Advanced Studies in Contemporary Mathematics
28 (2018), No. 2, pp. 261 - 268
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.2.261

THE CERTAIN SUMMATION INTEGRAL TYPE OPERATORS AND ITS INVERSE THEOREM

PRASHANTKUMAR PATEL AND VISHNU NARAYAN MISHRA

Abstract. In [1], Patel and Mishra introduced and discussed Stancu type generalization of integral modification of the well-known Baskakov operators with the weight function of Beta basis function. Simultaneous approximation results of these operators were established by Patel and Mishra [2]. The present paper deals with detail proof of inverse theorem of these operators.

2010 Mathematics Subject Classification. Primary 41A25, 41A30, 41A36.

Keywords and phrases. Inverse Theorem, Positive linear operators.

1. Introduction

In 2015, Patel and Mishra [1, 2] extended, the study of the Baskakov-Durrmeyer operators with three parameters, which was defined as follows: For $x \in [0,\infty)$,$\gamma > 0$, $0 \le \alpha \le \beta$

$$B_{n,\gamma}^{\alpha,\beta}(f,x) = \sum_{n=0}^{\infty} s_{n,k,\gamma}(x) \int_0^\infty u_{n,k,\gamma}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt + s_{n,0,\gamma}(0) f\left(\frac{\alpha}{n+\beta}\right),$$

where

$$s_{n,k,\gamma}(x) = \frac{\Gamma(\frac{n}{\gamma}+k)}{\Gamma(k+1)\Gamma(\frac{n}{\gamma})} \cdot \frac{(\gamma x)^k}{(1+\gamma x)^{\frac{n}{\gamma}+k}}$$

and

$$u_{n,k,\gamma}(t) = \frac{\gamma\Gamma(\frac{n}{\gamma}+k+1)}{\Gamma(k)\Gamma(\frac{n}{\gamma}+1)} \cdot \frac{(\gamma t)^{k-1}}{(1+\gamma t)^{\frac{n}{\gamma}+k+1}}.$$

Since the operators $B_{n,\gamma}^{\alpha,\beta}(f,\cdot)$ contains summation and integral sign, sometimes this type of operators known as summation-integral type operators. For particular case, i.e. $\alpha = \beta = 0$ and $\gamma = 1$, the operators $B_{n,1}^{0,0}(f,\cdot)$ reduce to the operators studied by Finta in [3]. Many other researchers work in this direction and obtain different approximation properties of many operators [4, 5, 8, 9, 10]. Details proof of inverse results of operators $B_{n,\gamma}^{\alpha,\beta}(f,x)$ are discussed in this manuscript.

Lemma 1.1 ([6]). *Consider $V_{n,m,\gamma}(x), m \in \mathbb{N} \cup \{0\}$ has*

$$\begin{aligned} V_{n,m,\gamma}(x) &= B_{n,\gamma}^{0,0}((t-x)^m, x) \\ &= \sum_{k=1}^{\infty} s_{n,k,\gamma}(x) \int_0^\infty u_{n,k,\gamma}(t)(t-x)^m dt + s_{n,0,\gamma}(0)(-x)^m, \end{aligned}$$

Then $V_{n,0,\gamma}(x) = 1, V_{n,1,\gamma}(x) = 0$ *and* $V_{n,2,\gamma}(x) = \frac{2x(1+\gamma x)}{n-\gamma}$, *and also the following recurrence relation holds:*

$$\begin{aligned}(n-\gamma m)V_{n,m+1,\gamma}(x) &= x(1+\gamma x)\left[(V_{n,m,\gamma})^{(1)}(x) + 2mV_{n,m-1,\gamma}(x)\right]\\ &\quad + m(1+2\gamma x)V_{n,m,\gamma}(x).\end{aligned}$$

