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GENERALITIES CONCERNING IRREDUCIBLE PSEUDOREPRESENTATIONS OF GROUPS

A. I. Shtern

Abstract. We obtain general results concerning irreducible pseudorepresentations of groups. Applications to unbounded pseudorepresentations admitting an unbounded irreducible subpseudorepresentation are indicated.

§ 1. Introduction

Recall that a mapping π of a given group G into the family of operators on a Banach space E is said to be a *quasirepresentation* of G if

$$\|\pi(g_1g_2) - \pi(g_1) - \pi(g_2)\| \le \varepsilon, \qquad g_1, g_2 \in G,$$

for some ε, which is usually assumed to be sufficiently small and is referred to as a *defect* of π and a quasirepresentation of G is said to be a *pseudorepresentation* of G if $\pi(g^n)$ is similar to $\pi(g)^n$, $n \in \mathbb{N}$, with the help of an operator sufficiently close to the identity operator; see [1–4].

Some preliminary results concerning irreducible Banach space pseudorepresentations of groups (in the case of irreducible finite-dimensional locally bounded pseudorepresentations of connected locally compact groups) were obtained in [5].

A pseudorepresentation π on a Banach space E is said to be *algebraically irreducible* if there is no nontrivial (not necessarily closed) vector subspace invariant with respect to all operators $\pi(g)$, $g \in G$, and *irreducible* if there is no nontrivial closed vector subspace invariant with respect to all operators $\pi(g)$, $g \in G$.

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§ 2. Preliminaries

Lemma. *Let G be a group and let π be a pseudorepresentation of G. If π is (algebraically) irreducible, then either π is an unbounded ordinary representation of G or π is a bounded algebraically irreducible pseudorepresentation.*

Proof. As is well known [1–4], the obvious identity

$$(\pi(ghk) - \pi(gh)\pi(k)) + (\pi(gh) - \pi(g)\pi(h))\pi(k) \\ + \pi(g)(\pi(h)\pi(k) - \pi(hk)) + (\pi(g)\pi(hk) - \pi(ghk)) = 0$$

holds for all $g, h, k \in G$ and implies that the linear span of the images of the operators $\pi(gh) - \pi(g)\pi(h)$, $g, h \in G$, is invariant with respect to all operators $\pi(k)$, $k \in G$. By assumption, this linear span is either equal to E (in which case it follows from the Banach–Steinhaus theorem that π is bounded) or the linear span is zero, which automatically means that π is an (unbounded) ordinary representation (cf. [6]).

§ 3. Main result

The following generalization of the lemma holds.

Theorem. *Let G be a group and let π be a pseudorepresentation of G. If π is irreducible and bounded on the linear span of the images of the defect operators*

$$\pi(gh) - \pi(g)\pi(h), \qquad g, h \in G,$$

then either π is an ordinary representation of G or π is a bounded irreducible pseudorepresentation of G.

Proof. The condition of the theorem is obviously a generalization of the condition of the lemma. If the linear span of the images of the defect operators

$$\pi(gh) - \pi(g)\pi(h), \qquad g, h \in G,$$

is zero, then π is an ordinary representation by definition; if this linear span is nonzero, then the closure of this vector subspace is invariant with respect to π, and, since π is irreducible, it follows that the linear span in question is dense in the pseudorepresentation space, and hence the pseudorepresentation is bounded on the whole pseudorepresentation space, as was to be proved.

§ 4. Application

The following result is an almost immediate consequence of the theorem. We say that a Banach space pseudorepresentation π is an ε-pseudorepresentation if the defect of π does not exceed ε.

Corollary. *Let G be an amenable group, let π be a pseudorepresentation of G, with sufficiently small defect ε, in a Banach space E dual to some Banach space E_*, and let F be a π-invariant subspace of E closed with respect to the E_*-topology on E and such that the restriction of the pseudorepresentation π to F and the quotient pseudorepresentation defined by π on E/F are irreducible, satisfy the conditions of the previous theorem, and are unbounded. Let I be a left invariant mean on G and let*

$$\tau(g)) = I_h(\pi(gh)\pi(h^{-1})), \qquad g \in G.$$

Then the mapping ρ defined by the formula

$$\rho(g) = \tau(g) - \pi(g)\tau(e) + \pi(g), \qquad g \in G,$$

is an ordinary representation of G close to π (the distance between $\rho(g)$ and $\pi(g)$ does not exceed $\varepsilon(1 + \|\pi(g)\|)$, $g \in G$).

Proof. It follows from the condition of the theorem that the defect operators

$$\pi(gh) - \pi(g)\pi(h), \qquad g, h \in G,$$

act from E to F, where F belongs to the kernel of every defect operator. Therefore, the products of any two defect operators of the form

$$\pi(gh) - \pi(g)\pi(h), \qquad g, h \in G,$$

are equal to the zero operator, and hence the theorem of [7] can be applied. This theorem establishes the desired result almost immediately; only the distance between π and ρ is to be estimated. It immediately follows from the definition of defect that

$$\|\tau(g) - \pi(g)\| \le \varepsilon$$

for all $g \in G$, and therefore

$$\begin{aligned}\|\rho(g) - \pi(g)\| &= \|\tau(g) - \pi(g)\tau(e)\| \\ &\le \|\tau(g) - \pi(g)\| + \|\pi(g) - \pi(g)\tau(e)\| \\ &\le \varepsilon + \|\pi(g)\|\|1 - \tau(e)\|,\end{aligned}$$

where

$$\|1 - \tau(e)\| \le \varepsilon,$$

which completes the proof of the corollary.

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Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN),
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru