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Index matrices with elements index matrices

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Abstract. Index Matrices (IMs) are extensions of the ordinary matrices. They are also object of extensions and modifications, e.g., extended index matrices. In the present research, we describe extended index matrices, havind as elements whole index matrices.

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1 Introduction

The concept of Index Matrix (IM) was introduced in 1984, but the full description of the research over them was published in [2] exactly 30 years later. In this book, different extensions and modifications of the concept of an IM are described. One of them is an extended index matrix (EIM). Over an EIM two types of hierarchical operators are defined that modify a given EIM containing as an element whole IM, to a new IM. In the present paper, we discuss the possibility a given EIM to have as elements IMs with different dimensions. We show some basic properties and representations of these matrices.

Firstly, we give the definition of an EIM.

Let $\mathcal{I}$ be a fixed set of indices,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \leq j \leq n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1 \leq n \leq \infty} \mathcal{I}^n.$$

Let $\mathcal{X}$ be a fixed set of some objects. In the particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, etc.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be fixed.

An EIM with index sets K and L ($K, L \subset \mathcal{I}^*$) and elements from set $\mathcal{X}$ is called the object (see, [1, 2]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_j} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i & a_{k_i,l_1} & \ldots & a_{k_i,l_j} & \ldots & a_{k_i,l_n} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_j} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, for $1 \le i \le m$, and $1 \le j \le n$: $a_{k_i,l_j} \in \mathcal{X}$.

In [2], for the IMs $A = [K, L, \{a_{k_i,l_j}\}], B = [P, Q, \{b_{p_r,q_s}\}]$, operations that are analogous to the usual matrix operations of addition and multiplication are defined, as well as other, specific ones. Here, we give only two of these operations, that will be used below.

Addition

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ & \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ & \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ & \\ 0, & \text{otherwise} \end{cases}$$

Structural subtraction

$$A \ominus B = [K - P, L - Q, \{c_{t_u,v_w}\}],$$

where "–" is the set–theoretic difference operation and

$$c_{t_u,v_w} = a_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

Multiplication with a constant α

$$\alpha A = [K, L, \{\alpha a_{k_i,l_j}\}],$$

where α is a constant.

In [2], two **hierarchical operators** are defined. They are applicable onto EIM, when their elements are not only numbers, variables, etc, but also whole (new) IMs.

Let A be an EIM and let its element a_{k_f,e_g} be an IM by itself:

$$a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}],$$

where

$$K \cap P = L \cap Q = \emptyset.$$

The first hierarchical operator is

$$A|(a_{k_f,l_g}) = [(K - \{k_f\}) \cup P, (L - \{l_g\}) \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K - \{k_f\} \text{ and } v_w = l_j \in L - \{l_g\} \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}.$$

Let us assume that in the case when a_{k_f,l_g} is not an element of IM A, then

$$A|(a_{k_f,l_g}) = A.$$

From the first definition of a hierarchical operator it follows that

$$A|(a_{k_f,l_g})$$

$$= \begin{array}{c|cccccccccc} & l_1 & \dots & l_{g-1} & q_1 & \dots & q_u & l_{g+1} & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_{g-1}} & 0 & \dots & 0 & a_{k_1,l_{g+1}} & \dots & a_{k_1,l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{f-1} & a_{k_{f-1},l_1} & \dots & a_{k_{f-1},l_{g-1}} & 0 & \dots & 0 & a_{k_{f-1},l_{g+1}} & \dots & a_{k_{f-1},l_n} \\ p_1 & 0 & \dots & 0 & b_{p_1,q_1} & \dots & b_{p_1,q_v} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_u & 0 & \dots & 0 & b_{p_u,q_1} & \dots & b_{p_u,q_v} & 0 & \dots & 0 \\ k_{f+1} & a_{k_{f+1},l_1} & \dots & a_{k_{f+1},l_{g-1}} & 0 & \dots & 0 & a_{k_{f+1},l_{g+1}} & \dots & a_{k_{f+1},l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_m & a_{k_m,l_1} & \dots & a_{k_m,l_{g-1}} & 0 & \dots & 0 & a_{k_m,l_{g+1}} & \dots & a_{k_m,l_n} \end{array}.$$

From this form of the IM $A|(a_{k_f,l_g})$ we see that for the hierarchical operator the following equality holds.

