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n-Dimensional Extended Index Matrices Part 1

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Abstract. Index Matrices (IMs) are extensions of the ordinary matrices. They are also object of extensions and modifications, e.g., extended index matrices. In the present research, we describe extended index matrices, havingd as elements whole index matrices.

Keywords: Function, Index matrix, Operation, Operator, Relation.
AMS Classification: 11C20.

1 Introduction

The concept of n-dimensional Index Matrix (IM) was introduced in 1984 for the case of $n = 2$ (see [1, 2, 4]), and for the case $n = 3$ (see [3, 4]). In [4, 5] the Extended IM (EIM) was introduced as an extension of the standard IM changing its elements with arbitrary objects, over which some arithmetic operations can be defined. In [7, 9, 11], the 2-dimensional IM operations and relations were extended for the 3-dimensional EIM.

Here, for the first time, we discuss the general case, when n is an arbitrary natural number greater than 3.

Firstly, we give the definition of a 2-dimensional EIM (2-DEIM).

Let $\mathcal{I}$ be a fixed set of indices,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \leq j \leq n)(i_j \in \mathcal{I})\}$$

and

$$\mathcal{I}^* = \bigcup_{1 \leq n \leq \infty} \mathcal{I}^n.$$

Let $\mathcal{X}$ be a fixed set of some objects. In particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, etc.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be fixed.

A 2-dimensional EIM with index sets K and L ($K, L \subset \mathcal{I}^*$) and elements from set $\mathcal{X}$ is called the object (see, [5, 4]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_j} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1} & \ldots & a_{k_i,l_j} & \ldots & a_{k_i,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_j} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, for $1 \le i \le m$, and $1 \le j \le n$: $a_{k_i,l_j} \in \mathcal{X}$.

The 3-dimensional EIM has more complex form. In [4], for brevity, it is called 3-dimensional IM, but it would more correct, if it were called there 3-dimensional EIM, as it is done in the papers [7, 9, 11].

Following [3, 4], we call "3D-IM" with index sets K, L and M ($K, L, H \subset \mathcal{I}$) the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}]$$

$$\equiv \left\{ \begin{array}{c|ccccc} h_g & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \ldots & a_{k_1,l_j,h_g} & \ldots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \ldots & a_{k_i,l_j,h_g} & \ldots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \ldots & a_{k_m,l_j,h_g} & \ldots & a_{k_m,l_n,h_g} \end{array} | h_g \in H \right\}$$

$$\equiv \left\{ \begin{array}{c|ccccc} h_1 & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1,h_1} & \ldots & a_{k_1,l_j,h_1} & \ldots & a_{k_1,l_n,h_1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1,h_1} & \ldots & a_{k_i,l_j,h_1} & \ldots & a_{k_i,l_n,h_1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1,h_1} & \ldots & a_{k_m,l_j,h_1} & \ldots & a_{k_m,l_n,h_1} \end{array} \right.,$$

$$
\begin{array}{c|ccccc}
h_2 & l_1 & \ldots & l_j & \ldots & l_n \\
\hline
k_1 & a_{k_1,l_1,h_2} & \ldots & a_{k_1,l_j,h_2} & \ldots & a_{k_1,l_n,h_2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
k_i & a_{k_i,l_1,h_2} & \ldots & a_{k_i,l_j,h_2} & \ldots & a_{k_i,l_n,h_2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
k_m & a_{k_m,l_1,h_2} & \ldots & a_{k_m,l_j,h_2} & \ldots & a_{k_m,l_n,h_2}
\end{array},
$$

$$
\ldots, \left.
\begin{array}{c|ccccc}
h_f & l_1 & \ldots & l_j & \ldots & l_n \\
\hline
k_1 & a_{k_1,l_1,h_f} & \ldots & a_{k_1,l_j,h_f} & \ldots & a_{k_1,l_n,h_f} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
k_i & a_{k_i,l_1,h_f} & \ldots & a_{k_i,l_j,h_f} & \ldots & a_{k_i,l_n,h_f} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
k_m & a_{k_m,l_1,h_f} & \ldots & a_{k_m,l_j,h_f} & \ldots & a_{k_m,l_n,h_f}
\end{array}\right\},
$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$, $H = \{h_1, h_2, ..., h_f\}$, and for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq g \leq f : a_{k_i,l_j,h_g} \in \mathcal{X}$.

We see that the 2-DEIM has the form of *one* IM, the 3-DEIM – of f in number IMs. Now, we introduce the concept of an n-DEIM and we see that it has essentially more complex form.

2 Definition of an n-Dimensional Extended Index Matrix

An n-Dimensional EIM (n-DEIM) with index sets $K_1, K_2, ..., K_n$ ($K_1, K_2, ..., K_n \subseteq \mathcal{I}^*$) and elements from set $\mathcal{X}$ is called the object:

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

where $K_i = \{k_{i,1}, k_{i,2}, ..., k_{i,m_i}\}$, $m_i \geq 1$ and $a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}} \in \mathcal{X}$ for $1 \leq i \leq n$ and $1 \leq s_i \leq m_i$.

