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ON THE EXTENSION OF DEGENERATE STIRLING POLYNOMIALS OF THE SECOND KIND AND DEGENERATE BELL POLYNOMIALS

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Abstract. Recently, the degenerate stirling numbers and polynomials of the second kind and the degenerate Bell polynomials are introduced in [5]. In this paper, we study the extension of the degenerate stirling numbers and polynomials of the second kind and degenerate Bell polynomials. In addition, we will give some identities and relations for these numbers and polynomials.

1. Introduction

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined as

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t) = e_\lambda^1(t), \quad (\text{see } [1,2,9]). \tag{1.1}$$

Note that $\lim_{\lambda \longrightarrow 0} e_\lambda(t) = \lim_{\lambda \longrightarrow 0}(1+\lambda t)^{\frac{1}{\lambda}} = e^t$.
Let $log_\lambda(t)$ be the inverse function of $e_\lambda(t)$ which is given by

$$log_\lambda t = \frac{1}{\lambda}(t^\lambda - 1), \quad (\text{see } [7]). \tag{1.2}$$

From (1.1) and (1.2), we note that

$$log_\lambda\big(e_\lambda(t)\big) = e_\lambda\big(log_\lambda(t)\big) = t. \tag{1.3}$$

The degenerate stirling number of the first kind is defined by the generating function to be

$$\frac{1}{k!}\big(log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \quad (\text{see } [7]), \tag{1.4}$$

where $k \in \{0,1,2,3,\cdots\}$.

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In the viewpoint of inversion formula of (1.4), the degenerate stirling number of the second kind is defined by the generating function to be

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\quad \text{(see [6])}. \tag{1.5}$$

In [6], we note that the degenerate stirling polynomials of the second kind are given by the generating function to be

$$\begin{aligned}\frac{1}{k!}\big(e_\lambda(t)-1\big)^k e_\lambda^x(t)&=\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k(1+\lambda t)^{\frac{x}{\lambda}}\\&=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k|x)\frac{t^n}{n!}.\end{aligned} \tag{1.6}$$

When $x=0$, $S_{2,\lambda}(n,k)=S_{2,\lambda}(n,k|0)$, $(n,k\geq 0)$.

From (1.6), we easily derive the following equation.

$$(x)_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k)(x)_k,\quad (see[6,9]), \tag{1.7}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda),(n\geq 1)$.
Note that $\lim_{\lambda\longrightarrow 1}(x)_{n,\lambda}=(x)_n=x(x-1)\cdots(x-n+1)$, $(see[8])$.

In [4], the degenerate Bell polynomials are defined by the generating function to be

$$e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}=e^{x\left(e_\lambda^x(t)-1\right)}=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.8}$$

When $x=1$, $B_{n,\lambda}=B_{n,\lambda}(1)$ are called the degenerate Bell numbers.
By (1.5) and (1.8), we easily get

$$B_{n,\lambda}(x)=\sum_{k=0}^{n}x^kS_{2,\lambda}(n,k),\quad (n\geq 0),\quad \text{(see [4])}. \tag{1.9}$$

In particular,

$$B_{n,\lambda}=\sum_{k=0}^{n}S_{2,\lambda}(n,k),\quad (n\geq 0). \tag{1.10}$$

As is well known, the stirling polynomials of the second kind are defined as

$$\frac{1}{k!}(e^t-1)^ke^{xt}=\sum_{n=k}^{\infty}S_2(n,k|x)\frac{t^n}{n!},\quad \text{(see [5, 10, 11])}. \tag{1.11}$$

Note that

$$\begin{aligned}\frac{1}{k!}(e^t-1)^k e^{xt} &= \lim_{\lambda\longrightarrow 0}\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k(1+\lambda t)^{\frac{x}{\lambda}}\\ &= \sum_{n=k}^{\infty}\lim_{\lambda\longrightarrow 0}S_{2,\lambda}(n,k|x)\frac{t^n}{n!},\quad (k\geq 0).\end{aligned} \tag{1.12}$$

By (1.11) and (1.12), we get

$$\lim_{\lambda\longrightarrow 0}S_{2,\lambda}(n,k|x)=S_2(n,k|x),\quad (n,k\geq 0). \tag{1.13}$$

For $r\in\mathbb{N}\cup\{0\}$, the extended stirling polynomials of the second kind are defined by the generating function to be

$$\frac{1}{k!}e^{xt}(e^t-1+rt)^k=\sum_{n=k}^{\infty}S_2^{(r)}(n,k|x)\frac{t^n}{n!},\quad \text{(see [5])}. \tag{1.14}$$

When $x=0$, $S_2^{(r)}(n,k)=S_2^{(r)}(n,k|0)$ are called the extended stirling numbers of the second kind.

