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RELATIONS BETWEEN THE FIRST AND SECOND ZAGREB INDICES OF SUBDIVISION GRAPHS

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Abstract. The first and second Zagreb indices of a graph are two of the topological invariants used in molecular calculations by Mathematicians and Chemists. First Zagreb index and multiplicative Zagreb indices, all versions of Zagreb indices of subdivision graphs, Zagreb indices of the line graphs of the subdivision graphs, Zagreb indices of subdivision graphs of double graphs, multiplicative Zagreb indices of graph operations were calculated and as a generalisation, the authors determined the multiplicative Zagreb indices of the r-subdivision of double graphs. In this paper, we obtain numerous new relations between the first and second Zagreb indices of the subdivision graphs of certain graph types.

1. Introduction

Let $G = (V, E)$ be a simple graph, that is a connected graph without loops and multiple edges, with $| V(G) |= n$ vertices and $| E(G) |= m$ edges. For a vertex $v \in V(G)$, the degree of v is denoted by $d_G(v)$. In particular, a vertex with degree one is called a pendant vertex. As usual, the path, cycle, star, complete, bipartite and tadpole graphs are denoted by $P_n, C_n, S_n, K_n, K_{t,s}$ and $T_{t,s}$, respectively.

The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each of its edges by a path of length 2, or equivalently by inserting an additional vertex into each edge of G. Naturally, the size and order of the subdivision graph $S(G)$ are larger than the size and order of the original graph G and it is interesting and also useful to find formulae for the topological graph indices of subdivision graph $S(G)$ in terms of the topological indices of G. Here we handled this problem for the first and second Zagreb indices as a

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continuation of [13].

Several topological graph indices have been defined and studied by many mathematicians and chemists as most graphs are generated from molecules by replacing atoms with vertices and bonds with edges. These indices are used to determine several properties, mostly physical and chemical, of the corresponding graph. Two of the most important topological graph indices are called the first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively:

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{u,v \in E(G)} d_G(u) d_G(v).$$

They were first defined 45 years ago by Gutman and Trinajstic [4]. There are hundreds of recent papers on these two indices giving some formulae, inequalities and comparision with other topological graph indices, e.g. see [7]. In [5] and [8], the difference of the first and second Zagreb indices were calculated. In [13], the first and second Zagreb indices for subdivision graphs of some well-known graphs were given as follows:

Theorem 1.1.

$$M_1(S(G)) = \begin{cases} 8n-10 & \text{if } G = P_n,\ n \geq 2 \\ 8n & \text{if } G = C_n,\ n > 2 \\ (n-1)(n+4) & \text{if } G = S_n,\ n \geq 2 \\ n^3 - n & \text{if } G = K_n,\ n \geq 2 \\ ts(t+s+4) & \text{if } G = K_{t,s},\ t,s \geq 1 \\ 2(4t+4s+1) & \text{if } G = T_{t,s},\ t \geq 3,\ s \geq 1 \end{cases}$$

and

$$M_2(S(G)) = \begin{cases} 8n-12 & \text{if } G = P_n,\ n \geq 2 \\ 8n & \text{if } G = C_n,\ n > 2 \\ 2n(n-1) & \text{if } G = S_n,\ n \geq 2 \\ 2n(n-1)^2 & \text{if } G = K_n,\ n \geq 2 \\ 2ts(t+s) & \text{if } G = K_{t,s},\ t,s \geq 1 \\ 4(2t+2s+1) & \text{if } G = T_{t,s},\ t \geq 3,\ s \geq 1. \end{cases}$$

Several calculations are made with Zagreb indices: In [3], first Zagreb index and multiplicative Zagreb indices; in [11], all versions of Zagreb indices of subdivision graphs; in [9], the Zagreb indices of the line graphs of the subdivision graphs; in [13], Zagreb indices of subdivision graphs of double graphs; in [2] and [6], the multiplicative Zagreb indices of graph operations were calculated.

