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HERMITE BASED POLY-BERNOULLI POLYNOMIALS WITH A q-PARAMETER

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ABSTRACT. We introduce the Hermite based poly-Bernoulli polynomials with a q parameter and give some of their basic properties including not only addition property, but also derivative properties and integral representations. We also define the Hermite based λ-Stirling polynomials of the second kind, and then provide some relations. Moreover, we derive several correlations and identities including the Hermite-Kampé de Fériet (or Gould-Hopper) family of polynomials, the Hermite based poly-Bernoulli polynomials with a q parameter and the Hermite based λ-Stirling polynomials of the second kind.

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1. INTRODUCTION

Special polynomials and numbers possess a lot of importance in many fields of mathematics, physics, engineering and other related disciplines including the topics such as differential equations, mathematical analysis, functional analysis, mathematical physics, quantum mechanics. One of the most considerable polynomials in the theory of special polynomials is the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials (see [1]) and another is Bernoulli polynomials (see [14], [20]). Nowadays, these type polynomials and their several generalizations have been studied and used by many mathematicians and physicists, (see [1-21] and the references therein). Araci et al. [2] introduced a new concept of the Apostol Hermite-Genocchi polynomials by using the modified Milne-Thomson's polynomials and also derived several implicit summation formulae and general symmetric identities arising from different analytical means and generating function method. Bretti et al. [4] defined multidimensional extensions of the Bernoulli and Appell polynomials by using the Hermite-Kampé de Fériet polynomials and gave the differential equations, satisfying by the corresponding $2D$ polynomials, derived from exploiting the factorization method. Bayad et al. [3] considered poly-Bernoulli polynomials and numbers and proved a collection of extremely important and fundamental identities satisfied by the

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poly-Bernoulli polynomials and numbers. Cenkci et al. [5] considered poly-Bernoulli numbers and polynomials with a q parameter and developed some arithmetical and number theoretical properties. Dattoli et al. [6] applied the method of generating function to introduce new forms of Bernoulli numbers and polynomials, which were exploited to derive further classes of partial sums involving generalized many indices and several variable polynomials. Khan et al. [7] introduced the Hermite poly-Bernoulli polynomials and numbers of the second kind and examined some of their applications in combinatorics, number theory and other fields of mathematics. Kim et al. [8] introduced higher-order poly-Bernoulli type polynomials, poly-Bernoulli mixed type polynomials [9], degenerate poly-Bernoulli polynomials [10], fully degenerate poly-Bernoulli polynomials with a q parameter. They also derived many interesting properties. Kurt et al. [12] studied the Hermite-Kampé de Fériet based the second kind Genocchi polynomials and presented some relationships among them. Ozarslan [15] introduced a unified family of Hermite-based Apostol-Bernoulli, Euler and Genocchi polynomials and then, acquired some symmetry identities between these polynomials and the generalized sum of integer powers. Ozarslan also gave explicit closed-form formulae for this unified family and proved a finite series relation between this unification and $3d$-Hermite polynomials. Pathan [16] defined a new class of generalized Hermite-Bernoulli polynomials and derived several implicit summation formulae and symmetric identities by using different analytical means applied to generating functions. Pathan et al. [17] introduced a new class of generalized polynomials associated with the modified Milne-Thomson's polynomials $\Phi_n^{(\alpha)}(x,v)$ of degree n and order α and provided some of their properties.

The usual notations $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers and the set of all nonnegative integers, respectively, in what follows.

The outline of this paper is as follows. Section 2 contains the definitions of the Hermite based poly-Bernoulli polynomials ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ with a q parameter and the Hermite based λ-Stirling polynomials $S_2^{(\lambda,j)}(n,m;x,y)$ of the second kind, and then provides some properties and relationships for ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ and $S_2^{(\lambda,j)}(n,m;x,y)$. Section 3 examines several correlations including the Hermite-Kampé de Fériet polynomials, the Hermite based poly-Bernoulli polynomials with a q parameter and the Hermite based λ-Stirling polynomials of the second kind.