Theorem (see [2]). Let $A = [K, L, \{a_{k_i,l_j}\}]$ be an IM and let $a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}]$ be its element. Then

$$A|(a_{k_f,l_g}) = (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f,l_g}.$$

In [2], the first hierarchical operator is modified so that all the information from the IMs, participating in it, be preserved. The new – second – form of this operator for the above defined IM A and its fixed element a_{k_f,l_g}, is

$$A|^*(a_{k_f,l_g})$$

$$= \begin{array}{c|ccccccccc}
 & l_1 & \dots & l_{g-1} & q_1 & \dots & q_u & l_{g+1} & \dots & l_n \\ \hline
k_1 & a_{k_1,l_1} & \dots & a_{k_1,l_{g-1}} & a_{k_1,l_g} & \dots & a_{k_1,l_g} & a_{k_1,l_{g+1}} & \dots & a_{k_1,l_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
k_{f-1} & a_{k_{f-1},l_1} & \dots & a_{k_{f-1},l_{g-1}} & a_{k_{f-1},l_g} & \dots & a_{k_{f-1},l_g} & a_{k_{f-1},l_{g+1}} & \dots & a_{k_{f-1},l_n} \\
p_1 & a_{k_f,l_1} & \dots & a_{k_f,l_{g-1}} & b_{p_1,q_1} & \dots & b_{p_1,q_v} & a_{k_f,l_{g+1}} & \dots & a_{k_f,l_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_u & a_{k_f,l_1} & \dots & a_{k_f,l_{g-1}} & b_{p_u,q_1} & \dots & b_{p_u,q_v} & a_{k_f,l_{g+1}} & \dots & a_{k_f,l_n} \\
k_{f+1} & a_{k_{f+1},l_1} & \dots & a_{k_{f+1},l_{g-1}} & a_{k_{f+1},l_g} & \dots & a_{k_{f+1},l_g} & a_{k_{f+1},l_{g+1}} & \dots & a_{k_{f+1},l_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
k_m & a_{k_m,l_1} & \dots & a_{k_m,l_{g-1}} & a_{k_m,l_g} & \dots & a_{k_m,l_g} & a_{k_m,l_{g+1}} & \dots & a_{k_m,l_n}
\end{array}.$$

Now, the following assertion is valid.

Theorem (see [2]). Let $A = [K, L, \{a_{k_i,l_j}\}]$ be an IM and let $a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}]$ be its element. Then

$$A|^*(a_{k_f,l_g})$$

$$= (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f,l_g} \oplus [P, L - \{l_g\}, \{c_{x,l_j}\}] \oplus [K - \{k_f\}, Q, \{d_{k_i,y}\}],$$

where for each $x \in P$ and for each $l_j \in L - \{l_g\}$,

$$c_{x,l_j} = a_{k_f,l_j}$$

and for each $k_i \in K - \{k_f\}$ and for each $y \in Q$,

$$d_{k_i,y} = a_{k_i,l_g}.$$

In [2], other representations of IMs $A|(a_{k_f,l_g})$ and $A|^*(a_{k_f,l_g})$ are given, using other operations defined over IMs.

2 Main results

Here, we discuss an EIM each element of which is an IM and an operator that modify this EIM to a standard IM, if no one of the IMs that are elements of the basic EIM is an EIM.

Let $A = [K, L, \{a_{k_i,l_j}\}]$ be an EIM, where $K = \{k_1, k_2, ..., k_m\}, L = \{l_1, l_2, ..., l_n\}$, and for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{X}$. Let each its element a_{k_f,e_g} be an IM by itself:

$$a_{k_f,l_g} = [P_{k_f,l_g}, Q_{k_f,l_g}, \{b_{k_f,l_g,p_u,q_v}\}],$$

where $1 \leq f \leq m$, $1 \leq g \leq n$, $1 \leq u \leq r_{f,g}$, $1 \leq v \leq s_{f,g}$ and

$$P_{k_f,l_g} = \{p_{k_f,l_g,1}, ..., p_{k_f,l_g,r_{f,g}}\},$$

$$Q_{k_f,l_g} = \{q_{k_f,l_g,1}, ..., q_{k_f,l_g,s_{f,g}}\},$$

$$K \cap P_{k_f,l_g} = L \cap Q_{k_f,l_g} = \emptyset \tag{1}$$

and for every four indices $k_f, k_h \in \mathcal{I}^*$ and $l_g, l_i \in \mathcal{I}^*$:

$$P_{k_f,l_g} \cap P_{k_h,l_i} = Q_{k_f,l_g} \cap Q_{k_h,l_i} = \emptyset. \tag{2}$$