Remark 1.2. *For all* $m \in \mathbb{N}$; $0 \le \alpha \le \beta$; *we have the following recursive relation for the images of the monomials* t^m *under* $B^{\alpha,\beta}_{n,\gamma}(t^m, x)$ *in terms of* $B_{n,\gamma}(t^j, x)$; $j = 0, 1, 2, \ldots, m$ *as*

$$B^{\alpha,\beta}_{n,\gamma}(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{n^j \alpha^{m-j}}{(n+\beta)^m} B^{0,0}_{n,\gamma}(t^j, x).$$

Also,

$$B^{\alpha,\beta}_{n,\gamma}((t-x)^m, x) = \sum_{k=0}^{m} \binom{m}{k} (-x)^{m-k} B^{\alpha,\beta}_{n,\gamma}(t^k, x).$$

One can prove that, for each $x \in (0, \infty)$

$$\begin{aligned}B^{\alpha,\beta}_{n,\gamma}(t^m, x) &= \frac{n^m \Gamma\left(\frac{n}{\gamma}+m\right)\Gamma\left(\frac{n}{\gamma}-m+1\right)}{(n+\beta)^m \Gamma\left(\frac{n}{\gamma}+1\right)\Gamma\left(\frac{n}{\gamma}\right)} x^m\\ &\quad + \frac{mn^{m-1}\Gamma\left(\frac{n}{\gamma}+m-1\right)\Gamma\left(\frac{n}{\gamma}-m+1\right)}{(n+\beta)^m \Gamma\left(\frac{n}{\gamma}+1\right)\Gamma\left(\frac{n}{\gamma}\right)}\\ &\quad \times \left[n(m-1) + \alpha\left(\frac{n}{\gamma}-m+1\right)\right] x^{m-1}\\ &\quad + \frac{\alpha m(m-1)n^{m-2}\Gamma\left(\frac{n}{\gamma}+m-2\right)\Gamma\left(\frac{n}{\gamma}-m+2\right)}{(n+\beta)^m \Gamma\left(\frac{n}{\gamma}+1\right)\Gamma\left(\frac{n}{\gamma}\right)}\\ &\quad \times \left[n(m-2) + \frac{\alpha\left(\frac{n}{\gamma}-m+2\right)}{2}\right] x^{m-2} + O(n^{-2}).\end{aligned}$$

Lemma 1.3. *If* f *has* r^{th} *derivative on* $[0, \infty)$ *with* $f^{(r-1)} = O(t^v)$,$v > 0$ *as* $t \to \infty$; *then for* $r = 1, 2, 3, \ldots$ *and* $n > v + r$, *we obtain*

$$\begin{aligned}\left(B^{\alpha,\beta}_{n,\gamma}\right)^{(r)}(f, x) &= \frac{n^r \Gamma\left(\frac{n}{\gamma}+r\right)\Gamma\left(\frac{n}{\gamma}-r+1\right)}{(n+\beta)^r \Gamma(\frac{n}{\gamma}+1)\Gamma(\frac{n}{\gamma})}\\ &\quad \times \sum_{k=0}^{\infty} s_{n+\gamma r,k,r}(x) \int_0^{\infty} u_{n-\gamma r,k+r,\gamma}(t) f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right) dt.\end{aligned}$$

2. Main Theorem

Lemma 2.1 ([7])**.** *The following equality is true.*

$$\{x(1+\gamma x)^r\} D^r [s_{n,k,\gamma}(x)] = \sum_{\substack{2i+j \le r\\ i,j \ge 0}} n^i (k-nx)^j Q_{i,j,r,\gamma}(x) s_{n,k,\gamma}(x),$$

where $D \equiv \frac{d}{dx}$*, for the polynomials* $Q_{i,j,\gamma}(x)$*, which does not dependent on* n *and* k*.*

Consider C_0 as the set of all continuous functions on the interval $(0,\infty)$ having a compact support and C_0^r as the class of r times continuously differentiable functions with $C_0^r \subset C_0$.The generalized Zygmund class $Liz(\xi,1,a,b) = \{f:$ there exist a constant M such that $\omega_2(f,\delta) \leq M\delta, \delta > 0\}$, where

$$\omega_2(f,\delta) = \sup_{\substack{|t-x|\leq\delta \\ t\in[a,b]}} |f(x+2h) - 2f(x+h) + f(x)|.$$