2. Preliminary Results

The exponential generating function for the Hermite-Kampé de Fériet (or Gould-Hopper) family of polynomials is (see [4])

$$\sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!} = e^{xt+yt^j}, \tag{1}$$

where $j \in \mathbb{N}$ with $j \geq 2$. In the case $j = 1$, the corresponding $2D$ polynomials are simply expressed by the Newton binomial formula. Upon setting

$j=2$ in (1) gives the two variable Hermite polynomials $H_n^{(2)}(x,y)$ and the mentioned polynomials have been used to define $2D$ extensions of some special polynomials, such as Bernoulli and Euler polynomials (see [6]).

Recalling that the polynomials $H_n^{(j)}(x,y)$ are the native solution of the generalized heat equation:

$$\frac{\partial}{\partial t}F(x,y)=\frac{\partial^j}{\partial x^j}F(x,y) \text{ with } F(x,0)=x^n$$

and satisfies the following formula

$$H_n^{(j)}(x,y)=n!\sum_{r=0}^{\left\lfloor \frac{n}{j}\right\rfloor}\frac{x^{n-jr}y^r}{r!\,(n-jr)!},$$

where $\lfloor\cdot\rfloor$ is Gauss' notation, and represents the maximum integer which does not exceed a number in the square bracket.

For $k\in\mathbb{Z}$ with $k>1$, the k-th polylogarithm function is defined by

$$Li_k(t)=\sum_{m=1}^{\infty}\frac{t^m}{m^k}\quad (t\in\mathbb{C}\text{ with } |t|<1). \tag{2}$$

We always assume $|t|<1$ along this paper. When $k=1$, $Li_1(t)=-\log(1-t)$. In the case $k\leq 0$, $Li_k(t)$ are the rational functions:

$$Li_0(t)=\frac{t}{1-t},\ Li_{-1}(t)=\frac{t}{(1-t)^2},\ Li_{-2}(t)=\frac{t^2+t}{(1-t)^3},$$
$$Li_{-3}(t)=\frac{t^3+4t^2+t}{(1-t)^4},\cdots.$$

Further information about poly-logarithm function and polynomials including poly-logarithm function (see, e.g., [3, 5, 7-11, 13]).

2.1. **The Hermite based poly-Bernoulli polynomials with a q parameter.** Let $n,k\in\mathbb{Z}$ in conjunction with $n\geq 0$ and $k>0$ and let $q\in\mathbb{R}-\{0\}$. We introduce the Hermite based poly-Bernoulli polynomials with a q parameter via the following exponential generating function defined by

$$\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}=\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}e^{xt+yt^j}. \tag{3}$$

Upon setting $x=y=0$, we then get ${}_H\mathcal{B}_{n,q}^{(k,j)}(0,0)=:{}_H\mathcal{B}_{n,q}^{(k,j)}$ which is called the poly-Bernoulli numbers with a q parameter (see [5]).

Some special cases of ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ are listed by the following consecutive remarks.

Remark 1. *Letting $y=0$, we have ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,0):={}_H\mathcal{B}_{n,q}^{(k)}(x)$ called poly-Bernoulli polynomials with a q parameter, cf.* [5].

Remark 2. *In the case $q=1$, ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ reduces to the the Hermite based poly-Bernoulli polynomials ${}_H\mathcal{B}_{n}^{(k,j)}(x,y)$, cf.* [15].

Remark 3. *When $q=1$ and $y=0$, we obtain the poly-Bernoulli polynomials $B_n(x)$, cf.* [14], [19], [20].

Remark 4. *When $q=k=1$ and $y=0$, we obtain the usual Bernoulli polynomials $B_n(x)$, cf.* [14] [19], [20].

The addition formula for ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ is provided in the following proposition.

