha}^{-}f \right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (x-t)^{\alpha-1} t^{-\alpha-\eta} f(t)dx, \tag{8}$$

where $\alpha, \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ (see [11]).
The generalized k-Bessel function defined in [10] as:

$$W_{v,c}^{k}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk+v+k)n!} \left(\frac{z}{2}\right)^{2n+\frac{v}{k}}, \tag{9}$$

where $k > 0$, $v > -1$, and $c \in \mathbb{R}$ and $\Gamma_k(z)$ is the k-gamma function defined in [1] as:

$$\Gamma_k(z) = \int_{0}^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt, z \in \mathbb{C}. \tag{10}$$

By inspection the following relation holds:

$$\Gamma_k(z+k) = z\Gamma_k(z) \tag{11}$$

and

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right). \tag{12}$$

If $k \to 1$ and $c = 1$, then the generalized k-Bessel function defined in (9) reduces to the well known classical Bessel function J_v defined in [3]. For

further detail about k-Bessel function and its properties (see [4]-[6]).
The generalized hypergeometric function ${}_pF_q(z)$ is defined in [2] as:

$$
{}_pF_q(z) = \; {}_pF_q, \left[\begin{array}{ll} (\alpha_1), (\alpha_2), \cdots (\alpha_p) & \\ & ; z \\ (\beta_1), (\beta_2), \cdots (\beta_q) & \end{array} \right]
$$

$$
= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \tag{13}
$$

where $\alpha_i, \beta_j \in \mathbb{C}$; $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$ and $b_j \neq 0, -1, -2, \cdots$ and $(z)_n$ is the Pochhammer symbols. The gamma function is defined as:

$$
\Gamma(\mu) = \int_0^{\infty} t^{\mu - 1} e^{-t} dt, \mu \in \mathbb{C}, \tag{14}
$$

$$
\Gamma(z + n) = (z)_n \Gamma(z), z \in \mathbb{C}, \tag{15}
$$

and beta function is defined as:

$$
B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt. \tag{16}
$$

Also, the following identity of Gauss hypergeometric function holds:

$$
{}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}; \Re(c - a - b) > 0, \tag{17}
$$

(see [2], [11]).
The Wright type hypergeometric function is defined (see [12]-[14]) by the following series as:

$$
{}_p\Psi_q(z) = \; {}_p\Psi_q \left[\begin{array}{ll} (\alpha_i, A_i)_{1,p} & \\ & ; z \\ (\beta_j, B_j)_{1,q} & \end{array} \right]
$$

$$
= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!} \tag{18}
$$

where β_r and μ_s are real positive numbers such that

$$
1 + \sum_{s=1}^{q} \beta_s - \sum_{r=1}^{p} \alpha_r > 0.
$$

Equation (18) differs from the generalized hypergeometric function ${}_pF_q(z)$ defined (13) only by a constant multiplier. The generalized hypergeometric function ${}_pF_q(z)$ is a special case of ${}_p\Psi_q(z)$ for $A_i = B_j = 1$, where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$:

(19)

$$
\frac{1}{\prod_{j=1}^{q} \Gamma(\beta_j)} \; {}_pF_q \left[\begin{array}{ll} (\alpha_1), \cdots (\alpha_p) & \\ & ; z \\ (\beta_1), \cdots (\beta_q) & \end{array} \right] = \frac{1}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \; {}_p\Psi_q \left[\begin{array}{ll} (\alpha_i, 1)_{1,p} & \\ & ; z \\ (\beta_j, 1)_{1,q} & \end{array} \right].
$$

For various properties of this function see [7].
Lemma 1.1 (A. A. Kilbas and N. Sebastian [8]) Let $\alpha, \beta, \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\lambda > \max[0, \beta - \mathfrak{n}]$, then the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1} \right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda+\eta-\beta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)} x^{\lambda-\beta-1} \tag{20}$$

Lemma 1.2 (A. A. Kilbas and N. Sebastian [8]) Let $\alpha, \beta, \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\lambda > \max[0, \beta - \mathfrak{n}]$, then the following relation holds:

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1} \right)(x) = \frac{\Gamma(\eta-\lambda+1)\Gamma(\beta-\lambda+1)}{\Gamma(1-\lambda)\Gamma(\alpha+\beta+\eta-\lambda+1)} x^{\lambda-\beta-1} \tag{21}$$

In the same paper, they define the following left and right sided Erdélyi-Kober fractional integral as:

$$\left(K_{\eta,\alpha}^{+} t^{\lambda-1} \right)(x) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\alpha+\eta)} x^{\lambda-1}, \tag{22}$$

where $\Re(\alpha) > 0$, $\Re(\lambda) > -\Re(\eta)$, and

$$\left(K_{\eta,\alpha}^{-} t^{\lambda-1} \right)(x) = \frac{\Gamma(\eta-\lambda+1)}{\Gamma(\alpha+\eta-\lambda+1)} x^{\lambda-1}, \tag{23}$$

where $\Re(\lambda) < 1 + \Re(\eta)$.

2. Representation of Generalized fractional integrals in term of Wright functions

In this section, we introduce the generalized left-sided fractional integration (3) of the k-Bessel functions (9). It is given by the following result.

Theorem 2.1. *Assume that α, β, η, λ, $v \in \mathbb{C}$ be such that*

$$\Re(v) > -1, \Re(\alpha) > 0, \Re(\lambda+v) > \max[0, \Re(\beta-\eta)], \tag{24}$$

then the following result holds:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t) \right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}} \times {}_2\Psi_3 \left[\begin{matrix} (\frac{\lambda}{k}+\frac{v}{k}, 2), (\frac{\lambda}{k}+\frac{v}{k}+\eta-\beta, 2k) \\ (\frac{\lambda}{k}+\frac{v}{k}-\beta, 2), (\frac{\lambda}{k}+\frac{v}{k}+\alpha+\eta, 2), (\frac{v}{k}+1, 1) \end{matrix} \middle| -\frac{cx^2}{4k} \right]. \tag{25}$$

Proof. Note that the condition

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 0$$

is satisfied so therefore ${}_2\Psi_3(z)$ is defined. Now, from (3) and (9), we have

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t) \right)(x) = \sum_{n=0}^{\infty} \frac{(-c)^n (\frac{1}{2})^{\frac{v}{k}+2n}}{\Gamma_k(v+k+nk) n!} \left(I_{0+,k}^{\alpha,\beta,\eta} t^{\frac{\lambda+v}{k}+2n-1} \right)(x)$$

By equation (24) and for any $n = 0, 1, 2, \cdots$, $\Re(\lambda + v + 2nk) \geq \Re(\lambda + v) > \max[0, \Re(\beta - \eta)]$. Applying equation (21), we obtain

$$\left(I^{\alpha,\beta,\eta}_{0+,k} t^{\frac{\lambda}{k}-1} W^k_{v,c}(t)\right)(x) = \frac{x^{\frac{\lambda+v}{k}-\beta-1}}{2^{\frac{v}{k}}} \times \sum_{n=0}^{\infty} \frac{\Gamma(\frac{v}{k}+\frac{\lambda}{k}+2n)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\eta-\beta+2n)}{\Gamma(\frac{v}{k}+\frac{\lambda}{k}-\beta+2n)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\alpha+\eta+2n)\Gamma_k(\frac{v}{k}+1+n)k^{\frac{v}{k}}} \times \frac{(-cx^2)^n}{(4k)^n n!} \tag{26}$$

By equation (18), we obtain

$$\left(I^{\alpha,\beta,\eta}_{0+,k} t^{\frac{\lambda}{k}-1} W^k_{v,c}(t)\right)(x) = \frac{x^{\frac{v}{k}+\frac{\lambda}{k}-\beta-1}}{(2k)^{\frac{v}{k}}} {}_2\Psi_3\left[\begin{matrix} (\frac{v}{k}+\frac{\lambda}{k},2),(\frac{v}{k}+\frac{\lambda}{k}+\eta-\beta,2) \\ (\frac{v}{k}+\frac{\lambda}{k}-\beta,2),(\frac{v}{k}+\frac{\lambda}{k}+\alpha+\eta,2),(\frac{v}{k}+1,1) \end{matrix} \,\Big|\, -\frac{cx^2}{4k}\right].$$

This is the required proof of (25). □

Corollary 2.2. *Assume that* α, λ, $v \in \mathbb{C}$ *be such that* $\Re(v) > -1$, $\Re(\alpha) > 0$, $\Re(\lambda + v) > 0$, *then the following result holds:*

$$\left(I^{\alpha}_{0+} t^{\frac{\lambda}{k}-1} W^k_{v,c}(t) \right)(x) = \frac{x^{\frac{v}{k}+\frac{\lambda}{k}+\alpha-1}}{(2k)^{\frac{v}{k}}} \times {}_1\Psi_2\left[\begin{matrix} (v+\lambda,2k) \\ (\frac{v}{k}+\frac{\lambda}{k}+\alpha,2),(\frac{v}{k}+1,k) \end{matrix} \,\Big|\, -\frac{cx^2}{4k}\right]. \tag{27}$$

Proof. By substituting $\beta = -\alpha$ in (25), we obtain the required result. □

Corollary 2.3. *Assume that* α, η, λ, $v \in \mathbb{C}$ *be such that* $\Re(v) > -1$, $\Re(\alpha) > 0$, $\Re(\lambda + v) > 0$, *then the following formula holds:*

$$\left(K^{+}_{\alpha,\eta} t^{\frac{\lambda}{k}-1} W^k_{v,c}(t) \right)(x) = \frac{x^{\frac{v}{k}+\frac{\lambda}{k}-1}}{(2k)^{\frac{v}{k}}} \times {}_1\Psi_2\left[\begin{matrix} (\frac{v}{k}+\frac{\lambda}{k}+\eta,2) \\ (\frac{v}{k}+\frac{\lambda}{k}+\alpha+\eta,2),(\frac{v}{k}+1,1) \end{matrix} \,\Big|\, -\frac{cx^2}{4k}\right]. \tag{28}$$

Proof. By setting $\beta = 0$ in (25), we get the desired result. □

Theorem 2.4. *Assume that* α, β, η, λ, $v \in \mathbb{C}$ *and* $k > 0$ *be such that*

$$\Re(v) > -1, \Re(\alpha) > 0, \Re(\lambda - v) < 1 + \min[\Re(\beta), \Re(\eta)], \tag{29}$$

then the following result holds:

$$\left(I^{\alpha,\beta,\eta}_{0-} t^{\frac{\lambda}{k}-1} W^k_{v,c}(\tfrac{1}{t}) \right)(x) = \frac{x^{\frac{\lambda-v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}$$

$$\times\ {}_2\Psi_3\left[\begin{array}{c}(1+\beta-\frac{\lambda}{k}+\frac{v}{k},2),(1-\frac{\lambda}{k}+\frac{v}{k}+\eta,2)\\ (1-\frac{\lambda}{k}+\frac{v}{k},2),(1+\beta+\alpha+\eta-\frac{\lambda}{k}+\frac{v}{k},2),(\frac{v}{k}+1,k)\end{array}\Big| -\frac{c}{4x^2}\right]. \tag{30}$$

Proof. Note that the condition

$$1+\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>0$$

$(i.e., 1>-1)$ is satisfied so therefore ${}_2\Psi_3(z)$ is defined. Now, from (4) and (9), we have

$$\left(I_{0-}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\tfrac{1}{t})\right)(x)=\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{\frac{v}{k}+2n}}{\Gamma_k(v+k+nk)n!}\left(I_{0-}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}+\frac{v}{k}-2n-1}\right)(x)$$