The new (third) hierarchical operator is defined by

$$A|^* =$$

$$\begin{array}{l|cccccccc}
 & q_{k_1,l_1,1} & \dots & q_{k_1,l_1,s_{1,1}} & \dots & q_{k_m,l_n,1} & \dots & q_{k_m,l_n,s_{m,n}} \\ \hline
p_{k_1,l_1,1} & a_{k_1,l_1,1,1} & \dots & a_{k_1,l_1,1,s_{1,1}} & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{k_1,l_1,r_{1,1}} & a_{k_1,l_1,r_{1,1},1} & \dots & a_{k_1,l_1,r_{1,1},s_{1,1}} & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{k_m,l_n,1} & 0 & \dots & 0 & \dots & a_{k_m,l_n,1,1} & \dots & a_{k_m,l_n,1,s_{m,n}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{k_m,l_n,r_{m,n}} & 0 & \dots & 0 & \dots & a_{k_m,l_n,r_{m,n},1} & \dots & a_{k_m,l_n,r_{m,n},s_{m,n}}
\end{array}.$$

We see that conditions (1) and (2) are important for the correctness of the definition.

For example, if

$$A = \begin{array}{c|cc}
 & l_1 & l_2 \\ \hline
k_1 & a_{1,1} & a_{1,2} \\
k_2 & a_{2,1} & a_{2,2} \\
k_3 & a_{3,1} & a_{3,2}
\end{array},$$

where

$$a_{1,1} = \begin{array}{c|cc} & q_{1,1} & q_{1,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} \\ p_{1,2} & b_{2,1} & b_{2,2} \\ p_{1,3} & b_{3,1} & b_{3,2} \end{array}, \quad a_{1,2} = \begin{array}{c|ccc} & q_{2,1} & q_{2,2} & q_{2,3} \\ \hline p_{2,1} & c_{1,1} & c_{1,2} & c_{1,3} \\ p_{2,2} & c_{2,1} & c_{2,2} & c_{2,3} \end{array},$$

$$a_{2,1} = \begin{array}{c|c} & q_{3,1} \\ \hline p_{3,1} & d_{1,1} \end{array}, \quad a_{2,2} = \begin{array}{c|cc} & q_{4,1} & q_{4,2} \\ \hline p_{4,1} & e_{1,1} & e_{1,2} \\ p_{4,2} & e_{2,1} & e_{2,2} \\ p_{4,3} & e_{3,1} & e_{3,2} \end{array},$$

then

$$A|^{*} = \begin{array}{c|cccccccc} & q_{1,1} & q_{1,2} & q_{2,1} & q_{2,2} & q_{2,3} & q_{3,1} & q_{4,1} & q_{4,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,2} & b_{2,1} & b_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{1,3} & b_{3,1} & b_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{2,1} & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 \\ p_{2,2} & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 \\ p_{3,1} & 0 & 0 & 0 & 0 & 0 & d_{1,1} & 0 & 0 \\ p_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} \\ p_{4,2} & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} \\ p_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & e_{3,1} & e_{3,2} \end{array}.$$

Now, keeping condition (1), we can change condition (2) with the weaker one: for every four indices $k_f, k_h \in \mathcal{I}^*$ and $l_g, l_i \in \mathcal{I}^*$:

$$P_{k_f,l_g} \times Q_{k_f,l_g} \cap P_{k_h,l_i} \times Q_{k_h,l_i} = \emptyset,$$

i.e., there is no pair of indices that is found in two different a-IMs in the given EIM A, where $\times$ is the standard Cartesian product.

In this case, if there are two or more p-indices in $A|^*$ that coincide, all members of the rows with these indices are written on the respective places in the first row with coinciding indices. Obviously, there are no two elements, that stay on one and the same place, because they are members of different IMs.

After this, if there are two or more q-indices in $A|^*$ that coincide, all members of the columns with these indices are written on the respective places in the first column with coinciding indices. Obviously, again, there are no two elements, that stay on one and the same place.