We denoted $Lip^*(\xi,a,b)$ by $Liz(\xi,1,a,b)$. Suppose that

$$G^{(r)} = \{h : h \in C_0^{r+2},\ \text{supp } h \subset [a',b'],\ \text{where } [a',b'] \subset (a,b)\}.$$

The Peetre's K-functionals are defined as

$$K_r(\xi,f) = \inf_{h\in G^{(r)}} \left[\|f^{(r)} - h^{(r)}\|_{C[a',b']} + \xi\left\{\|h^{(r)}\|_{C[a',b']} + \|h^{(r+2)}\|_{C[a',b']}\right\}\right],$$

$0 < \xi \leq 1$, where f is r^{th} times continuously differentiable function with *supp* $f \subset [a',b']$.

For $0 < \xi < 2$, $C_0^r(\xi,1,a,b) = \left\{f : \sup_{0<\xi\leq 1} \xi^{\frac{-\xi}{2}} K_r(\xi,f,a,b) < C\right\}$. We denote $C_\mu[0,\infty) = \{f \in C[0,\infty) : \exists M > 0, \mu > 0 \ni |f(t)| \leq Mt^\mu\}$. Then the space $(C_\mu[0,\infty), \|\cdot\|_\mu)$ form a norm linear space with norm $\|f\|_\mu = \sup_{0\leq t<\infty} |f(t)|t^{-\mu}$.

Lemma 2.2. *Let* $0 < a' < a'' < b'' < b' < b < \infty$ *and* $f^{(r)} \in C_0$ *with supp* $f \subset [a'',b'']$ *&* $f \in C_0^r(\xi,1,a',b')$*,* $f^{(r)} \in Liz(\xi,1,a',b')$ *i.e.* $f^{(r)} \in Lip^*(\xi,a',b')$*, where* $Lip^*(\xi,a',b')$ *denotes the Zygmund class satisfying* $K_r(\delta,f) \leq C_1\delta^{\xi/2}$*.*

Theorem 2.3. *Let* $f \in C_\mu[0,\infty)$ *for some* $\mu > 0$ *and* $0 < a < a_1 < b_1 < b < \infty$*. Then for* n *sufficiently large, we obtain*

$$\|\left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)}(f,\cdot) - f^{(r)}\|_{C[a_1,b_1]} \leq P_1\omega_2\left(f^{(r)}, n^{-\frac{1}{2}}, [a_1,b_1]\right) + P_2 n^{-1}\|f\|_\mu,$$

where $P_1 = P_1(r)$ *and* $P_2 = P_2(r,f)$*.*

Theorem 2.4. *Let* $f \in C_\mu[0,\infty)$ *and* $(r+2)^{th}$ *derivative of* f *exists at* $x \in (0,\infty)$*, then*

$$\begin{aligned}\lim_{n\to\infty} n\left(\left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)}(f,x) - f^{(r)}(x)\right) &= r\left(\gamma(r-1) - \beta\right) f^{(r)}(x) \\ &\quad + \left[r\gamma(1+2x) + \alpha - \beta x\right] f^{(r+1)}(x) \\ &\quad + x(1+\gamma x) f^{(r+2)}(x).\end{aligned}$$

The proof of theorem 2.3 and theorem 2.4 was discussed in [1].

Theorem 2.5. *Assume that* $0 < \xi < 2, 0 < a_1 < a_2 < b_2 < b_1 < \infty$*. Suppose* $f \in C_\mu[0,\infty)$*. Then (i) implies (ii), where (i) and (ii) stated as*

(1) $\|\left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)}(f,\cdot) - f^{(r)}\|_{C[a_1,b_1]} = O\left(n^{-\frac{\xi}{2}}\right)$,

(2) $f^{(r)} \in Lip^*(\xi,a_2,b_2)$,

where $Lip^*(\xi, a_2, b_2)$ *denotes the Zygmund class satisfying* $\omega_2(f,\delta,a_2,b_2) \leq M_2\delta^{\xi}$.

