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A NOTE ON DEGENERATE STIRLING NUMBERS AND THEIR APPLICATIONS

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Abstract. In this paper, we introduce degenerate Stirling numbers of the first kind which are different from the Carlitz's type degenerate Stirling numbers of the first kind. Then we derive some identities involving these numbers and other degenerate special numbers like degenerate Stirling numbers of the second kind, degenerate Bernoulli numbers, degenerate Euler numbers, and degenerate Daehee numbers of the second kind.

1. Introduction

Stirling numbers are defined as the coefficients in an expansion of positive integral powers of a variable in terms of factorial powers, and vice versa:

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k,\ (n \geq 0), \tag{1.1}$$

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,\ (n \geq 0), \quad (\text{see}[1,2,3]). \tag{1.2}$$

where $(x)_n = x(x-1)\cdots(x-(n-1))$, $(n \geq 1)$, $(x)_0 = 1$, (see [3]). The numbers $S_1(n,k)$ and $S_2(n,k)$ are the Stirling numbers of the first kind and of the second kind.

The generating functions of (1.1) and (1.2) are respectively given by

$$\frac{1}{k!}\Big(\log(1+t)\Big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!},\ (k \geq 0), \tag{1.3}$$

and

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!},\ (k \geq 0). \tag{1.4}$$

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As is well known, the Bernoulli and Euler numbers are respectively defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}, \tag{1.5}$$

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}, \quad (\text{see } [2,3,9]). \tag{1.6}$$

The Daehee numbers are given by the generating function

$$\frac{\log(1+t)}{t}=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}, \quad (\text{see } [3,4,8,9]). \tag{1.7}$$

From (1.5) and (1.7), we note that

$$D_n=\sum_{k=0}^{n}S_1(n,k)B_k,\ (n\geq 0), \tag{1.8}$$

and

$$B_n=\sum_{k=0}^{n}S_2(n,k)D_k,\ (n\geq 0), \quad (\text{see } [3,4]). \tag{1.9}$$

For $\lambda\in\mathbb{R}$, L. Carlitz introduced the degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ and the degenerate Euler polynomials $\mathcal{E}_{n,\lambda}(x)$, $(n\geq 0)$, which are given by

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.10}$$

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \quad (\text{see } [1,2]). \tag{1.11}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x)=B_n(x)$, and $\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}(x)=E_n(x)$, where $B_n(x)$ and $E_n(x)$ are the Bernoulli and Euler polynomials.

When $x=0$, $\beta_{n,\lambda}=\beta_{n,\lambda}(0)$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}(0)$ are called the degenerate Bernoulli and Euler numbers.

Now, we define the degenerate exponential function as follows:

$$e_\lambda(t)=(1+\lambda t)^{\frac{1}{\lambda}}=\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!}, \quad (\text{see } [1,2,7]). \tag{1.12}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n\geq 1)$. Note that $\lim_{\lambda\to 0}e_\lambda(t)=e^t$.

Recently, the λ-Stirling numbers of the first kind are defined as

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{1,\lambda}(n,l)x^l, \ (n \geq 0). \tag{1.13}$$

Note that $\lim_{\lambda\to 1} S_{1,\lambda}(n,l) = S_1(n,l)$, $(n \geq l \geq 0)$.

In this paper, we introduce the degenerate Stirling numbers of the first kind which are different from Carlitz's type degenerate Stirling numbers of the first kind and λ-Stirling numbers of the first kind. Then we derive some identities involving these numbers and other degenerate special numbers like degenerate Stirling numbers of the second kind, degenerate Bernoulli numbers, degenerate Euler numbers, and degenerate Daehee numbers of the second kind.

2. The degenerate Stirling numbers of the first kind

Now, we define the degenerate logarithmic function as the inverse to the degenerate exponential function $e_\lambda(t)$ as follows:

$$\log_\lambda t = \frac{t^\lambda - 1}{\lambda}, \ (\lambda \neq 0). \tag{2.1}$$

From (1.12) and (2.1), we note that

$$e_\lambda(\log_\lambda t) = t = \log_\lambda(e_\lambda(t)), \ \lim_{\lambda\to 0} \log_\lambda t = \log t. \tag{2.2}$$

In the view of (1.3), for $n \geq k \geq 0$, we define the degenerate Stirling numbers of the first kind by

$$\frac{1}{k!}\Big(\log_\lambda(1+t)\Big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.2), we easily get

$$\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k), \ (n \geq k \geq 0). \tag{2.4}$$

where $S_1(n,k)$ are the Stirling numbers of the first kind.