By equation (29) and for any $k>0$ and $n=0,1,2,\cdots$, $\Re(\lambda-v-2n-1)\leq 1+\Re(\lambda-v-1)<1+\min[\beta,\Re(\eta)]$. Applying equation (21), we obtain

$$\left(I_{0-}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\frac{1}{t})\right)(x)=\frac{x^{\frac{\lambda}{k}+\frac{v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\beta-\frac{\lambda}{k}+\frac{v}{k}+1+2n)\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1+2n)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k}+2n)\Gamma(\alpha+\beta+\eta-\frac{\lambda}{k}+\frac{v}{k}+1+2n)\Gamma(\frac{v}{k}+1+n)}$$
$$\times\frac{(-c)^n}{(4kx^2)^n n!} \tag{31}$$

By equation (18), we obtain

$$\left(I_{0-}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\frac{1}{t})\right)(x)=\frac{x^{\frac{\frac{\lambda}{k}-\frac{v}{k}-\beta}{k}-1}}{(2k)^{\frac{v}{k}}}$$
$$\times{}_2\Psi_3\left[\begin{array}{c}(\beta-\frac{\lambda}{k}+\frac{v}{k}+1,2),(\eta-\frac{\lambda}{k}+\frac{v}{k}+1,2)\\ (1-\frac{\lambda}{k}+\frac{v}{k},2),(\alpha+\beta+\eta-\frac{\lambda}{k}+\frac{v}{k}+1,2),(\frac{v}{k}+1,1)\end{array}\Big| -\frac{c}{4kx^2}\right].$$

This is the required proof of (30). □

Corollary 2.5. *Assume that α, η λ, $v\in\mathbb{C}$ and $k>0$ be such that $\Re(v)>-1$, $0<\Re(\alpha)<1-\Re(\lambda-v)$, then the following result holds:*

$$\left(I_{0+}^{\alpha}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\tfrac{1}{t})\right)(x)=\frac{x^{\frac{\lambda}{k}-\frac{v}{k}+\alpha-1}}{(2k)^{\frac{v}{k}}}$$
$$\times\ {}_1\Psi_2\left[\begin{array}{c}(1-\alpha-\frac{\lambda}{k}+\frac{v}{k},2)\\ (1-\frac{\lambda}{k}+\frac{v}{k},2),(\frac{v}{k}+1,1)\end{array}\Big| -\frac{c}{4kx^2}\right]. \tag{32}$$

Corollary 2.6. *Assume that α, η, λ, $v\in\mathbb{C}$ and $k>0$ be such that $\Re(v)>-1$, $\Re(\alpha)>0$, $\Re(\lambda+v)<1+\max[0,\Re(\eta)]$, then the following formula holds:*

$$\left(K_{\alpha,\eta}^{-}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\tfrac{1}{t})\right)(x)=\frac{x^{\frac{\lambda}{k}-\frac{v}{k}-1}}{(2k)^{\frac{v}{k}}}$$

$$\times\ {}_1\Psi_2\left[\begin{matrix}(1+-\frac{\lambda}{k}+\frac{v}{k}+\eta,2)\\(1-\frac{\lambda}{k}+\frac{v}{k}+\alpha+\eta,2),(\frac{v}{k}+1,1)\end{matrix}\ \Big|-\frac{c}{4kx^2}\right]. \tag{33}$$

3. Representation in terms of generalized hypergeometric function

In this section, we introduce the generalized fractional integrals of k-Bessel function in term of generalized hypergeometric function. First we consider the following well known results:

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2});z\in\mathbb{C} \tag{34}$$

and

$$(z)_{2n}=2^{2n}(\frac{z}{2})_n(\frac{z+1}{2})_n,z\in\mathbb{C},n\in\mathbb{N}. \tag{35}$$

We represent the following theorems containing the generalized hypergeometric function.