In [5], the extended Bell polynomials are also defined by the generating function to be

$$e^{x(e^t-1+rt)}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.15}$$

When $x=1$, $B_n^{(r)}=B_n^{(r)}(1)$ are called the extended Bell numbers. From (1.14) and (1.15), we note that

$$B_n^{(r)}(x)=\sum_{m=0}^{n}x^m S_2^{(r)}(n,m),\quad (n\geq 0),\quad \text{(see [5])}. \tag{1.16}$$

It is not difficult to show that $\lim_{r\longrightarrow 0}B_n^{(r)}(x)=B_n(x)$ where $B_n(x)$ are ordinary Bell polynomials which are defined by the generating function to be

$$e^{x(e^t-1)}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \text{(see [8, 10, 11])}. \tag{1.17}$$

Note that $\lim_{r\longrightarrow 0}B_n^{(r)}(x)=B_n(x),\quad (n\geq 0)$.

For $n,k\geq 0$, it is well known that the associated stirling number of the first kind is defined by the generating function to be

$$\frac{1}{k!}\big(log(1+t)-t\big)^k=\sum_{n=2k}^{\infty}(-1)^{n-k}S^*(n,k)\frac{t^n}{n!}\quad \text{(see [3, 11])}. \tag{1.18}$$

In [7], the degenerate associated stirling number of the first kind is given by

$$\frac{1}{k!}\big(log_{\lambda}(1+t)-t\big)^{k}=\sum_{n=2k}^{\infty}(-1)^{n-k}S_{\lambda}^{*}(n,k)\frac{t^{n}}{n!}. \tag{1.19}$$

Note that $\lim_{\lambda\longrightarrow 0}S_{\lambda}^{*}(n,k)=S^{*}(n,k),\quad (n,k\geq 0)$.
In this paper, we study the extension of the degenerate stirling numbers and polynomials of the second kind and the degenerate Bell polynomials.
In addition, we give some formulae for these numbers and polynomials which are derived from the generating functions.

2. extension of degenerate stirling polynomials and degenerate Bell polynomials

In this section, we assume that r is integer. For $k\geq 0$, we define the degenerate stirling-like polynomials of the second kind which are given by the generating function to be

$$\begin{aligned}\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^{k}(1+\lambda t)^{\frac{x}{\lambda}}&=\frac{1}{k!}\big(e_{\lambda}(t)-1+rt\big)e_{\lambda}^{x}(t)\\&=\sum_{n=k}^{\infty}S_{2,\lambda}^{(r)}(n,k|x)\frac{t^{n}}{n!}.\end{aligned} \tag{2.1}$$

Note that $S_{2,\lambda}^{(r)}(n,k|x)$ is the extension of $S_{2,\lambda}(n,k|x)$. It is easy to show that $\lim_{r\longrightarrow 0}S_{2,\lambda}^{(r)}(n,k|x)=S_{2,\lambda}(n,k|x)$. When $x=0$, $S_{2,\lambda}^{(r)}(n,k)=S_{2,\lambda}^{(r)}(n,k|0)$ are called the degenerate stirling-like numbers of the second kind.