Proof. The proof of the theorem divided in two cases.
Case I. When $0 < \xi \leq 1$. Choose $a', a'', b', b' > 0$ such that $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$.
Assume that $h \in C_0^{\infty}$ together with *supp* $h \subset [a'', b'']$ and $h(x) = 1$ on the interval $[a_2, b_2]$. For $x \in [a', b']$ with $D \equiv \frac{d}{dx}$. By linearity property, we get

$$\begin{aligned}\left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)}(fh,x) - (fh)^{(r)}(x) &= D^r\left(B_{n,\gamma}^{\alpha,\beta}\left((fh)(t) - (fh)(x)\right), x\right) \\ &= D^r\left(B_{n,\gamma}^{\alpha,\beta}\left(f(t)(h(t) - h(x)), x\right)\right) \\ &\quad + D^r\left(B_{n,\gamma}^{\alpha,\beta}\left(h(x)(f(t) - f(x)), x\right)\right) \\ &= J_1 + J_2.\end{aligned}$$

Let us consider

$$W_{n,\gamma}(x,t) = \sum_{k=1}^{\infty} s_{n,k,\gamma}(x)u_{n,k,\gamma}(t) + (1+\gamma x)^{-\frac{n}{\gamma}}\delta(t),$$

$\delta(t)$ being the Dirac delta function. Using the Leibnitz formula, we have

$$\begin{aligned}J_1 &= \frac{\partial^r}{\partial x^r}\int_0^{\infty} W_{n,\gamma}(x,t) f\left(\frac{nt+\alpha}{n+\beta}\right)\left[h\left(\frac{nt+\alpha}{n+\beta}\right) - h(x)\right]dt \\ &= \sum_{i=0}^{r}\binom{r}{i}\int_0^{\infty} W_{n,\gamma}^{(i)}(x,t)\frac{\partial^{r-i}}{\partial x^{r-i}}\left[f\left(\frac{nt+\alpha}{n+\beta}\right)\left[h\left(\frac{nt+\alpha}{n+\beta}\right) - h(x)\right]\right]dt \\ &= -\sum_{i=0}^{r-1}\binom{r}{i}h^{(r-i)}(x)\left(B_{n,\gamma}^{\alpha,\beta}\right)^{(i)}(f,x) \\ &\quad \times \int_0^{\infty} W_{n,\gamma}^{(r)}(x,t) f\left(\frac{nt+\alpha}{n+\beta}\right)\left[h\left(\frac{nt+\alpha}{n+\beta}\right) - h(x)\right]dt \\ &= J_3 + J_4.\end{aligned}$$

Applying theorem 1, we have

$$\begin{aligned}J_3 &= -\sum_{i=0}^{r-1}\binom{n}{i}h^{(r-i)}(x)f^{(i)}(x) + O\left(n^{-\frac{\xi}{2}}\right) \\ &= -(fh)^{(r)}(x) + h(x)f^{(r)}(x) + O\left(n^{-\frac{\xi}{2}}\right),\end{aligned}$$

uniformly in $x \in [a', b']$. By Taylor's expansion of $f(t)$ and $h(t)$, we have

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + O(t-x)^r$$

and

$$h(t) = \sum_{i=0}^{r+1}\frac{h^{(i)}(x)}{i!}(t-x)^i + O(t-x)^{r+1}.$$

Substituting the above expansions in J_4 and using theorem 2, the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} J_4 &= \sum_{i=0}^{r} \frac{h^{(i)}(x) f^{(r-i)}(x)}{i!(r-i)!} r! + O\left(n^{-1/2}\right) \\ &= \sum_{i=0}^{r} \binom{r}{i} h^{(i)}(x) f^{(r-i)}(x) + O\left(n^{-\xi/2}\right) \\ &= (fh)^{(r)}(x) - h(x) f^{(r)}(x) + O\left(n^{-\xi/2}\right), \end{aligned}$$