Theorem 3.1. *Assume that α, β, η, λ, $v\in\mathbb{C}$ be such that*

$$\Re(v)>-1,\Re(\alpha)>0,\Re(\lambda+v)>\max[0,\Re(\beta-\eta)], \tag{36}$$

and let $\frac{\lambda}{k}+\frac{v}{k}$, $\frac{\lambda}{k}+\frac{v}{k}+\eta-\beta\neq 0,-1,\cdots$, then the following result holds:

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^k(t)\right)(x)=\frac{x^{\frac{\lambda}{k}+\frac{v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}\frac{\Gamma(\frac{\lambda}{k}+\frac{v}{k})\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\eta-\beta)}{\Gamma(\frac{\lambda}{k}+\frac{v}{k}-\beta)\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\alpha+\eta)\Gamma(\frac{v}{k}+1)}$$

(37)

$$\times{}_4F_5\left[\begin{matrix}\frac{\lambda}{2k}+\frac{v}{2k},\frac{\lambda}{2k}+\frac{v}{2k}+\frac{1}{2},\frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta-\beta}{2},\frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta-\beta+1}{2}\\ \frac{v}{k}+1,\frac{\lambda}{2k}+\frac{v}{2k}-\frac{\beta}{2},\frac{\lambda}{2k}+\frac{v}{2k}-\frac{\beta+1}{2},\frac{\lambda}{k}+\frac{v}{k}+\frac{\alpha+\eta}{2},\frac{\lambda}{2k}+\frac{v}{2k}+\frac{\alpha+\eta+1}{2}\end{matrix}\ \Big|-\frac{cx^2}{4k}\right].$$

Proof. Note that ${}_4F_5$ defined in (37) exit as the series is absolutely convergent. Now, using (15) with $z=\frac{v}{k}+1$ and (26) and applying (35) with z being replaced by $\frac{\lambda}{k}+\frac{v}{k}$, $\frac{\lambda}{k}+\frac{v}{k}+\eta-\beta$ and $\frac{\lambda}{k}+\frac{v}{k}+\alpha+\eta$, we have

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^k(t)\right)(x)=\frac{x^{\frac{\lambda+v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\frac{v}{k}+\frac{\lambda}{k})\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\eta-\beta)}{\Gamma(\frac{v}{k}+\frac{\lambda}{k}-\beta)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\alpha+\eta)\Gamma(\frac{v}{k}+1)}$$

$$\times\frac{(\frac{v}{k}+\frac{\lambda}{k})_{2n}(\frac{v}{k}+\frac{\lambda}{k}+\eta-\beta)_{2n}}{(\frac{v}{k}+\frac{\lambda}{k}-\beta)_{2n}(\frac{v}{k}+\frac{\lambda}{k}+\alpha+\beta)_{2n}}\frac{(-cx^2)^n}{(4k)^n n!}$$

$$=\frac{x^{\frac{\lambda+v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}\frac{\Gamma(\frac{v}{k}+\frac{\lambda}{k})\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\eta-\beta)}{\Gamma(\frac{v}{k}+\frac{\lambda}{k}-\beta)\Gamma(\frac{v}{k}+\frac{\lambda}{k}+\alpha+\eta)\Gamma(\frac{v}{k}+1)}$$

$$\times\sum_{n=0}^{\infty}\frac{(\frac{v}{2k}+\frac{\lambda}{2k})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{1}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\eta-\beta}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\eta-\beta}{2})_n}{(\frac{v}{k}+1)(\frac{v}{2k}+\frac{\lambda}{2k}-\frac{\beta}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}-\frac{\beta}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\alpha+\eta}{2})_n(\frac{v}{2k}+\frac{\lambda}{2k}+\frac{\alpha+\eta+1}{2})_n}$$

$$\times\frac{(-cx^2)^n}{(4k)^n n!}.$$

Thus, in accordance with equation (13), we get the required result (37). □

Corollary 3.2. *Assume that α, λ, $v \in \mathbb{C}$ be such that $\Re(v) > -1$, $\Re(\alpha) > 0$, $\Re(\lambda + v) > 0$ and $\frac{\lambda}{k} + \frac{v}{k} = 0, -1, \cdots$, then the following result holds:*