As a generalisation of subdivision graphs, r-subdivision graphs were defined and studied in [12] and [10]. In this paper, we obtain some comperative results between the first and second Zagreb indices of the subgraphs in four cases: i) Comparision between $M_1(S(G))$ and $M_2(S(G))$ for the same graph G; ii) comparision between $M_1(S(G))$ and $M_2(S(G'))$ for two different graphs G and G'; iii) comparision between $M_1(S(G))$ and $M_1(S(G'))$ for two different graphs G and G'; and finally iv) comparision between $M_2(S(G))$ and $M_2(S(G'))$ for two different graphs G and G' where the graphs under consideration are the path, cycle, star, complete, complete bipartite and tadpole graphs. In all comparisions, the relation between two indices is given in terms of the number n except for those including complete bipartite graphs in which case the relation also includes the number $m' = ts$ of edges of $K_{t,s}$. Similar results were obtained in [9] for the Zagreb indices of the line graphs of the subdivision graphs, and in [2] for the multiplicative Zagreb indices of graph operations. Also in [13], the same authors calculated the Zagreb indices and multiplicative Zagreb indices of subdivision graphs of double graphs. Ascioglu and Cangul calculated the sigma index and forgotten index of the subdivision and r-subdivision graphs in [1].

2. Relations between the first and second Zagreb indices of subgraphs

Let G be a simple connected graph. We first compare the first and second Zagreb indices of the subdivision graph of G:

2.1. Comparision between $M_1(S(G))$ and $M_2(S(G))$.

Theorem 2.1. *The following relations hold between the first and second Zagreb indices of the subdivision graph of G:*

$$M_2(S(G)) - M_1(S(G)) = \begin{cases} -2 & \textit{if } G = P_n,\ n \geq 2 \\ 0 & \textit{if } G = C_n,\ n > 2 \\ (n-1)(n-4) & \textit{if } G = S_n,\ n \geq 2 \\ n(n-1)(n-3) & \textit{if } G = K_n,\ n \geq 2 \\ ts(t+s-4) & \textit{if } G = K_{t,s},\ t,s \geq 1 \\ 2 & \textit{if } G = T_{t,s},\ t \geq 3,\ s \geq 1. \end{cases}$$

Proof. From Theorem 1.1, we have the first and second Zagreb indices of subdivision graphs of given graphs. So as a result of this theorem, we can obtain the difference $M_2(S(G))-M_1(S(G))$ for all the given graphs. For the complete graph K_n, we have

$$\begin{aligned} M_2(S(K_n)) - M_1(S(K_n)) &= 2n(n-1)^2 - n^3 - n \\ &= n^3 - 4n^2 + 3n \\ &= n(n-1)(n-3). \end{aligned}$$

If we find the roots of this equation, we have $M_2(S(K_n)) - M_1(S(K_n)) < 0$ if $n = 2$; $= 0$ if $n = 3$ and > 0 if $n > 3$. If we follow the similar proocedures for the other graph types, we get the desired results. □

The following two corollaries are easy to prove by means of the above results:

Corollary 2.1.1. *The difference $M_2(S(G)) - M_1(S(G))$ can also be stated in a different way for S_n, K_n and $K_{t,s}$:*

$$M_2(S(S_n)) - M_1(S(S_n)) < 0 \text{ if } n = 2,3;\ = 0 \text{ if } n = 4;\ \text{ and } > 0 \text{ if } n > 4,$$

$$M_2(S(K_n)) - M_1(S(K_n)) < 0 \text{ if } n = 2;\ = 0 \text{ if } n = 3;\ \text{ and} > 0 \text{ if } n > 3$$

and

$$M_2(S(K_{t,s}))-M_1(S(K_{t,s})) < 0 \text{ if } t+s = 2,3;\ = 0 \text{ if } t+s = 4;\ \text{ and} > 0 \text{ if } t+s > 4.$$

Corollary 2.1.2. *The difference $2M_1(S(G)) - M_2(S(G))$ is given as follows for S_n, K_n and $K_{t,s}$:*

$$2M_1(S(G)) - M_2(S(G)) = \begin{cases} 8(n-1) & \text{if } G = S_n, n \geq 2 \\ 4n(n-1) & \text{if } G = K_n, n \geq 2 \\ 8ts & \text{if } G = K_{t,s}, t, s \geq 1. \end{cases}$$

Secondly, we compare these two indices for two different graphs:

2.2. **Comparision between $M_1(S(G))$ and $M_2(S(G'))$.**

Theorem 2.2. *a)* $M_1(S(P_n)) = \begin{cases} M_2(S(C_n)) - 10 & \text{for } n \geq 1 \\ 4M_2(S(K_{t,s}))/m' - 10 & \text{for } t, s \geq 1 \\ M_2(S(T_{t,s})) - 14 & \text{for } t \geq 3, s \geq 1, \end{cases}$

$$M_1(S(P_n)) > \begin{cases} M_2(S(S_n)) & \text{for } n = 2,3 \\ M_2(S(K_n)) & \text{for } n = 2 \end{cases}$$

and

$$M_1(S(P_n)) < \begin{cases} M_2(S(S_n)) & \textit{for } n > 3 \\ M_2(S(K_n)) & \textit{for } n \geq 3. \end{cases}$$

b) $$M_1(S(C_n)) = \begin{cases} M_2(S(P_n)) + 12 & \textit{for } n \geq 2 \\ M_2(S(S_n)) & \textit{for } n = 5 \\ M_2(S(K_n)) & \textit{for } n = 3 \\ 4M_2(S(K_{t,s}))/m' & \textit{for } t, s \geq 1 \\ M_2(S(T_{t,s})) - 4 & \textit{for } t \geq 3, s \geq 1 \end{cases}$$

and

$$M_1(S(C_n)) < \begin{cases} M_2(S(S_n)) & \textit{for } n > 5 \\ M_2(S(K_n)) & \textit{for } n > 3 \end{cases},$$

$$M_1(S(C_n)) > M_2(S(S_n)) \quad \textit{for} \quad 2 < n < 5.$$

c) $$M_1(S(S_n)) = \begin{cases} [M_2(S(P_n))^2 + 48M_2(S(P_n)) + 176]/64 & \textit{for } n \geq 2 \\ [2M_2(S(K_{t,s}))^2 + 3m' M_2(S(K_{t,s}))]/2(m')^2 - 4 & \textit{for } t, s \geq 1 \end{cases},$$

$$M_1(S(S_n)) < \begin{cases} M_2(S(C_n)) & \textit{for } 2 < n \leq 5 \\ M_2(S(K_n)) & \textit{for } n \geq 3 \\ M_2(S(T_{t,s})) & \textit{for } 4 \leq n = t + s \leq 6 \end{cases}$$

and

$$M_1(S(S_n)) > \begin{cases} M_2(S(C_n)) & \textit{for } n > 5 \\ M_2(S(K_n)) & \textit{for } n = 2 \\ M_2(S(T_{t,s})) & \textit{for } n = t + s > 6 \end{cases}.$$

d) $$M_1(S(K_n)) > \begin{cases} M_2(S(P_n)) & \textit{for } n \geq 2 \\ M_2(S(C_n)) & \textit{for } n > 3 \\ M_2(S(S_n)) & \textit{for } n \geq 2 \\ M_2(S(T_{t,s})) & \textit{for } n = t + s \geq 4 \end{cases}$$

and

$$M_1(S(K_n)) = \begin{cases} M_2(S(C_n)) & \textit{for } n = 3 \\ [M_2(S(K_{t,s}))^3 - 4(m')^2 M_2(S(K_{t,s}))]/8(m')^3 & \textit{for } t, s \geq 1 \end{cases}.$$

$$e)\ M_1(S(K_{t,s})) = \begin{cases} m^{'}\left[M_2(S(P_n)) + 44\right]/8 & for\ n \geq 2 \\ m^{'}\left[M_2(S(C_n)) + 32\right]/8 & for\ n \geq 2 \\ m^{'}\left[M_2(S(S_n)) + 8m\right]/2m & for\ n \geq 2 \\ m^{'}\left[M_2(S(T_{t,s})) + 28\right]/8 & for\ t \geq 3, s \geq 1 \end{cases}.$$

$$f)\ M_1(S(T_{t,s})) = \begin{cases} M_2(S(P_n)) + 12 & for\ n \geq 2 \\ M_2(S(C_n)) + 2 & for\ n \geq 2 \\ 4M_2(S(K_{t,s}))/m^{'} + 2 & for\ t, s \geq 1 \end{cases},$$

$$M_1(S(T_{t,s})) < \begin{cases} M_2(S(S_n)) & for\ n > 5 \\ M_2(S(K_n)) & for\ n \geq 4 \end{cases}$$

and

$$M_1(S(T_{t,s})) > M_2(S(S_n)) \quad for \quad n = 4, 5.$$

Proof. We prove c). The others follow by means of combinatorial calculations.