Proposition 2.1. *We have*

$$ {}_H\mathcal{B}_{n,q}^{(k,j)}(x_1+x_2,y_1+y_2)=\sum_{m=0}^{n}\binom{n}{m}{}_H\mathcal{B}_{n-m,q}^{(k,j)}(x_1,y_1)H_m^{(j)}(x_2,y_2). \tag{4}$$

Proof. In view of (3), with the series manipulation procedure, we see

$$\begin{aligned}\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x_1+x_2,y_1+y_2)\frac{t^n}{n!} &= \frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}e^{(x_1+x_2)t+(y_1+y_2)t^j}\\ &= \frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}e^{x_1t+y_1t^j}e^{x_2t+y_2t^j}\\ &= \left(\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}H_n^{(j)}(x,y)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_H\mathcal{B}_{n-m,q}^{(k,j)}(x_1,y_1)H_m^{(j)}(x_2,y_2)\right)\frac{t^n}{n!},\end{aligned}$$

which implies the asserted result (4). □

An immediate output of Proposition 2.1 is stated in the following corallary.

Corollary 2.2. *The following identity holds true:*

$$ {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)=\sum_{m=0}^{n}\binom{n}{m}{}_H\mathcal{B}_{n-m,q}^{(k,j)}H_m^{(j)}(x,y). \tag{5}$$

The derivative properties of ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ are stated in the following proposition.

Proposition 2.3. *Each of the following formulas holds true:*

$$\frac{\partial_H\mathcal{B}_{n,q}^{(k,j)}(x,y)}{\partial x}=n\,{}_H\mathcal{B}_{n-1,q}^{(k,j)}(x,y)\ \text{ and }\ \frac{\partial_H\mathcal{B}_{n,q}^{(k,j)}(x,y)}{\partial y}=\{n\}_j\ {}_H\mathcal{B}_{n-j,q}^{(k,j)}(x,y),$$

where $\{n\}_j=n(n-1)(n-2)\cdots(n-j+1)$ is called the falling factorial function, cf. [20].

Proof. The proof follows from (3). So we omit them. □

The integral represenations of ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ are stated in the following proposition.

Proposition 2.4. *The following equalities hold true:*

$$\int_{\upsilon}^{\mu} {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\,dx = \frac{{}_H\mathcal{B}_{n+1,q}^{(k,j)}(\mu,y) - {}_H\mathcal{B}_{n+1,q}^{(k,j)}(\upsilon,y)}{n+1}$$

and

$$\int_{\gamma}^{\zeta} {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\,dy = \frac{{}_H\mathcal{B}_{n+j,q}^{(k,j)}(x,\zeta) - {}_H\mathcal{B}_{n+j,q}^{(k,j)}(x,\gamma)}{(n+1)_j},$$

where $(n)_j = n(n+1)(n+2)\cdots(n+j-1)$ *is called the Pochhammer symbol, cf.* [20].

Proof. Using the derivative properties of ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ given in Proposition 2.3, we easily get the asserted results. So, we omit them. □

We have the following proposition.

Proposition 2.5. *The following formula is valid:*

$$\text{(6)}\qquad {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y) = \sum_{m=0}^{\left\lfloor \frac{n}{j} \right\rfloor} {}_H\mathcal{B}_{n-jm,q}^{(k)}(x)\frac{\{n\}_m y^m}{(n-jm)!}.$$

Proof. By (3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!} &= \frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}e^{xt}e^{yt^j} \\
&= \sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(k)}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} y^n\frac{t^{jn}}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\left\lfloor \frac{n}{j} \right\rfloor} {}_H\mathcal{B}_{n-jm,q}^{(k)}(x)\frac{t^{n-jm}}{(n-jm)!}y^m\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\left\lfloor \frac{n}{j} \right\rfloor} {}_H\mathcal{B}_{n-jm,q}^{(k)}(x)\frac{(n)_m y^m}{(n-jm)!}\right)\frac{t^n}{n!},
\end{aligned}$$

which implies the desired result (6). □

2.2. **The Hermite based λ-Stirling polynomials of the second kind.** We introduce the Hermite based λ-Stirling polynomials of the second kind defined by

$$\text{(7)}\qquad \sum_{n=0}^{\infty} S_2^{(\lambda,j)}(n,m;x,y)\frac{t^n}{n!} = \frac{\left(\lambda e^t-1\right)^m}{m!}e^{xt+yt^j}.$$

Several specific circumstances of ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ are listed via the following consecutive remarks.