From (2.1), (1.3), and (1.4), we note that

$$\begin{aligned}
\frac{1}{k!}\Big(\log_\lambda(1+t)\Big)^k &= \frac{\lambda^{-k}}{k!}\Big((1+t)^\lambda - 1\Big)^k = \frac{\lambda^{-k}}{k!}\Big(e^{\lambda\log(1+t)} - 1\Big)^k \\
&= \lambda^{-k}\sum_{l=k}^{\infty} S_2(l,k)\lambda^l \frac{1}{l!}\Big(\log(1+t)\Big)^l \\
&= \lambda^{-k}\sum_{l=k}^{\infty} S_2(l,k)\lambda^l \sum_{n=l}^{\infty} S_1(n,l)\frac{t^n}{n!} \\
&= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n} \lambda^{l-k} S_2(l,k)S_1(n,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

Therefore, by (2.3) and (2.5), we obtain the following lemma.

Theorem 2.1. *For $n, k \geq 0$ with $n \geq k$, we have*

$$S_{1,\lambda}(n,k) = \sum_{l=k}^{n} \lambda^{l-k} S_2(l,k) S_1(n,l).$$

Now, we observe that

$$x^n = \Big(e_\lambda\big(\log(1+x)\big) - 1\Big)^n, \ (n \geq 0). \tag{2.6}$$

In [5], the degenerate Stirling numbers of the second kind were defined by the generating function

$$\frac{1}{k!}\Big(e_\lambda(t) - 1\Big)^k = \frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}} - 1\Big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \ (k \geq 0). \tag{2.7}$$

Thus, we note that

$$\begin{aligned}
\Big(e_\lambda\big(\log_\lambda(1+x)\big) - 1\Big)^n &= n!\frac{1}{n!}\Big(e_\lambda\big(\log_\lambda(1+x)\big) - 1\Big)^n \\
&= n!\sum_{k=n}^{\infty} S_{2,\lambda}(k,n)\frac{1}{k!}\Big(\log_\lambda(1+x)\Big)^k \\
&= n!\sum_{k=n}^{\infty} S_{2,\lambda}(k,n)\sum_{m=k}^{\infty} S_{1,\lambda}(m,k)\frac{x^m}{m!} \\
&= n!\sum_{m=n}^{\infty}\left(\sum_{k=n}^{m} S_{2,\lambda}(k,n)S_{1,\lambda}(m,k)\right)\frac{x^m}{m!}.
\end{aligned} \tag{2.8}$$

Therefore, by (2.6) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $m, n \in \mathbb{N}$, we have*

$$\sum_{k=n}^{m} S_{2,\lambda}(k,n)S_{1,\lambda}(m,k) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m > n. \end{cases}$$

From (1.10), we have

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}\frac{t^n}{n!} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} = \frac{\log_\lambda\Big(e_\lambda(t) - 1 + 1\Big)}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}. \tag{2.9}$$

By (2.1), we easily get

$$\begin{aligned}\log_\lambda(1+t)&=\frac{1}{\lambda}\Big((1+t)^\lambda-1\Big)=\frac{1}{\lambda}\left(\sum_{l=0}^{\infty}(\lambda)_l\frac{t^l}{l!}-1\right)\\&=\frac{1}{\lambda}\sum_{l=1}^{\infty}(\lambda)_l\frac{t^l}{l!.}\end{aligned}\tag{2.10}$$

Thus, by (2.10), we get

$$\begin{aligned}\frac{\log_\lambda(e_\lambda(t)-1+1)}{(1+t)^{\frac{1}{\lambda}}-1}&=\frac{1}{\lambda\Big((1+\lambda t)^{\frac{1}{\lambda}}-1\Big)}\sum_{l=1}^{\infty}(\lambda)_l\frac{\big(e_\lambda(t)-1\big)^l}{l!}\\&=\frac{1}{\lambda}\sum_{l=0}^{\infty}\frac{(\lambda)_{l+1}}{l+1}\frac{1}{l!}\Big((1+\lambda t)^{\frac{1}{\lambda}}-1\Big)^l\\&=\frac{1}{\lambda}\sum_{l=0}^{\infty}\frac{(\lambda)_{l+1}}{l+1}\sum_{n=l}^{\infty}S_{2,\lambda}(n,l)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}\sum_{l=0}^{n}\frac{(\lambda)_{l+1}}{l+1}S_{2,\lambda}(n,l)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.11}$$

Therefore, by (1.10) and (2.11), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, we have*

$$\beta_{n,\lambda}=\frac{1}{\lambda}\sum_{l=0}^{n}\frac{(\lambda)_{l+1}}{l+1}S_{2,\lambda}(n,l).$$

From (1.11), we note that

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}\frac{t^n}{n!}=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\frac{2}{e^{\frac{1}{\lambda}\log(1+\lambda t)}+1}.\tag{2.12}$$

It is not difficult to show that

$$\frac{1}{e^{\log(1+\lambda t)^{\frac{1}{\lambda}}}+1}=\frac{1}{2}\sum_{m=0}^{\infty}W_m\Big(-\frac{1}{2}\Big)\frac{1}{m!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m,\tag{2.13}$$

where $W_m(x)=\sum_{n=0}^{m}S_2(m,n)n!x^n$.