uniformly in $x \in [a', b']$. Applying the Leibnitz formula, we obtain

$$\begin{aligned} J_2 &= \sum_{i=0}^{r} \binom{r}{i} \int_0^\infty W_{n,\gamma}^{(i)}(x,t) \frac{\partial^{r-i}}{\partial x^{r-i}} \left[h(x) \left(f\left(\frac{nt+\alpha}{n+\beta}\right) - f(x) \right) \right] dt \\ &= \sum_{i=0}^{r} \binom{r}{i} h^{(r-i)}(x) (B_{n,\gamma}^{\alpha,\beta})^{(i)}(f,x) - (fh)^{(r)}(x) \\ &= \sum_{i=0}^{r} \binom{r}{i} h^{(r-i)}(x) f^{(i)}(x) - (fh)^{(r)}(x) + O(n^{-\xi/2}) \\ &= O\left(n^{-\xi/2}\right), \end{aligned}$$

uniformly in $x \in [a', b']$. Finally combing the estimates of J_1 to J_4, we have

$$\left\| \left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)} (fh, \cdot) - (fh)^{(r)} \right\|_{C[a',b']} = O\left(n^{-\xi/2}\right).$$

Thus by lemma 3 and 4, we get $(fh)^{(r)} \in Lip^*(\xi, a', b')$, which give that $f^{(r)} \in Lip^*(\xi, a_2, b_2)$ as $h(x) = 1$ on the interval $[a_2, b_2]$. For the case $0 < \xi \le 1$, the result is proved.

Case II. When $1 < \zeta < 2$. If $a_1^*, b_1^*, a_2^*, b_2^* > 0$ with $a_1 < a_1^* < a_2^* < a_2 < b_2 < b_2^* < b_1^* < b_1$. If $\delta > 0$ then $1 - \delta < 1$. Therefore by Case I, we obtain $f^{(r)} \in Lip^*(1-\delta, a_1^*, b_1^*)$. Assume that $h \in C_0^\infty$ with $h(x) = 1$ on $[a_2, b_2]$ and *supp* $h \subset (a_2^*, b_2^*)$. If $\chi_2(t) = 1$ if $t \in [a_1^*, b_1^*]$ and $\chi_2(t) = 0$ if $t \notin [a_1^*, b_1^*]$, we obtain

$$\begin{aligned} &\left\| \left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)} (f\,h, x) - (f\,h)^{(r)}(x) \right\|_{C[a_2^*, b_2^*]} \\ &\le \left\| D^r \left(B_{n,\gamma}^{\alpha,\beta} \left(h(x) \left(f(t) - f(x) \right) \right), x \right) \right\|_{C[a_2^*, b_2^*]} \\ &\quad + \left\| D^r \left(B_{n,\gamma}^{\alpha,\beta} \left(f(t) \left(h(t) - h(x) \right) \right), x \right) \right\|_{C[a_2^*, b_2^*]} = P_1 + P_2. \end{aligned}$$

Using linearity property, Leibniz theorem, theorem 2 and the hypothesis that (i) holds, we get

$$\begin{aligned} P_1 &= \left\| D^{(r)}\left[h(x)\, B_{n,\gamma}^{(\alpha,\beta)(f,x)} - (f\ h)(x)\, B_{n,\gamma}^{\alpha,\beta}(1,x)\right]\right\|_{C[a_2^*,b_2^*]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} h^{(r-i)}(x) \left(B_{n,\gamma}^{\alpha,\beta}\right)^{(i)}(f,x) - (f\ h)^{(r)}(x) \right\|_{C[a_2^*,b_2^*]} \\ &= \left\| \sum_{i=0}^{r} \binom{r}{i} h^{(r-i)}(x)\, f^{(i)}(x) - (f\ h)^{(r)}(x) \right\|_{C[a_2^*,b_2^*]} + O\left(n^{-\frac{\zeta}{2}}\right) \\ &= O\left(n^{-\frac{\zeta}{2}}\right). \end{aligned}$$