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t)\right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}+\alpha-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\frac{\lambda}{k}+\frac{v}{k})}{\Gamma(\frac{\lambda}{k}+\frac{v}{k}-\beta)\Gamma(\frac{v}{k}+1)}$$

$$(38) \qquad \times_2 F_3 \left[\begin{matrix} \frac{\lambda}{2k}+\frac{v}{2k}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{1}{2}, \\ \frac{v}{k}+1, \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\beta}{2}, \frac{\lambda}{2k}+\frac{v}{2k}-\frac{\beta+1}{2} \end{matrix} \mid -\frac{cx^2}{4k} \right].$$

Proof. By substituting $\beta = -\alpha$ in (37), we obtain the required result. □

Corollary 3.3. *Assume that α, η, λ, $v \in \mathbb{C}$ be such that $\Re(v) > -1$, $\Re(\alpha) > 0$, $\Re(\lambda + v) > 0$ and let $\frac{\lambda}{k} + \frac{v}{k} + \eta - \beta \neq 0, -1, \cdots$, then the following result holds:*

$$\left(K_{\alpha,\eta}^{+} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(t)\right)(x) = \frac{x^{\frac{\lambda}{k}+\frac{v}{k}-1}}{(2k)^{\frac{v}{k}}} \frac{\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\eta)}{\Gamma(\frac{\lambda}{k}+\frac{v}{k}+\alpha+\eta)\Gamma(\frac{v}{k}+1)}$$

$$(39) \qquad \times\, {}_2F_3 \left[\begin{matrix} \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\eta+1}{2} \\ \frac{v}{k}+1, \frac{\lambda}{k}+\frac{v}{k}+\frac{\alpha+\eta}{2}, \frac{\lambda}{2k}+\frac{v}{2k}+\frac{\alpha+\eta+1}{2} \end{matrix} \mid -\frac{cx^2}{4k} \right].$$

Proof. By setting $\beta = 0$ in (37), we get the desired result. □

Theorem 3.4. *Assume that α, β, η, λ, $v \in \mathbb{C}$ and $k > 0$ be such that*

$$(40) \qquad \Re(v) > -1, \Re(\alpha) > 0, \Re(\lambda - v) < 1 + \min[\Re(\beta), \Re(\eta)],$$

and let $\frac{\beta-\lambda}{k} + \frac{v}{k} + 1$, $\eta - \frac{\lambda}{k} + \frac{v}{k} + 1 \neq 0, -1, \cdots$, then the following result holds:

$$\left(I_{0-}^{\alpha,\beta,\eta} t^{\frac{\lambda}{k}-1} W_{v,c}^{k}(\frac{1}{t})\right)(x) = \frac{x^{\frac{\lambda}{k}-\frac{v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}$$

$$\times \frac{\Gamma(\beta-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\alpha+\beta+\eta-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\frac{v}{k}+1)}$$

$$(41)$$

$$\times_4 F_5 \left[\begin{matrix} \frac{\beta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k}, \frac{\beta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k}, \frac{\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k}, \frac{\eta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k} \\ \frac{v}{k}+1, \frac{1}{2}-\frac{\lambda}{2k}+\frac{v}{2k}, 1-\frac{\lambda}{2k}+\frac{v}{2k}, \frac{\alpha+\beta+\eta+1}{2}-\frac{\lambda}{k}+\frac{v}{k}, \frac{\alpha+\eta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k} \end{matrix} \mid -\frac{c}{4kx^2} \right].$$