From (2.1), we note that

$$\begin{aligned}
\frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\Big)^k &= \frac{1}{k!}\Big(e_\lambda(t)-1+e_\lambda\big(log_\lambda(rt+1)\big)-1\Big)^k\\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\big(e_\lambda(t)-1\big)^{k-l}\Big(e_\lambda\big(log_\lambda(rt+1)\big)-1\Big)^l\\
&=\sum_{l=0}^{k}\Big(\frac{1}{(k-l)!}\big(e_\lambda(t)-1\big)^{k-l}\Big)\Big(\frac{1}{l!}\big(e_\lambda(log_\lambda(rt+1))-1\big)^l\Big)\\
&=\sum_{l=0}^{k}\Big\{\Big(\sum_{m=k-l}^{\infty}S_{2,\lambda}(m,k-l)\frac{t^m}{m!}\Big)\Big(\sum_{j=l}^{\infty}S_{2,\lambda}(j,l)\frac{1}{j!}\\
&\quad\times\big(log_\lambda(rt+1)\big)^j\Big)\Big\}\\
&=\sum_{l=0}^{k}\Big\{\Big(\sum_{m=k-l}^{\infty}S_{2,\lambda}(m,k-l)\frac{t^m}{m!}\Big)\Big(\sum_{j=l}^{\infty}S_{2,\lambda}(j,l)\sum_{i=j}^{\infty}\\
&\quad\times S_{1,\lambda}(i,j)r^i\frac{t^i}{i!}\Big)\Big\}\\
&=\sum_{l=0}^{k}\Big\{\sum_{n=k}^{\infty}\Big(\sum_{i=l}^{n-k+l}\sum_{j=l}^{i}S_{2,\lambda}(j,l)S_{1,\lambda}(i,j)S_{2,\lambda}(n-i,k-l)\\
&\quad\times\binom{n}{i}r^i\Big)\frac{t^n}{n!}\Big\}\\
&=\sum_{n=k}^{\infty}\Big\{\sum_{l=0}^{k}\sum_{i=l}^{n-k+l}\sum_{j=l}^{i}S_{2,\lambda}(j,l)S_{1,\lambda}(i,j)S_{2,\lambda}(n-i,k-l)\\
&\quad\times\binom{n}{i}r^i\Big\}\frac{t^n}{n!}.
\end{aligned}\tag{2.2}$$

On the other hand,

$$\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^k=\sum_{n=k}^{\infty}S_{2,\lambda}^{(r)}(n,k)\frac{t^n}{n!}.\tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n,k\geq 0$ with $n\geq k$, we have*

$$S_{2,\lambda}^{(r)}(n,k)=\sum_{l=0}^{k}\sum_{i=l}^{n-k+l}\sum_{j=l}^{i}S_{2,\lambda}(j,l)S_{1,\lambda}(i,j)S_{2,\lambda}(n-i,k-l)\binom{n}{i}r^i.$$

Note that $\lim_{\lambda \longrightarrow 0} S_{2,\lambda}^{(r)}(n,k) = S_{2,\lambda}(n,k), \quad (n,k \geq 0)$.

By (2.1), we get

$$\begin{aligned}\frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\Big)^k (1+\lambda t)^{\frac{x}{\lambda}} &= \Big(\sum_{m=k}^{\infty} S_{2,\lambda}^{(r)}(m,k)\frac{t^m}{m!}\Big)\Big(\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\Big) \\ &= \sum_{n=k}^{\infty}\Big(\sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}^{(r)}(m,k)(x)_{n-m,\lambda}\Big) \\ &\times \frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

On the other hand,

$$\frac{1}{k!}\Big((1+\lambda t)^{\frac{1}{\lambda}} - 1\Big)^k (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=k}^{\infty} S_{2,\lambda}^{(r)}(n,k|x)\frac{t^n}{n!}. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following theorem.

Theorem 2.2. *For $n,k \geq 0$ with $n \geq k$, we have*

$$S_{2,\lambda}(n,k|x) = \sum_{m=k}^{n}\binom{n}{m}S_{2,\lambda}^{(r)}(m,k)(x)_{n-m,\lambda}.$$

Now, we define the degenerate Bell-like polynomials which are given by the generating function to be

$$e^{x\left(e_\lambda(t)-1+rt\right)} = e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\right)} = \sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.6}$$

Note that $B_{n,\lambda}^{(r)}(x)$ is the extension of $B_{n,\lambda}(x), \quad (n \geq 0)$.
Indeed, $\lim_{r \longrightarrow 0} B_{n,\lambda}^{(r)}(x) = B_{n,\lambda}(x), \quad (n \geq 0)$.
When $x = 1$, $B_{n,\lambda}^{(r)} = B_{n,\lambda}^{(r)}(1)$ are called the degenerate Bell-like numbers, $(n \geq 0)$.