Remark 5. *Upon setting* $\lambda = 1$, *we have* $S_2^{(1,j)}(n,m;x,y) := S_2^{(j)}(n,m;x,y)$ *called the Hermite based Stirling polynomials of the second kind.*

Remark 6. *Letting* $y = 0$*,* $S_2^{(\lambda,j)}(n,m;x,y)$ *reduces to the* $S_2^{\lambda}(n,m;x)$ *called the weighted* λ*-Stirling numbers of the second kind, cf.* [5].

Remark 7. *When* $\lambda = 1$ *and* $y = 0$*,* $S_2^{(j)}(n,m;x,y)$ *reduces to the* $S_2(n,m;x)$ *called the weighted Stirling numbers of the second kind, cf.* [5].

Remark 8. *Setting* $x = y = 0$*, we have* $S_2^{(\lambda,j)}(n,m;0,0) := S_2^{\lambda}(n,m)$ *called the familiar* λ*-Stirling numbers of the second kind, cf.* [5].

Remark 9. *In the case* $\lambda = 1$ *and* $x = y = 0$*, we have* $S_2^{(1,j)}(n,m;0,0) :=$ $S_2(n,m)$ *which are called the familiar Stirling numbers of the second kind, cf.* [7], [20].

We give some relations and properties belonging to the Hermite based λ-Stirling polynomials of the second kind by the following consecutive propositions.

Proposition 2.6. *We have*

$$\begin{aligned} S_2^{(\lambda,j)}(n,m;x,y) &= \sum_{l=0}^{n}\binom{n}{l}S_2^{\lambda}(l,m)\,H_{n-l}^{(j)}(x,y) \\ S_2^{(\lambda,j)}(n,m;x,y) &= \sum_{l=0}^{n}\binom{n}{l}S_2^{\lambda}(l,m;0,y)\,x^{n-l} \\ S_2^{(\lambda,j)}(n,m;x,y) &= \sum_{l=0}^{\left\lfloor \frac{n}{j}\right\rfloor} S_2^{\lambda}(l,m;x,0)\,\frac{\{n\}_l\, y^l}{(n-jl)!}. \end{aligned}$$

Proposition 2.7. *We have*

$$S_2^{\left(q^{-1},j\right)}(n+1,m+1;x,y) = \frac{1}{q}\sum_{l=0}^{n}\binom{n}{l}S_2^{q^{-1}}(l,m;x,y).$$

Proposition 2.8. *Each of the following differential formulas holds true:*

$$\frac{\partial}{\partial x}S_2^{(\lambda,j)}(n,m;x,y) = n\ S_2^{(\lambda,j)}(n-1,m;x,y)$$

and

$$\frac{\partial}{\partial y}S_2^{(\lambda,j)}(n,m;x,y) = \{n\}_j\,S_2^{(\lambda,j)}(n-j,m;x,y).$$

Proposition 2.9. *The following integral representations hold true:*

$$\int_{\upsilon}^{\mu} S_2^{(\lambda,j)}(n,m;x,y)\,dx = \frac{S_2^{(\lambda,j)}(n+1,m;\mu,y) - S_2^{(\lambda,j)}(n+1,m;\upsilon,y)}{n+1}$$

and

$$\int_{\gamma}^{\zeta} S_2^{(\lambda,j)}(n,m;x,y)\,dy = \frac{S_2^{(\lambda,j)}(n+j,m;x,\zeta) - S_2^{(\lambda,j)}(n+j,m;x,\gamma)}{(n+1)_j}.$$

3. Main Results

This part includes our main results.

A correlation between ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ and $H_n^{(j)}(x,y)$ is given by the following theorem.

Theorem 3.1. *The polynomials* ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ *and* $H_n^{(j)}(x,y)$ *satisfy the following relation:*

$$\begin{aligned}(8)\qquad &\sum_{m=0}^{n}\binom{n}{m}\,{}_H\mathcal{B}_{n-m,q}^{(k,j)}(x,y)\,q^m-{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\\ &=\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r\,H_n^{(j)}(x+q-rq,y).\end{aligned}$$

Proof. By (3), we write

$$\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}=\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}-1}e^{(x+q)t+yt^j}.$$

Then, we consider

$$(9)\qquad \left(e^{qt}-1\right)\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}=qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{(x+q)t+yt^j}.$$