Proof. Using (15) with $z = \frac{v}{k}+1$ and (31) and applying (35) with z being replaced by $\beta-\frac{\lambda}{k}+\frac{v}{k}+1$, $1-\frac{\lambda}{k}+\frac{v}{k}$ and $\beta-\frac{\lambda}{k}+\frac{v}{k}+\alpha+\eta+1$, we have

$$\left(I_{0-}^{\alpha,\beta,\eta}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\frac{1}{t})\right)(x) = \frac{x^{\frac{\lambda}{k}-\frac{v}{k}-\beta-1}}{(2k)^{\frac{v}{k}}}$$
$$\frac{\Gamma(\beta-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\alpha+\beta+\eta-\frac{\lambda}{k}+\frac{v}{k}+1)\Gamma(\frac{v}{k}+1)}$$
$$\times\sum_{n=0}^{\infty}\frac{(\frac{\beta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\beta}{2}-\frac{\lambda}{2k}+\frac{v}{2k}+1)_n(\frac{\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\eta}{2}-\frac{\lambda}{2k}+\frac{v}{2k}+1)_n}{(\frac{v}{k}+1)_n((\frac{1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n)(1-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\alpha+\beta+\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k})_n(\frac{\alpha+\beta+\eta}{2}-\frac{\lambda}{2k}+\frac{v}{2k}+1)_n}$$
$$\times\frac{(-c)^n}{(4kx^2)^n n!}.$$

By equation (13), we obtain the required given in (41). □

Corollary 3.5. *Assume that α, η λ, $v\in\mathbb{C}$ and $k>0$ be such that $\Re(v)>-1$, $0<\Re(\alpha)<1-\Re(\lambda-v)$, and let $\frac{\lambda}{k}-\frac{v}{k}+\alpha\neq 1,2,\cdots$ then the following result holds:*

$$\left(\ I_{0+}^{\alpha}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\tfrac{1}{t})\ \right)(x) = \frac{x^{\frac{\lambda}{k}-\frac{v}{k}+\alpha-1}}{(2k)^{\frac{v}{k}}}\frac{\Gamma(-\alpha-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\frac{v}{k}+1)}$$
$$\times\ {}_2F_3\left[\begin{matrix}\frac{-\beta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k},\frac{-\alpha+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k}, \\ \frac{v}{k}+1,\frac{1}{2}-\frac{\lambda}{2k}+\frac{v}{2k},1-\frac{\lambda}{2k}+\frac{v}{2k},\end{matrix}\ \middle|-\frac{c}{4kx^2}\right]. \tag{42}$$

Corollary 3.6. *Assume that α, η, λ, $v\in\mathbb{C}$ and $k>0$ be such that $\Re(v)>-1$, $\Re(\alpha)>0$, $\Re(\lambda+v)<1+\max[0,\Re(\eta)]$ and let $\frac{\lambda}{k}-\frac{v}{k}-\eta\neq 1,2,\cdots$, then the following formula holds:*

$$\left(\ K_{\eta,\alpha}^{-}t^{\frac{\lambda}{k}-1}W_{v,c}^{k}(\tfrac{1}{t})\ \right)(x) = \frac{x^{\frac{\lambda}{k}-\frac{v}{k}-1}}{(2k)^{\frac{v}{k}}}\frac{\Gamma(\eta-\frac{\lambda}{k}+\frac{v}{k}+1)}{\Gamma(1-\frac{\lambda}{k}+\frac{v}{k})\Gamma(\frac{v}{k}+1)}$$
$$\times\ {}_2F_3\left[\begin{matrix}\frac{\eta+1}{2}-\frac{\lambda}{2k}+\frac{v}{2k},\frac{\eta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k} \\ \frac{v}{k}+1,\frac{\alpha+\eta+1}{2}-\frac{\lambda}{k}+\frac{v}{k},\frac{\alpha+\eta+2}{2}-\frac{\lambda}{2k}+\frac{v}{2k}\end{matrix}\ \middle|-\frac{c}{4kx^2}\right]. \tag{43}$$

Corollary 3.5 and 3.6 follow from theorem 3.4 in respective cases $\beta=-\alpha$ and $\beta=0$.

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