From (2.1) and (2.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} &= e^{x\left((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\right)} = \sum_{m=0}^{\infty} x^m \frac{1}{m!}\Big((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\Big)^m \\ &= \sum_{m=0}^{\infty} x^m \sum_{n=m}^{\infty} S_{2,\lambda}^{(r)}(n,m)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n} x^m S_{2,\lambda}^{(r)}(n,m)\Big) \\ &\times \frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Therefore, by comparing the coefficients on the both sides of (2.7), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$B_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} x^m S_{2,\lambda}^{(r)}(n,m).$$

In particular,

$$B_{n,\lambda}^{(r)} = \sum_{m=0}^{n} S_{2,\lambda}^{(r)}(n,m).$$

For $r = -1$, by replacing t by $\log_\lambda(1+t)$, we get

$$\begin{aligned}\sum_{m=k}^{\infty} S_{2,\lambda}^{(-1)}(m,k)\frac{1}{m!}\big(\log_\lambda(1+t)\big)^m &= \frac{1}{k!}\left(e_\lambda\big(\log_\lambda(1+t)\big) - 1 - log_\lambda(1+t)\right)^k \\ &= \frac{(-1)^k}{k!}\big(\log_\lambda(1+t) - t\big)^k = \sum_{n=2k}^{\infty}(-1)^n S_\lambda^*(n,k)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

On the other hand,

$$\begin{aligned}\sum_{m=k}^{\infty} S_{2,\lambda}^{(-1)}(m,k)\frac{1}{m!}\big(\log_\lambda(1+t)\big)^m &= \sum_{m=k}^{\infty} S_{2,\lambda}^{(-1)}(m,k)\sum_{n=m}^{\infty} S_{1,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n} S_{2,\lambda}^{(-1)}(m,k)S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.4. *For $n, k \geq 0$, we have*

$$\sum_{m=k}^{n} S_{2,\lambda}^{(-1)}(m,k)S_{1,\lambda}(n,m) = \begin{cases}(-1)^n S_\lambda^*(n,k), & \text{if } n \geq 2k,\\ 0, & \text{otherwise.}\end{cases}$$

Now, we observe that

$$\begin{aligned}\sum_{n=k}^{\infty} S_{2,\lambda}^{(r)}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\big)^k = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l} r^l t^l \big((1+\lambda t)^{\frac{1}{\lambda}} - 1\big)^{k-l} \\ &= \sum_{l=0}^{k}\frac{r^l}{l!}t^l\frac{1}{(k-l)!}\big((1+\lambda t)^{\frac{1}{\lambda}} - 1\big)^{k-l} = \sum_{l=0}^{k}\frac{r^l}{l!}\sum_{n=k}^{\infty} S_{2,\lambda}(n-l,k-l)\frac{t^n}{(n-l)!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{k}\binom{n}{l} S_{2,\lambda}(n-l,k-l)r^l\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

By (2.6), we easily get

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}^{(r)} \frac{t^n}{n!} &= e^{(1+\lambda t)^{\frac{1}{\lambda}}-1+rt} = e^{(1+\lambda t)^{\frac{1}{\lambda}}-1} \cdot e^{rt} \\ &= \left(\sum_{m=0}^{\infty} B_{m,\lambda} \frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} r^l \frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} B_{m,\lambda} r^{n-m}\right)\frac{t^n}{n!,}.\end{aligned} \tag{2.11}$$

Therefore, by Theorem 3, (2.10) and (2.11), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$ we have*

$$B_{n,\lambda}^{(r)} = \sum_{m=0}^{n}\binom{n}{m} B_{m,\lambda} r^{n-m} = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{l} S_{2,\lambda}(n-l, m-l) r^l.$$

From (1.6), we note that

$$\begin{aligned}\sum_{n=k}^{\infty} S_{2,\lambda}(n,k|x)\frac{t^n}{n!} &= \frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1+rt-rt\right)^k (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{l=0}^{k}\sum_{n=k}^{\infty}(-r)^l\binom{n}{l} S_{2,\lambda}^{(r)}(n-l, k-l|x)\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{k}\binom{n}{l}(-1)^l r^l S_{2,\lambda}^{(r)}(n-l, k-l|x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

Therefore, by comparing the coefficients on the both sides of (2.12), we obtain the following theorem.