For example, for the IM A from the above example, if

$$a_{1,1} = \begin{array}{c|cc} & q_{1,1} & q_{1,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} \\ p_{1,2} & b_{2,1} & b_{2,2} \\ p_{1,3} & b_{3,1} & b_{3,2} \end{array},$$

$$a_{1,2} = \begin{array}{c|ccc} & q_{2,1} & q_{2,2} & q_{2,3} \\ \hline p_{1,2} & c_{1,1} & c_{1,2} & c_{1,3} \\ p_{2,2} & c_{2,1} & c_{2,2} & c_{2,3} \end{array},$$

$$a_{2,1} = \begin{array}{c|c} & q_{3,1} \\ \hline p_{3,1} & d_{1,1} \end{array},$$

$$a_{2,2} = \begin{array}{c|cc} & q_{3,1} & q_{4,2} \\ \hline p_{4,1} & e_{1,1} & e_{1,2} \\ p_{4,2} & e_{2,1} & e_{2,2} \\ p_{4,3} & e_{3,1} & e_{3,2} \end{array},$$

then

$$A|^* = \begin{array}{c|ccccccc} & q_{1,1} & q_{1,2} & q_{2,1} & q_{2,2} & q_{2,3} & q_{3,1} & q_{4,2} \\ \hline p_{1,1} & b_{1,1} & b_{1,2} & 0 & 0 & 0 & 0 & 0 \\ p_{1,2} & b_{2,1} & b_{2,2} & c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 \\ p_{1,3} & b_{3,1} & b_{3,2} & 0 & 0 & 0 & 0 & 0 \\ p_{2,2} & 0 & 0 & c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 \\ p_{3,1} & 0 & 0 & 0 & 0 & 0 & d_{1,1} & 0 \\ p_{4,1} & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} \\ p_{4,2} & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} \\ p_{4,3} & 0 & 0 & 0 & 0 & 0 & e_{3,1} & e_{3,2} \end{array}.$$

The following assertions are valid.

Theorem 1. Let $A = [K, L, \{a_{k_i,l_j}\}]$ be an EIM and let each its element a_{k_f,l_g} be an IM. Then

$$A|^* = \sum_{\substack{1 \le f \le m \\ 1 \le g \le n}} a_{k_f,l_g},$$

where symbol $\sum$ denotes the generalization of operation addition $\oplus$.

Theorem 2. Let A be an EIM and α be a fixed real (complex) number. Then

$$(\alpha A)|^* = \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \alpha a_{k_1,l_1} & \dots & \alpha a_{k_1,l_j} & \dots & \alpha a_{k_1,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & \alpha a_{k_i,l_1} & \dots & \alpha a_{k_i,l_j} & \dots & \alpha a_{k_i,l_n} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & \alpha a_{k_m,l_1} & \dots & \alpha a_{k_m,l_j} & \dots & \alpha a_{k_m,l_n} \end{array} =$$

	$q_{k_1,l_1,1}$	$\dots$	$q_{k_1,l_1,s_{1,1}}$	$\dots$	$q_{k_m,l_n,1}$	$\dots$	$q_{k_m,l_n,s_{m,n}}$
$p_{k_1,l_1,1}$	$\alpha a_{k_1,l_1,1,1}$	$\dots$	$\alpha a_{k_1,l_1,1,s_{1,1}}$	$\dots$	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{k_1,l_1,r_{1,1}}$	$\alpha a_{k_1,l_1,r_{1,1},1}$	$\dots$	$\alpha a_{k_1,l_1,r_{1,1},s_{1,1}}$	$\dots$	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{k_m,l_n,1}$	0	$\dots$	0	$\dots$	$\alpha a_{k_m,l_n,1,1}$	$\dots$	$\alpha a_{k_m,l_n,1,s_{m,n}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p_{k_m,l_n,r_{m,n}}$	0	$\dots$	0	$\dots$	$\alpha a_{k_m,l_n,r_{m,n},1}$	$\dots$	$\alpha a_{k_m,l_n,r_{m,n},s_{m,n}}$

.

3 Conclusion

The hierarchical operators from the three types can be used for different aims. For example, in the near future they will be applied to cognitive maps, to extension of the IMs in InterCriteria Analysis (ICrA, see [3]) and others.

For example, the application to the ICrA is related to the possibility to have criteria consisting of multiple subcriteria (or subindicators) against which we also have the objects measured. It would be then useful to compare the results of ICrA as applied to the subcriteria and to the respective generalized criteria.

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