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A REMARK CONCERNING THE INHOMOGENEITY IN KATO'S 1953 THEOREM FOR ABSTRACT EVOLUTION EQUATIONS

A. I. Shtern

Abstract. We improve membership relations in the famous Tosio Kato theorem concerning strong solutions of abstract evolution equations in Banach spaces.

§ 1. Introduction

In 1953, Kato [1] proved his famous theorem concerning the solution of the abstract evolution equation in a Banach space E,

$$dx(t)/dt = A(t)x(t) + f(t), \qquad t \in [a, b], \quad x\colon [a, b] \to E, \tag{1}$$

with the condition $x(a) = x_0 \in E$. The homogeneous equation ($f \equiv 0$) was solved under the following conditions: every operator $A(t)$, $t \in [a, b]$, is an infinitesimal generator of a strongly continuous semigroup $T_t(s)$, $s \in [0, +\infty)$, these operators have the same dense domain $D \subset E$, and there are real numbers μ and $c \geq 1$ such that

$$\|T_{t_1}(s_1) \cdots T_{t_n}(s_n)\| \leq c \exp(\mu \sum_{k=1}^{n} s_k)$$

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for any

$$t_1, \dots, t_n \in [a,b], \quad t_1 \le \dots \le t_n, \quad s_1, \dots, s_n \in [0,+\infty),$$

and for any positive integer n, the operator-valued function

$$A(t)A(s)^{-1}, \qquad (t,s) \in [a,b] \times [a,b],$$

is weakly differentiable with respect to $t \in [a,b]$ for any $s \in [a,b]$ and the derivative is strongly continuous with respect to $t \in [a,b]$ for some $s \in [a,b]$. Kato [1] proved the existence of the corresponding evolution operator, i.e., an operator-valued function $U(t,s)$, $a \le s \le t \le b$, satisfying the following conditions:

(A) every value $U(t,s)$ of the function U is a bounded linear operator on E for any t, s, $a \le s \le t \le b$,

(B) the function $(t,s) \mapsto U(t,s)$ is jointly strongly continuous, and

$$U(t,t) = I (= I_E) \quad \text{for any} \quad t \in [a,b],$$

(C) $U(t,r)U(r,s) = U(t,s)$ for any $a \le s \le r \le t \le b$,

(D) for any $x \in D$, the function x given by the rule $x(t) = U(t,a)x$, $t \in [a,b]$, is strongly differentiable,

$$dx(t)/dt = A(t)x(t) \quad \text{for any} \quad t \in [a,b],$$

and the function $t \mapsto A(t)x(t)$, $t \in [a,b]$, is strongly continuous on $[a,b]$,

(E) the function $(t,s) \mapsto A(t)U(t,s)A(s)^{-1}$ is jointly strongly continuous for any (t,s), $a \le s \le t \le b$.

§ 2. Preliminaries

Kato [1] proved also that, if f is strongly continuous, $f(s) \in D$ for every $s \in [a,b]$, and $A(r)f(s)$ is strongly continuous with respect to $s \in [a,b]$ for some $r \in [a,b]$, then the expression

$$x(t) = U(t,a)x_0 + \int_a^t U(t,s)f(s)\,ds \tag{2}$$

is well defined, the right-hand side of (2) belongs to D for every $t \in [a,b]$, is strongly continuously differentiable with respect to $t \in [a,b]$, and gives a strong solution of equation (1) with the condition $x(a) = x_0 \in D$. Moreover, $A(t)x(t)$, $t \in [a,b]$, is strongly continuous.

The above assumptions concerning the family of operators $\{A(t), t \in [a,b]\}$ are called below the Kato assumptions. We use below the terminology and notation of [1].

§ 3. Main results

As was proved in [2], an analog of the last cited result of Kato remains valid if f is representable as an indefinite integral of a E-valued integrable function. Since the publication [2] contained a lot of misprints, we represent here the proof of the corresponding result. We also prove a theorem weakening the conditions of the last cited result of Kato (concerning a D-valued right-hand side of the inhomogeneous equation (1)).

Lemma 3. *Let the assumptions of Theorem 1 hold, let U be the corresponding evolution operator, and let $f\colon [a,b] \to E$ be a strongly continuously differentiable function. Then the function $u\colon [a,b] \to E$ given by the rule*

$$u(t) = \int_a^t U(t,r)f(r)\,dr, \qquad t \in [a,b], \tag{3}$$

takes the values in the domain D of $A(t)$ for any $t \in [a,b]$ and is strongly continuously differentiable on $[a,b]$. Moreover, the function $t \mapsto A(t)g(t)$, $t \in [a,b]$, is strongly continuous.