Let LHS denotes left-hand side of (9). Then

$$\begin{aligned}LHS&=\sum_{n=0}^{\infty}q^n\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}\\ (10)\qquad&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\,{}_H\mathcal{B}_{n-m,q}^{(k,j)}(x,y)\,q^m-{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Let RHS denotes the right-hand side of (9). Then, by (1), we get

$$\begin{aligned}RHS&=q\sum_{m=1}^{\infty}\frac{\left(\frac{1-e^{-qt}}{q}\right)^m}{m^k}e^{(x+q)t+yt^j}=q\sum_{m=0}^{\infty}\frac{\left(\frac{1-e^{-qt}}{q}\right)^{m+1}}{(m+1)^k}e^{(x+q)t+yt^j}\\ &=\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{(x+q-rq)t+yt^j}\\ &=\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r\sum_{n=0}^{\infty}H_n^{(j)}(x+q-rq,y)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r\,H_n^{(j)}(x+q-rq,y)\right)\frac{t^n}{n!}.\end{aligned}$$

LHS and RHS yield the desired result (8). □

A correlation between ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ and $S_2^{(j)}(n,m;x,y)$ is stated in the following theorem.

Theorem 3.2. *The following correlation holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}\ {}_H\mathcal{B}_{n-m,q}^{(k,j)}(x,y)\,q^m - {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y) = \sum_{m=0}^{\infty}\frac{(-1)^{n+m+1}}{(m+1)^k}q^{n-m}(m+1)!\ S_2^{(j)}\left(n,m;-\frac{x}{q}-1,(-1)^j\frac{y}{q^j}\right). \tag{11}$$

Proof. By (3), we write

$$\left(e^{qt}-1\right)\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!} = qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{(x+q)t+yt^j}.$$

Using the series manipulation procedure, LHS of (7):

$$LHS = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\ {}_H\mathcal{B}_{n-m,q}^{(k,j)}(x,y)\,q^m - {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\right)\frac{t^n}{n!}.$$

By (9), RHS of (7):

$$\begin{aligned} RHS &= q\sum_{m=1}^{\infty}\frac{\left(\frac{1-e^{-qt}}{q}\right)^m}{m^k}e^{(x+q)t+yt^j} = q\sum_{m=0}^{\infty}\frac{\left(\frac{1-e^{-qt}}{q}\right)^{m+1}}{(m+1)^k}e^{(x+q)t+yt^j} \\ &= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^k}q^{-m}\left(e^{-qt}-1\right)^{m+1}e^{(x+q)t+yt^j} \\ &= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)^k}q^{-m}(m+1)!\sum_{n=0}^{\infty}S_2^{(j)}\left(n,m;-\frac{x}{q}-1,(-1)^j\frac{y}{q^j}\right)\frac{(-qt)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(-1)^{n+m+1}}{(m+1)^k}q^{n-m}(m+1)!\ S_2^{(j)}\left(n,m;-\frac{x}{q}-1,(-1)^j\frac{y}{q^j}\right)\right)\frac{t^n}{n!}. \end{aligned}$$

LHS and RHS gives the desired result (10). □

From (2), we readily derive

$$\frac{d}{dt}Li_k(f(t)) = \frac{f'(t)}{f(t)}Li_{k-1}(f(t)). \tag{12}$$

An relation for the Hermite based poly-Bernoulli polynomials with a q parameter is given by the following theorem.

Theorem 3.3. *We have*

$$\begin{aligned} &\sum_{m=0}^{n}\binom{n}{m}\ {}_H\mathcal{B}_{n+1-m,q}^{(k,j)}(x,y)\,q^m - {}_H\mathcal{B}_{n+1,q}^{(k,j)}(x,y) \\ &= q\ {}_H\mathcal{B}_{n,q}^{(k-1,j)}(x,y) + x\sum_{m=0}^{n}\binom{n}{m}\ {}_H\mathcal{B}_{n-m,q}^{(k,j)}(x,y)\,q^m - (x+q)\ {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y) \\ &+ yj\frac{(n-j+1)!}{n!}\left(\sum_{m=0}^{n-j+1}\binom{n-j+1}{m}\ {}_H\mathcal{B}_{n-j+1-m,q}^{(k,j)}(x,y)\,q^m - {}_H\mathcal{B}_{n+j-1,q}^{(k,j)}(x,y)\right). \end{aligned} \tag{13}$$