By Theorem 1.1, $8n = M_2(S(P_n)) + 12$. Therefore

$$\begin{aligned} M_1(S(S_n)) &= (n-1)(n+4) \\ &= \left(\frac{M_2(S(P_n)) + 4}{8}\right)\left(\frac{M_2(S(P_n)) + 44}{8}\right) \\ &= [M_2(S(P_n))^2 + 48M_2(S(P_n)) + 176]/64. \end{aligned}$$

Similarly, $M_1(S(S_n)) - M_2(S(C_n)) = n^2 - 5n - 4$ which has the real roots $(5 \pm \sqrt{41})/2$. Therefore, these two indices cannot be equal. When $2 < n \leq 5$, $M_1(S(S_n)) < M_2(S(C_n))$ and for n>5, $M_1(S(S_n)) > M_2(S(C_n))$. If we want to compare star and complete graphs, we have $M_2(S(K_n)) - M_1(S(S_n)) = 2n^3 - 5n^2 - n + 4$. Similarly to the previous comparision, we have the required inequalities.

Again by Theorem 1, $n = M_2(S(K_{t,s}))/2m'$. As $M_1(S(S_n)) = n^2 + 3n - 4$, we calculate that

$$M_1(S(S_n)) = \left(\frac{M_2(S(K_{t,s}))}{2m'}\right)^2 + \frac{3}{2m'}M_2(S(K_{t,s})) - 4$$

implying that $M_1(S(S_n)) = [4M_2(S(K_{t,s}))^2 + 6m^{'}M_2(S(K_{t,s})) - 16(m^{'})^2]/4(m^{'})^2$. Finally, $M_1(S(S_n)) - M_2(S(T_{t,s})) = n^2 - 5n - 8$. Calculating the roots of this equation, the required inequalities are obtained. □

Thirdly, we compare the first Zagreb indices for two different graphs:

2.3. **Comparision between** $M_1(S(G))$ **and** $M_1(S(G'))$**.** We have

Theorem 2.3. *a)* $M_1(S(P_n)) = \begin{cases} M_1(S(C_n)) - 10 & \textit{for } n \geq 2 \\ 8M_1(S(K_{t,s}))/m^{'} - 42 & \textit{for } t,s \geq 1 \\ M_1(S(T_{t,s})) - 12 & \textit{for } t \geq 3, s \geq 1 \\ M_1(S(S_n)) & \textit{for } n = 2,3 \\ M_1(S(K_n)) & \textit{for } n = 2 \end{cases}$

and

$$M_1(S(P_n)) < \begin{cases} M_1(S(S_n)) & \textit{for } n > 3 \\ M_1(S(K_n)) & \textit{for } n > 2 \end{cases}$$

b) $M_1(S(C_n)) = \begin{cases} 4M_1(S(K_{t,s}))/m^{'} & \textit{for } t,s > 0 \\ M_1(S(T_{t,s})) - 4 & \textit{for } t \geq 3, s \geq 1 \\ M_1(S(S_n)) & \textit{for } n = 5 \\ M_1(S(K_n)) & \textit{for } n = 3 \end{cases},$

$M_1(S(C_n)) > M_1(S(S_n)) \quad \textit{for } 2 < n < 5,$

and

$M_1(S(C_n)) < M_1(S(K_n)) \quad \textit{for } n > 3.$

c) $M_1(S(S_n)) = \begin{cases} M_1(S(K_n)) & \textit{for } n = 2 \\ [M_1(S(K_{t,s}))^2 - 2m^{'}M_1(S(K_{t,s}))]/2\left(m^{'}\right)^2 & \textit{for } t,s \geq 1 \end{cases}$

$M_1(S(S_n)) < M_1(S(K_n)) \quad \textit{for } n > 2,$

$M_1(S(S_n)) < M_1(S(T_{t,s})) \quad \textit{for } n = t+s = 4,5,$

$M_1(S(S_n)) > M_1(S(T_{t,s})) \quad \textit{for } n = t+s > 5$

$$M_1(S(K_n)) = \begin{cases} [M_1(S(K_{t,s}))^3 - 4m^{'}M_1(S(K_{t,s}))^2 + \\ 4(m^{'})^2M_1(S(K_{t,s}))]/4\left(m^{'}\right)^3 & \textit{for } t,s \geq 1 \\ > M_1(S(T_{t,s})) & \textit{for } n = t+s \geq 4 \end{cases}$$