By the Leibniz formula & theorem 1, we obtain

$$\begin{aligned} P_2 &= \left\| -\sum_{i=0}^{r-1} \binom{r}{i} h^{(r-i)}(x)\, B_{n,\gamma}^{\alpha,\beta}(f,x) \right. \\ &\quad \left. + \left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)}\left(f(t)\left(h(t)-h(x)\right)\chi_2(t), x\right) \right\|_{C[a_2^*,b_2^*]} + O\left(n^{-1}\right) \\ &= \left\| P_3 + P_4 \right\|_{C[a_2^*,b_2^*]} + O\left(n^{-1}\right). \end{aligned}$$

Using theorem 2, we get

$$\begin{aligned} P_3 &= \sum_{i=0}^{r} \binom{r}{i} h^{(r-i)}(x)\, f^{(i)}(x) + O\left(n^{-\frac{\zeta}{2}}\right) \\ &= -(f\ h)^{(r)}(x) + h(x)\ f^{(r)}(x) + O\left(n^{-\frac{\zeta}{2}}\right), \end{aligned}$$

uniformly in $x \in [a_2^*, b_2^*]$. Applying Taylor's expansion of $f(t)$, we have

$$\begin{aligned} P_4 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_{n,\gamma}^{(r)}(x,t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^i \left[h\left(\frac{nt+\alpha}{n+\beta}\right) - h(x)\right] \chi(t)\ dt \\ &+ \int_0^\infty W_{n,\gamma}^{(r)}(x,t) \left[\frac{f^{(r)}(\xi) - f^{(r)}(x)}{r!}\right] \left(\frac{nt+\alpha}{n+\beta} - x\right)^r \left[h\left(\frac{nt+\alpha}{n+\beta}\right) - h(x)\right] \chi(t)\ dt \\ &= P_5 + P_6, \end{aligned}$$

where $\xi \in [t, x]$. Using theorem 2, we obtain

$$\begin{aligned} P_5 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_{n,\gamma}^{(r)}(x,t) \left(\frac{nt+\alpha}{n+\beta} - x\right)^i \left[h\left(\frac{nt+\alpha}{n+\beta}\right) - h(x)\right] dt + O\left(n^{-1}\right) \\ &= P_7 + O\left(n^{-1}\right) \end{aligned}$$

uniformly in $x \in [a_2^*, b_2^*]$. Again using Taylor's expansion of $h \in C_0^\infty$, we get

$$P_7 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_{n,\gamma}^{(r)}(x,t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^i [h(x)$$

$$\left. + \sum_{j=1}^{r+2} \frac{h^{(j)}(x)}{j!} \left(\frac{nt+\alpha}{n+\beta} - x \right)^j + \epsilon(t,x) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{r+2} - h(x) \right] dt$$

where $\epsilon(t,x) \to 0$ as $t \to x$)

$$= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=1}^{r+2} \frac{h^{(j)}(x)}{j!} \int_0^\infty W_{n,\gamma}^{(r)}(x,t) \left(\frac{(nt+\alpha)}{n+\beta} - x \right)^{i+j} dt$$

$$+ \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_{n,\gamma}^{(r)}(x,t)\, \epsilon(x,t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^{i+r+2} dt$$

$$= P_8 + P_9.$$

Since $\int_0^\infty W_{n,\gamma}^{(r)}(x,t) \left(\frac{nt+\alpha}{n+\beta} - x \right)^k dt = 0 \ \forall \ k < r$. Therefore by theorem 2, we obtain

$$P_8 = \sum_{j=1}^{r} \binom{r}{j} h^{(j)}(x)\, f^{(r-j)}(x) + O\left(n^{-1}\right) \quad uniformly\ in \quad x \in [a_2^*, b_2^*]$$

$$= (h\, f)^{(r)}(x) - h(x)\, f^{(r)}(x) + O\left(n^{-1}\right).$$

By some simple computation we can show that $P_9 = O\left(n^{-\frac{\zeta}{2}}\right)$ uniformly in $x \in [a_2^*, b_2^*]$. Applying lemma 3, the mean value theorem and Schwarz inequality, we obtain