Proof. In the light of (12), we get

$$\frac{d}{dt}Li_k\left(\frac{1-e^{-qt}}{q}\right)=\frac{q}{e^{qt}-1}Li_{k-1}\left(\frac{1-e^{-qt}}{q}\right).$$

Differentiating both sides of (9) with respect to t, we derive

$$\frac{d}{dt}\left(e^{qt}\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}\right)=qe^{qt}\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}$$
$$+e^{qt}\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n+1,q}^{(k,j)}(x,y)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n+1,q}^{(k,j)}(x,y)\frac{t^n}{n!}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left(qLi_k\left(\frac{1-e^{-qt}}{q}\right)e^{(x+q)t+yt^j}\right)\\
=\;&\frac{q^2}{e^{qt}-1}Li_{k-1}\left(\frac{1-e^{-qt}}{q}\right)e^{(x+q)t+yt^j}\\
&+qLi_k\left(\frac{1-e^{-qt}}{q}\right)\left[(x+q)e^{(x+q)t+yt^j}+yjt^{j-1}e^{(x+q)t+yt^j}\right]\\
=\;&\frac{q^2}{e^{qt}-1}Li_{k-1}\left(\frac{1-e^{-qt}}{q}\right)e^{(x+q)t+yt^j}\\
&+e^{qt}(x+q)\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}-1}e^{(x+q)t+yt^j}-(x+q)\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}-1}e^{(x+q)t+yt^j}\\
&+e^{qt}yjt^{j-1}\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}-1}e^{(x+q)t+yt^j}-yjt^{j-1}\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{e^{qt}-1}e^{(x+q)t+yt^j}\\
=\;&q\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k-1,j)}(x,y)\frac{t^n}{n!}\\
&+(x+q)\sum_{n=0}^{\infty}q^n\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}-(x+q)\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}\\
&+yjt^{j-1}\sum_{n=0}^{\infty}q^n\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}-yjt^{j-1}\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!}\\
=\;&q\sum_{n=0}^{\infty}{}_{H}\mathcal{B}_{n,q}^{(k-1,j)}(x,y)\frac{t^n}{n!}\\
&+(x+q)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_{H}\mathcal{B}_{n-m,q}^{(k,j)}(x,y)q^m-{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\right)\frac{t^n}{n!}\\
&+yj\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_{H}\mathcal{B}_{n-m,q}^{(k,j)}(x,y)q^m-{}_{H}\mathcal{B}_{n,q}^{(k,j)}(x,y)\right)\frac{t^{n+j-1}}{n!},
\end{aligned}$$

which gives the desired result (13). □

Here, we give the following theorem.

Theorem 3.4. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then*

$$ {}_H\mathcal{B}_{n,q}^{(-k,j)}(x,y) = \sum_{m=0}^{\min(n,k)} (m!)^2 \ S_2^{(j)}\left(n,m;\frac{x}{q}+1,\frac{y}{q^j}\right) q^n \ S_2^{q^{-1}}(k,m;1). $$

Proof. By inspiring the proof given by Cenkci and Komatsu [5], by (3), we write

$$ \sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(-k,j)}(x,y)\frac{t^n}{n!} = \frac{q}{1-e^{-qt}}\sum_{m=0}^{\infty}(m+1)^k\left(\frac{1-e^{-qt}}{q}\right)^{m+1} e^{xt+yt^j}. $$