Theorem 2.6. *For $n, k \geq 0$ with $n \geq k$, we have*

$$S_{2,\lambda}(n,k|x) = \sum_{l=0}^{k}\binom{n}{l}(-1)^l r^l S_{2,\lambda}^{(r)}(n-l, k-l|x).$$

Now, we consider the inversion formula of Theorem 6. By (2.1), we get

$$\begin{aligned}\sum_{n=k}^{\infty} S_{2,\lambda}^{(r)}(n,k|x)\frac{t^n}{n!} &= \frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^k(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{l=0}^{k}\frac{r^l}{l!}t^l\frac{1}{(k-l)!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^{k-l}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{l=0}^{k}\frac{r^l}{l!}\sum_{n=k}^{\infty}S_{2,\lambda}(n-l,k-l|x)\frac{n!}{(n-l)!}\frac{t^n}{n!} \\ &= \sum_{n=k}^{\infty}\left(\sum_{l=0}^{k}\binom{n}{l}r^l S_{2,\lambda}(n-l,k-l|x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

Therefore, by comparing the coefficients on the both sides of (2.13), we obtain the following theorem.

Theorem 2.7. *For $n,k\geq 0$ with $n\geq k$, we have*

$$S_{2,\lambda}^{(r)}(n,k|x)=\sum_{l=0}^{k}\binom{n}{l}r^l S_{2,\lambda}(n-l,k-l|x).$$

In particular,

$$S_{2,\lambda}^{(r)}(n,k)=\sum_{l=0}^{k}\binom{n}{l}r^l S_{2,\lambda}(n-l,k-l).$$

For $m,n,k\geq 0$ with $n\geq k+m$, we have

$$\begin{aligned}&\frac{1}{m!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^m\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^k(1+\lambda t)^{\frac{x}{\lambda}} \\ &=\frac{(m+k)!}{m!k!}\frac{1}{(m+k)!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^{m+k}(1+\lambda t)^{\frac{x}{\lambda}} \\ &=\binom{m+k}{m}\sum_{n=m+k}^{\infty}S_{2,\lambda}^{(r)}(n,m+k|x)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

On the other hand,

$$\begin{aligned}&\frac{1}{m!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^m\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^k(1+\lambda t)^{\frac{x}{\lambda}} \\ &=\left(\sum_{l=m}^{\infty}S_{2,\lambda}^{(r)}(l,m)\frac{t^l}{l!}\right)\left(\sum_{j=k}^{\infty}S_{2,\lambda}^{(r)}(j,k|x)\frac{t^j}{j!}\right) \\ &=\sum_{n=k+m}^{\infty}\left(\sum_{l=m}^{n-k}\binom{n}{l}S_{2,\lambda}^{(r)}(l,m)S_{2,\lambda}^{(r)}(n-l,k|x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.8. *For $m, n, k \geq 0$ with $n \geq m+k$, we have*

$$\binom{m+k}{m} S_{2,\lambda}^{(r)}(n, m+k|x) = \sum_{l=m}^{n-k} \binom{n}{l} S_{2,\lambda}^{(r)}(l, m) S_{2,\lambda}^{(r)}(n-l, k|x).$$