By (2.12) and (2.13), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}\frac{t^n}{n!} &= \sum_{m=0}^{\infty}W_m\Big(-\frac{1}{2}\Big)\frac{1}{m!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \\ &= \sum_{m=0}^{\infty}W_m\Big(-\frac{1}{2}\Big)\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n}{n!}t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}W_m\Big(-\frac{1}{2}\Big)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}\end{aligned} \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,\lambda} = \sum_{m=0}^{n}W_m\Big(-\frac{1}{2}\Big)\lambda^{n-m}S_1(n,m).$$

From (2.7), we have

$$\begin{aligned}\sum_{m=0}^{\infty}(n)_{m,\lambda}\frac{t^m}{m!} &= (1+\lambda t)^{\frac{n}{\lambda}} = \Big((1+\lambda t)^{\frac{1}{\lambda}}-1+1\Big)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}}-1\Big)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}k!\sum_{m=k}^{\infty}S_{2,\lambda}(m,k)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}k!S_{2,\lambda}(m,k)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.15}$$

Thus, by (2.15), we get

$$(n)_{m,\lambda} = \sum_{k=0}^{n}\binom{n}{k}k!S_{2,\lambda}(m,k), \ \ (n,m \geq 0). \tag{2.16}$$

From (2.16), we note that

$$\begin{aligned}(1)_{m,\lambda}+(2)_{m,\lambda}+\cdots+(n)_{m,\lambda} &= \sum_{j=0}^{n}(j)_{m,\lambda} \\ &= \sum_{j=0}^{n}\left(\sum_{k=0}^{j}\binom{j}{k}k!S_{2,\lambda}(m,k)\right) \\ &= \sum_{k=0}^{n}S_{2,\lambda}(m,k)k!\sum_{j=k}^{n}\binom{j}{k}.\end{aligned} \tag{2.17}$$

It is easy to show that

$$\sum_{j=k}^{n}\binom{j}{k}=\sum_{j=k}^{n}\left(\binom{j+1}{k+1}-\binom{j}{k+1}\right)=\binom{n+1}{k+1}. \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$(1)_{m,\lambda}+(2)_{m,\lambda}+\cdots+(n)_{m,\lambda}=\sum_{k=0}^{n}\binom{n+1}{k+1}S_{2,\lambda}(m,k)k!.$$

Now, we define the degenerate Daehee numbers of the second kind by

$$\frac{\log_\lambda(1+t)}{t}=\sum_{n=0}^{\infty}d_{n,\lambda}\frac{t^n}{n!}. \tag{2.19}$$

By replacing t by $(1+\lambda t)^{\frac{1}{\lambda}}-1$, we get

$$\begin{aligned}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}&=\sum_{k=0}^{\infty}d_{k,\lambda}\frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}}-1\Big)^k\\&=\sum_{k=0}^{\infty}d_{k,\lambda}\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}d_{k,\lambda}S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Therefore, by (1.10) and (2.20), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$\beta_{n,\lambda}=\sum_{k=0}^{n}d_{k,\lambda}S_{2,\lambda}(n,k).$$

From (1.10), we have

$$\begin{aligned}\frac{\log_\lambda(1+t)}{t}&=\sum_{k=0}^{\infty}\beta_{k,\lambda}\frac{1}{k!}\Big(\log_\lambda(1+t)\Big)^k\\&=\sum_{k=0}^{\infty}\beta_{k,\lambda}\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_{1,\lambda}(n,k)\beta_{k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.21}$$

Therefore, by (2.19) and (2.21), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$d_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)\beta_{k,\lambda}.$$

Remark. It is well known that the Changhee numbers are defined by the generating function

$$\frac{2}{t+2} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}. \tag{2.22}$$

From (1.6) and (2.22), we can easily derive the following equations:

$$Ch_n = \sum_{k=0}^{n} S_1(n,k)E_k, \ (n \geq 0),$$

and

$$E_n = \sum_{k=0}^{n} S_2(n,k)Ch_k, \ (n \geq 0).$$

Now, we observe that

$$\begin{aligned}\frac{2}{t+2} &= \frac{2}{e_\lambda\big(\log_\lambda(1+t)\big)+1} = \sum_{k=0}^{\infty} \mathcal{E}_{k,\lambda}\frac{1}{k!}\Big(\log_\lambda(1+t)\Big)^k \\ &= \sum_{k=0}^{\infty} \mathcal{E}_{k,\lambda}\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \mathcal{E}_{k,\lambda}S_{1,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

From (2.22), we note that

$$\begin{aligned}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} &= \sum_{k=0}^{\infty} Ch_k \frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}}-1\Big)^k \\ &= \sum_{k=0}^{\infty} Ch_k \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} Ch_k S_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

Thus we have

$$\mathcal{E}_{n,\lambda} = \sum_{k=0}^{n} Ch_k S_{2,\lambda}(n,k), \ (n \geq 0).$$

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