Then, using (7), for $|z|<1$, we obtain

$$ \begin{aligned} & \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(-k,j)}(x,y)\frac{t^n}{n!}\frac{z^k}{k!} \\ = & \frac{q}{1-e^{-qt}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(m+1)^k\left(\frac{1-e^{-qt}}{q}\right)^{m+1}\frac{z^k}{k!}e^{xt+yt^j} \\ = & \frac{q}{1-e^{-qt}}\sum_{m=0}^{\infty}\left(\frac{1-e^{-qt}}{q}\right)^{m+1} e^{(m+1)z}e^{xt+yt^j} \\ = & \frac{q}{1-e^{-qt}}\sum_{m=0}^{\infty}\left(\frac{e^z\left(1-e^{-qt}\right)}{q}\right)^{m+1} e^{xt+yt^j} \\ = & \frac{qe^{z+qt}}{q-\left(e^{qt}-1\right)\left(e^z-q\right)}e^{xt+yt^j} \\ = & \sum_{m=0}^{\infty} e^{(x+q)t+yt^j}\left(e^{qt}-1\right)^m e^z\left(q^{-1}e^z-1\right)^m \\ = & \sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty} m! \ S_2^{(j)}\left(n,m;\frac{x}{q}+1,\frac{y}{q^j}\right)q^n\frac{t^n}{n!}\right]\cdot\left[m!\sum_{k=0}^{\infty} \ S_2^{q^{-1}}(k,m;1)\frac{z^k}{k!}\right] \\ = & \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}(m!)^2 \ S_2^{(j)}\left(n,m;\frac{x}{q}+1,\frac{y}{q^j}\right)q^n \ S_2^{q^{-1}}(k,m;1)\right)\frac{t^n}{n!}\frac{z^k}{k!}, \end{aligned} $$

which finalize this theorem. □

A correlation including $S_2^{(j)}(n,m;x,y)$ and ${}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)$ is given by the following theorem.

Theorem 3.5. *The following identity holds true:*

(14)
$$ {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y) = q\sum_{u=0}^{n}\binom{n}{u}\sum_{s=0}^{\infty}s^u\sum_{m=0}^{\infty}\frac{(m+1)!}{(m+1)^k}S_2^{(j)}\left(n-u,m+1;-\frac{x}{q},(-1)^j\frac{y}{q^j}\right)(-q)^{n-m-1}. $$

Proof. From (3) and utilizing (7), with the series manipulation procedure, we acquire

$$\sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!} = \frac{q}{1-e^{-qt}}\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}}{(m+1)^k}\left(e^{-qt}-1\right)^{m+1}e^{xt+yt^j}$$

$$= q\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-sqt)^u}{u!}\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}}{(m+1)^k}\left(e^{-qt}-1\right)^{m+1}e^{xt+yt^j}$$

$$= q\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}(-sq)^u\frac{t^u}{u!}\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}(m+1)!}{(m+1)^k}$$

$$\cdot\sum_{n=0}^{\infty}S_2^{(j)}\left(n,m+1;-\frac{x}{q},(-1)^j\frac{y}{q^j}\right)(-q)^n\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(q\sum_{s=0}^{\infty}(-sq)^n\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}(m+1)!}{(m+1)^k}\right)\frac{t^n}{n!}$$

$$\cdot\sum_{n=0}^{\infty}S_2^{(j)}\left(n,m+1;-\frac{x}{q},(-1)^j\frac{y}{q^j}\right)(-q)^n\frac{t^n}{n!},$$

which implies the desired result (14). □

We give the following theorem.

Theorem 3.6. *The following relation holds true:*

(15)

$$ {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y) = \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{l=0}^{m+1}\binom{m+1}{l}(-1)^l H_n^{(j)}(x-lq-sq,y).$$

Proof. By means of (1), (3) and (7), we derive

$$\sum_{n=0}^{\infty} {}_H\mathcal{B}_{n,q}^{(k,j)}(x,y)\frac{t^n}{n!} = \frac{q}{1-e^{-qt}}\sum_{m=0}^{\infty}\frac{(-q)^{-m-1}}{(m+1)^k}\left(e^{-qt}-1\right)^{m+1}e^{xt+yt^j}$$

$$= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{l=0}^{m+1}\binom{m+1}{l}(-1)^l e^{(x-lq-sq)t+yt^j}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{-m}}{(m+1)^k}\sum_{l=0}^{m+1}\binom{m+1}{l}(-1)^l H_n^{(j)}(x-lq-sq,y)\right)\frac{t^n}{n!}.$$

Thus, we arrive at the desired identity (15). □

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