$M_1(S(T_{t,s})) = 4\left(M_1(S(K_{t,s})) + m^{'}\right)/m^{'} \textit{ for } t,s \geq 1.$

Proof. We prove a). The others can similarly be proven.

$$M_1(S(P_n)) = M_1(S(C_n)) - 10$$

follows from Theorem 1.1 immediately. Recall that

$$M_1(S(S_n)) - M_1(S(P_n)) = (n-2)(n-3).$$

Therefore these two indices are equal when $n = 2$ and 3, which is clear visually as in these two cases, the graphs are equal. Also when $n > 3$, it follows that

$$M_1(S(P_n)) < M_1(S(S_n)).$$

Also

$$M_1(S(K_n)) - M_1(S(P_n)) = (n-2)(n^2 + 2n - 5)$$

is obtained from Theorem 1.1. Therefore these two indices are equal only when $n = 2$, and for all $n > 2$,

$$M_1(S(K_n)) > M_1(S(P_n)).$$

Now $M_1(S(P_n)) = 8n - 10$ and therefore

$$M_1(S(K_{t,s})) = ts(t + s + 4) = m'(n + 4) = m'\left(\frac{M_1(S(P_n)) + 10}{8} + 4\right)$$

which gives the required result. Finally, $M_1(S(T_{t,s})) = 2(4t + 4s + 1)$, and using the fact that $s + t = n$, we obtain $M_1(S(T_{t,s})) = 8n + 2$ and therefore

$$M_1(S(K_{t,s})) - M_1(S(P_n)) = 12.$$

□

Finally, we compare the second Zagreb indices for two different graphs.

2.4. **Comparision between $M_2(S(G))$ and $M_2(S(G'))$.**

Theorem 2.4. *a)*

$$M_2((S(P_n)) = \begin{cases} M_2((S(C_n)) - 12 & n \geq 2 \\ M_2(S(S_n)) & \textit{for } n = 2, 3 \\ M_2(S(K_n)) & \textit{for } n = 2 \\ [4M_2(S(K_{t,s})) - 12]/m' & \textit{for } t, s \geq 1 \\ M_2(S(T_{t,s})) - 16 & \textit{for } t \geq 3, s \geq 1 \end{cases}$$

and

$$M_2((S(P_n)) < \begin{cases} M_2(S(S_n)) & \textit{for } n > 3 \\ M_2(S(K_n)) & \textit{for } n > 2 \end{cases}.$$

b)

$$M_2((S(C_n)) = \begin{cases} M_2(S(S_n)) & \text{for } n = 5 \\ M_2(S(K_n)) & \text{for } n = 3 \\ 4M_2(S(K_{t,s}))/m' & \text{for } t, s \geq 1 \\ M_1(S(T_{t,s})) - 4 & \text{for } t \geq 3, s \geq 1 \end{cases},$$

$$M_2((S(C_n)) < \begin{cases} M_2(S(S_n)) & \text{for } n < 5 \\ M_2(S(K_n)) & \text{for } n > 3 \end{cases}$$

and

$$M_2((S(C_n)) > M_2(S(S_n)) \text{ for } n > 5.$$

c)

$$M_2((S(S_n)) = \begin{cases} M_2(S(K_n)) & \text{for } n = 2 \\ [M_2(S(K_{t,s}))^2 - 2m'M_2(S(K_{t,s}))]/2(m')^2 & \text{for } t, s \geq 1 \end{cases},$$

$$M_2((S(S_n)) < \begin{cases} M_2(S(K_n)) & \text{for } n > 2 \\ M_2(S(T_{t,s})) & \text{for } n = t + s = 4, 5 \end{cases}$$

and

$$M_2((S(S_n)) > M_2(S(T_{t,s})) \text{ for } \; n = t + s > 5.$$

d)

$$M_2(S(K_n)) = [M_2(S(K_{t,s}))/m'] \left[\begin{array}{c} [M_2(S(K_{t,s}))^2 - 4m'M_2(S(K_{t,s})) \\ +4(m')^2]/4(m')^2 \end{array} \right]$$

for $t, s \geq 1$ *and*

$$M_2(S(K_n)) > M_2(S(T_{t,s})) \text{ for } n = t + s \geq 4.$$

e)

$$M_2(S(K_{t,s})) = m'(M_2(S(T_{t,s})) - 4)/4 \text{ for } t \geq 3, s \geq 1.$$

Proof. The proof follows similarly. □

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