Proof of Lemma 3. Let

$$u(t) = \int_a^t U(t,s)f(s)\,ds, \qquad t \in [a,b].$$

Since the function

$$s \mapsto U(t,s)A(s)^{-1}f(s), \qquad s \in [a,b],$$

is obviously strongly continuously differentiable on $[a,b]$ and the immediate evaluation shows that

$$\begin{aligned}(U(t,s)A(s)^{-1}f(s))'_s &= -U(t,s)f(s)\\ &\quad + U(t,s)(A(s)^{-1})'f(s) + U(t,s)A(s)^{-1}f'(s)\end{aligned}$$

for any $s \in (a,t)$ and the functions on the left- and right-hand side are strongly continuous, it follows that

(4)

$$\begin{aligned}u(t) = \int_a^t U(t,s)f(s)\,ds &= -A(t)^{-1}f(t) + U(t,a)A(a)^{-1}f(a)\\ &\quad - \int_a^t U(t,s)A(s)^{-1}A'(s)A(s)^{-1}f(s)\,ds + \int_a^t U(t,s)A(s)^{-1}f'(s)\,ds.\end{aligned}$$

Since, by assumption, the function $A(t)A(s)^{-1}$ on $[a,b]\times[a,b]$ is weakly differentiable with respect to $t\in[a,b]$ for any $s\in[a,b]$ and the derivative is strongly continuous with respect to $t\in[a,b]$ for some (and hence for any) $s\in[a,b]$, it follows from (4) that the function u takes the values in the domain D and the function $t\mapsto A(t)u(t)$, $t\in[a,b]$, is strongly continuous. Moreover, evaluating the strong derivative of u, one can see that

$$u'(t)=A(t)u(t)+f(t),\qquad t\in[a,b],$$

which completes the proof of the assertions of Lemma 1.

Theorem 1. *Let the Kato assumptions concerning the family of operators* $\{A(t)\}, t\in[a,b]$ *hold, let* U *be the corresponding evolution operator, and let* f *be a strongly continuous* E*-valued function on* $[a,b]$ *such that*

$$f(t)=f(a)+\int_a^t \psi(r)\,dr,\qquad t\in[a,b],$$

for some scalar measurable E*-valued function* ψ *on* $[a,b]$ *with integrable norm. Then the function* $u\colon[a,b]\to E$ *given by* (3) *takes the values in the domain* D *of* $A(t)$ *for any* $t\in[a,b]$ *and is strongly continuously differentiable for any* $t\in[a,b]$*. Moreover, the function* $t\mapsto A(t)u(t)$, $t\in[a,b]$, *is strongly continuous, and* u *is a strong solution of* (1) *on* $[a,b]$*.*

Proof of Theorem 1. Let $\psi_n\colon[a,b]\to E$ be strongly continuous functions such that $\|\psi_n-\psi\|_{L^1([a,b],E)}\to 0$ as $n\to\infty$. Let f_n be the strongly continuous E-valued function on $[a,b]$ given by the rule

$$f_n(t)=f(a)+\int_a^t \psi_n(r)\,dr,\qquad t\in[a,b]. \tag{5}$$

Then, by Lemma 1, the functions $u_n\colon[a,b]\to E$ given by the rule

$$u_n(t)=\int_a^t U(t,r)f_n(r)\,dr,\qquad t\in[a,b], \tag{6}$$

take the values in the domain D of $A(t)$ for any $t\in[a,b]$ and are strongly continuously differentiable for any $t\in[a,b]$, the functions $t\mapsto A(t)u_n(t)$, $t\in[a,b]$, are strongly continuous, and the following representation holds:

$$\begin{aligned}u_n(t)=\int_a^t U(t,s)f_n(s)\,ds&=-A(t)^{-1}f_n(t)+U(t,a)A(a)^{-1}f_n(a)\\&-\int_a^t U(t,s)A(s)^{-1}A'(s)A(s)^{-1}f_n(s)\,ds+\int_a^t U(t,s)A(s)^{-1}f_n'(s)\,ds.\end{aligned}$$

Moreover, it follows from (5) that u_n tends to

$$u(t) = \int_a^t U(t,r)f(r)\,dr, \qquad t \in [a,b],$$

strongly and uniformly on $[a,b]$.

Further, it immediately follows from (6) that

$$\begin{aligned} u_n'(t) = {} & A(t)U(t,a)A(a)^{-1}f_n(a) \\ & - \int_a^t A(t)U(t,s)A(s)^{-1}A'(s)A(s)^{-1}f_n(s)\,ds \\ & + \int_a^t A(t)U(t,s)A(s)^{-1}f_n'(s)\,ds, \end{aligned}$$

and, on the other hand, the function

$$\begin{aligned} t \mapsto A(t)u_n(t) = {} & -f_n(t) + A(t)U(t,a)A(a)^{-1}f_n(a) \\ & - \int_a^t A(t)U(t,s)A(s)^{-1}A'(s)A(s)^{-1}f_n(s)\,ds \\ & + \int_a^t A(t)U(t,s)A(s)^{-1}f_n'(s)\,ds, \qquad t \in [a,b], \end{aligned}$$

is clearly strongly continuous and tends as $n \to \infty$ (uniformly on $[a,b]$) to the strongly continuous function

$$\begin{aligned} t \mapsto & -f(t) + A(t)U(t,a)A(a)^{-1}f(a) \\ & - \int_a^t A(t)U(t,s)A(s)^{-1}A'(s)A(s)^{-1}f(s)\,ds \\ & + \int_a^t A(t)U(t,s)A(s)^{-1}\psi(s)\,ds, \qquad t \in [a,b], \end{aligned}$$