Remark. We observe that

$$\begin{aligned}
&\frac{1}{m!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\right)^m \frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\right)^k \\
&= \left(\frac{1}{m!}\sum_{l=0}^{m}\binom{m}{l}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^{m-l} r^l t^l\right)\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^{k-j} r^j t^j\right) \\
&= \left(\sum_{n_1=m}^{\infty}\left(\sum_{l=0}^{m}\binom{n_1}{l} r^l S_{2,\lambda}(n_1 - l, m-l)\right)\frac{t^{n_1}}{n_1!}\right) \\
&\quad \times \left(\sum_{n_2=k}^{\infty}\left(\sum_{j=0}^{k}\binom{n_2}{j} r^j S_{2,\lambda}(n_2 - j, k-j)\right)\frac{t^{n_2}}{n_2!}\right) \\
&= \sum_{n=m+k}^{\infty}\left\{\sum_{n_1=m}^{n-k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{n_1}{l}\binom{n-n_1}{j} r^{l+j}\binom{n}{n_1}\right. \\
&\quad \left. \times S_{2,\lambda}(n_1 - l, m-l) S_{2,\lambda}(n - n_1 - j, k-j)\right\}\frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

It is easy to show that

$$\begin{aligned}
&\frac{1}{m!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\right)^m \frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + rt\right)^k \\
&= \binom{m+k}{m}\sum_{n=m+k}^{\infty} S_{2,\lambda}^{(r)}(n, m+k)\frac{t^n}{n!}.
\end{aligned} \tag{2.17}$$

By (2.16) and (2.17), we get

$$\begin{aligned}
&\binom{m+k}{m} S_{2,\lambda}^{(r)}(n, m+k|x) \\
&= \sum_{n_1=m}^{n-k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{n_1}{l}\binom{n-n_1}{j} r^{l+j}\binom{n}{n_1} S_{2,\lambda}(n_1 - l, m-l) S_{2,\lambda}(n - n_1 - j, k-j),
\end{aligned} \tag{2.18}$$

where $m, n, k \geq 0$ with $n \geq m+k$.

Now, we consider the inversion formula of (2.18). From (1.6), we note that

$$\begin{aligned}
&\frac{1}{m!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^m\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k\\
&=\frac{1}{m!}\big((1+\lambda t)^{\frac{1}{\lambda}}+rt-1-rt\big)^m\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt-rt\big)^k\\
&=\left(\frac{1}{m!}\sum_{l=0}^{m}\binom{m}{l}(-rt)^l\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^{m-l}\right)\\
&\quad\times\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\big((1+\lambda t)^{\frac{1}{\lambda}}-1+rt\big)^{k-j}(-rt)^j\right)\\
&=\left(\sum_{n_1=m}^{\infty}\sum_{l=0}^{m}\binom{n_1}{l}(-1)^l r^l S_{2,\lambda}^{(r)}(n_1-l,m-l)\frac{t^{n_1}}{n_1!}\right)\\
&\quad\times\left(\sum_{n_2=k}^{\infty}\sum_{j=0}^{k}\binom{n_2}{j}(-1)^j r^j S_{2,\lambda}^{(r)}(n_2-j,k-j)\frac{t^{n_2}}{n_2!}\right)\\
&=\sum_{n=m+k}^{\infty}\Bigg\{\sum_{n_1=m}^{n-k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{n_1}{l}\binom{n-n_1}{j}\binom{n}{n_1}(-1)^{l+j}r^{l+j}\\
&\quad\times S_{2,\lambda}^{(r)}(n_1-l,m-l)S_{2,\lambda}^{(r)}(n-n_1-j,k-j)\Bigg\}\frac{t^n}{n!}.
\end{aligned}\tag{2.19}$$

On the other hand

$$\begin{aligned}
&\frac{1}{m!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^m\frac{1}{k!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^k\\
&=\frac{(m+k)!}{m!k!}\frac{1}{(m+k)!}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^{m+k}\\
&=\binom{m+k}{m}\sum_{n=m+k}^{\infty}S_{2,\lambda}(n,m+k)\frac{t^n}{n!}.
\end{aligned}\tag{2.20}$$

Therefore, by (2.19) and (2.20), we get

$$\begin{aligned}
&\binom{m+k}{m}S_{2,\lambda}(n,m+k)\\
&=\sum_{n_1=m}^{n-k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{n_1}{l}\binom{n-n_1}{j}\binom{n}{n_1}(-1)^{l+j}r^{l+j}\\
&\quad\times S_{2,\lambda}^{(r)}(n_1-l,m-l)S_{2,\lambda}^{(r)}(n-n_1-j,k-j),
\end{aligned}$$

where $n,m,k\geq 0$ with $n\geq m+k$.

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