$$\|P_6\|_{C[a_2^*,b_2^*]} \le \sum_{\substack{2i+j \le r \\ i,j \ge 0}} (n)^{i+j} \left\| \frac{Q_{i,j,r,\gamma}}{[x(1+\gamma\, x)]^r} \int_0^\infty W_n(x,t) \left| \frac{nt+\alpha}{n+\beta} - x \right|^{\delta+r+1} \right.$$

$$\left. \frac{\left(\left|f^{(r)}(\xi) - f^{(r)}(x)\right|\right)}{r!} \left|h'(\eta)\right| \chi(t)\, dt \right\|_{C[a_2^*,b_2^*]} = O\left(n^{-\frac{\delta}{2}}\right),$$

where δ is chosen in such a way that $0 \le \delta \le 2 - \zeta$ and η lying between t and x. Now, combining the inequalities P_1 to P_9, we obtain

$$\left\| \left(B_{n,\gamma}^{\alpha,\beta}\right)^{(r)}(f\ h, x) - (f\ h)^{(r)}(x) \right\|_{C[a_2^*,b_2^*]} = O\left(n^{-\frac{\zeta}{2}}\right).$$

Since *supp* $fh \subset (a_2^*, b_2^*)$, Therefore by lemma 3 and 4, we get $(fh)^{(r)} \in Lip^*(\zeta, a_2^*, b_2^*)$, which gives $f^{(r)} \in Lip^*(\zeta, a_2, b_2)$ as $h(x) = 1$ on $[a_2, b_2]$. For case $1 < \zeta < 2$, the proof is completed. This completes the proof of the theorem. □

Acknowledgments

We thank the anonymous referees for valuable comments which led to improvement of the paper and the editor for giving us an opportunity to submit the revised version.

References

[1] P. Patel, V. N. Mishra, On Simultaneous Approximation of Modified Baskakov-Durrmeyer Operators, International Journal of Analysis, vol. 2015, Article ID 805395 (2015), 10 pages.

[2] P. Patel, V . N. Mishra, Approximation Properties of Certain Summation Integral Type Operators, Demonstratio Mathematica, 48(1) (2015) 77-90.

[3] Z. Finta, On Converse approximation theorem, J. Math. Anal. Appl., 312(1) (2005) 159-180.

[4] A.R. Gairola, Deepmala, L.N. Mishra, Rate of Approximation by Finite Iterates of q-Durrmeyer Operators, Proceedings of the National Academy of Sciences ndia Section A: Physical Sciences, 86(2) (2016) 229-234.

[5] G. Icoz, R. N. Mohapatra, Approximation properties by q-Durrmeyer-Stancu operators, Anal. Theory Appl., 29(4) (2013) 373-383.

[6] P. Patel, V. N. Mishra, A note on Simultaneous Approximation of some Integral Generalization of the Lupas operators, Asian Journal of Mathematics and Computer Research, 4(1) (2015) 28-44.

[7] Wang Li, The Voronovskaja type expansion formula of the modified Baskakov-Beta operators, J. Baoji University Arts Sci. (Natural Science Edition), 25 (2005) 94-97, 479-492.

[8] V N Mishra, K Khatri, and L N Mishra, Statistical approximation by Kantorovich-type discrete q-Beta operators, Advances in Difierence Equations, 345(1) (2013) doi:10.1186/1687-1847-2013-345.

[9] V N Mishra, K Khatri, L N Mishra and Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, 2013:586, (2013), doi:10.1186/1029-242X-2013-586.

[10] R B Gandhi, Deepmala and V N Mishra. Local and global results for modified SzászMirakjan operators. Mathematical Methods in the Applied Sciences 40(7) (2017) 2491-2504.

Department of Mathematics, St. Xaviers College (Autonomous), Ahmedabad-380 009 (Gujarat), India
E-mail address: prashant225@gmail.com

Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Madhya Pradesh, India
L. 1627 Awadh Puri Colony Beniganj Phase -III, Opposite-Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad-224 001 (Uttar Pradesh), India
E-mail address: vishnunarayanmishra@gmail.com; vishnu_narayanmishra@yahoo.co.in