while the functions u_n' tend as $n \to \infty$ (uniformly on $[a,b]$) to the strongly continuous function

$$\begin{aligned} t \mapsto & A(t)U(t,a)A(a)^{-1}f(a) \\ & - \int_a^t A(t)U(t,s)A(s)^{-1}A'(s)A(s)^{-1}f(s)\,ds \\ & + \int_a^t A(t)U(t,s)A(s)^{-1}\psi(s)\,ds, \qquad t \in [a,b], \end{aligned}$$

and, using the fact that every operator $A(t)$ is closed and applying the theorem concerning the differentiability of limits of function sequences having uniformly convergent derivatives, we obtain the assertion of Theorem 1.

Theorem 2. *Let the Kato assumptions concerning the family of operators* $\{A(t)\}, t \in [a, b]$ *hold, let* U *be the corresponding evolution operator, and let* f *be a* D*-valued function on* $[a, b]$ *such that, for some* r*,*

$$A(r)f(t) = \psi(t), \qquad t \in [a, b],$$

for some scalar measurable E*-valued function* ψ *on* $[a, b]$ *with integrable norm. Then the function* $u\colon [a, b] \to E$ *given by the rule* (3) *takes the values in the domain* D *of* $A(t)$ *for any* $t \in [a, b]$ *and is a strong solution of* (1) *on* $[a, b]$*. In particular, the function* $t \mapsto A(t)u(t)$*,* $t \in [a, b]$*, is strongly continuous.*

Proof of Theorem 2. As in the proof of Theorem 1, let $\psi_n\colon [a, b] \to E$ be strongly continuous functions such that $\|\psi_n - \psi\|_{L^1([a,b],E)} \to 0$ as $n \to \infty$. Let f_n be the strongly continuous E-valued function on $[a, b]$ given by the rule

$$f_n(t) = A^{-1}(t)\psi_n(t), \qquad t \in [a, b]. \tag{7}$$

Then, by Kato's Theorem 5 in [1], the functions $u_n\colon [a, b] \to E$ given by the rule

$$u_n(t) = U(t, a)x_0 + \int_a^t U(t, r)f_n(r)\, dr, \qquad t \in [a, b], \tag{8}$$

take the values in the domain D of $A(t)$ for any $t \in [a, b]$ and are strongly continuously differentiable for any $t \in [a, b]$, the functions $t \mapsto A(t)u_n(t)$, $t \in [a, b]$, are strongly continuous, and the following representation holds:

$$du_n(t)/dt = A(t)U(t, a)x_0 + f_n(t) + \int_a^t A(t)U(t, s)f_n(s)\, ds \tag{9}$$

(see [1], (3.37)), where $\int_a^t U(t, s)f_n(s)\, ds \in D$ for every $t \in [a, b]$ and

$$A(t)\int_a^t U(t, s)f_n(s)\, ds = \int_a^t A(t)U(t, s)f_n(s)\, ds.$$

Obviously, since $W(t, s) = A(t)U(t, s)A^{-1}(s)$ is jointly strongly continuous, it follows that $\int_a^t U(t, r)f(r)\, dr$ belongs to D and $A(t)\int_a^t U(t, r)f(r)\, dr = \int_a^t W(t, r)A(r)f(r)\, dr$; this function is clearly strongly continuous on $[a, b]$.

This implies that $A(t)u(t)$ is strongly continuous on $[a,b]$. It clearly follows from the obvious strong continuity of $u(t) = U(t,a)x_0 + \int_a^t U(t,r)f(r)\,dr$, $t \in [a,b]$, that, since $x_0 \in D$, the function $t \mapsto U(t,a)x_0$, $t \in [a,b]$, is strongly continuously differentiable on $[a,b]$, and the other summand on the right-hand side, $t \mapsto \int_a^t U(t,r)f(r)\,dr$, $t \in [a,b]$, is obviously right differentiable with the strong right derivative $A(t)\int_a^t U(t,r)f(r)\,dr + \lim(\Delta t)^{-1}\int(W(t+\Delta t,s) - W(t,s))\psi(s)\,ds$, where the limit obviously exists when ψ is replaced by an $L^1([a,b],E)$-close strongly continuous function, and the $L^1([a,b],E)$-convergent to ψ sequence of strongly continuous functions ψ_n defines a uniformly norm convergent sequence of approximants to the limit in question, and thus this limit is strongly continuous on $[a,b]$, as was to be proved.

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References

1. T. Kato, *Integration of the equation of evolution in a Banach space*, J. Math. Soc. Japan **5** (1953), 208–234.
2. A. I. Shtern, *Comments on 1953 Tosio Kato's Theorem Concerning Abstract Evolution Equations of Hyperbolic Type*, J. Appl. Funct. Differ. Equ. JAFDE **1** (2006), no. 1, 59–70.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis (FGU FNTs NIISI RAN),
Russian Academy of Sciences,
Moscow, 117312 Russia
E-mail: ashtern@member.ams.org, aishtern@mtu-net.ru