The n-DEIM A has the following $\frac{n(n-1)}{2}$ different representations as 2-DIM:

$$A = \begin{array}{c|ccc} \langle k_{3,s_3}, ..., k_{n,s_n} \rangle & k_{2,1} & \ldots & k_{2,m_2} \\ \hline k_{1,1} & a_{k_{1,1},k_{2,1},k_{3,s_3},...,k_{n,s_n}} & \ldots & a_{k_{1,1},k_{2,m_2},k_{3,s_3},...,k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,i} & a_{k_{1,i},k_{2,1},k_{3,s_3},...,k_{n,s_n}} & \ldots & a_{k_{1,i},k_{2,m_2},k_{3,s_3},...,k_{n,s_n}} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1,m_1} & a_{k_{1,m_1},k_{2,1},k_{3,s_3},...,k_{n,s_n}} & \ldots & a_{k_{1,m_1},k_{2,m_2},k_{3,s_3},...,k_{n,s_n}} \end{array}$$

$$= \ldots$$

$$= \begin{array}{c|cc} \langle k_{1,s_1}, ..., k_{n-2,s_{n-2}} \rangle & k_{n,1} & \ldots \\ \hline k_{n-1,1} & a_{k_{1,s_1},...,k_{n-2,s_{n-2}},k_{n-1,1},k_{n,1}} & \ldots \\ \vdots & \vdots & \ddots \\ k_{n-1,i} & a_{k_{1,s_1},...,k_{n-2,s_{n-2}},k_{n-1,i},k_{n,1}} & \ldots \\ \vdots & \vdots & \ddots \\ k_{n-1,m_{n-1}} & a_{k_{1,s_1},...,k_{n-2,s_{n-2}},k_{n-1,m_{n-1}},k_{n,1}} & \ldots \end{array}$$

3 Operations over n-Dimensional Extended Index Matrices

Here, we introduce the basic operations over n-DEIMs, by analogy with [3, 4].

First, we mention that in [4] it is shown that there are 6 $(= 3!)$ transpositions of a 3-DEIM. Obviously, in the general case of n-DEIM there will exist $n!$ transpositions, one basic and $n! - 1$ others.

Let us define

$$[K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]^{[j_1,j_2,...,j_n]}$$
$$= [K_{j_1}, K_{j_2}, ..., K_{j_n}, \{a_{k_{j_1,s_{j_1}},k_{j_2,s_{j_2}},...,k_{j_n,s_{j_n}}}\}].$$

Obviously, the basic transposition is

$$[K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]^{[1,2,...,n]} = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}].$$

Second, let us have two n-DEIMs

$$A = [K_1, K_2, ..., K_n, \{a_{k_{1,s_1},k_{2,s_2},...,k_{n,s_n}}\}]$$

and

$$B = [L_1, L_2, ..., L_n, \{b_{l_{1,t_1}, l_{2,t_2}, ..., l_{n,t_n}}\}].$$

Let everywhere below i be a natural number for which $1 \leq i \leq n$. Following [4] we introduce the following operations over A and B.

3.1 Operation "addition"

$$A \oplus_{(\circ)} B = [K_1 \cup L_1, K_2 \cup L_2, ..., K_n \cup L_n, \{c_{p_{1,q_1}, p_{2,q_2}, ..., p_{n,q_n}}\}],$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, ..., p_{n,q_n}} = \begin{cases} a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} \in K_i \\ & \text{and there is } i \text{ such that } k_{i,s_i} \notin L_i \\ b_{l_{1,t_1}, l_{2,t_2}, ..., l_{n,t_n}}, & \text{if for each } i\text{: } p_{i,q_i} = l_{i,t_i} \in K_i \\ & \text{and there is } i \text{ such that } l_{i,t_i} \notin K_i \\ a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} \\ \circ\, b_{l_{1,t_1}, l_{2,t_2}, ..., l_{n,t_n}}, & \text{if for each } i\text{: } p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i \\ \bot, & \text{otherwise} \end{cases}$$

Here and below, symbol $\bot$ denotes an undefined value. When the elements of A and B are natural, real or complex numbers, or elements of set $\{0, 1\}$, then $\bot$ can be 0; when these elements are propositions or predicates, it can be $false$; when they are intuitionistic fuzzy pairs (see [6]) it can be $\langle 0, 1 \rangle$.

3.2 Operation "termwise multiplication"

$$A \otimes_{(\circ)} B = [K_1 \cap L_1, K_2 \cap L_2, ..., K_n \cap L_n, \{c_{p_{1,q_1}, p_{2,q_2}, ..., p_{n,q_n}}\}],$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, ..., p_{n,q_n}} = a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} \circ b_{l_{1,t_1}, l_{2,t_2}, ..., l_{n,t_n}}$$

for each i, $p_{i,q_i} = k_{i,s_i} = l_{i,t_i} \in K_i \cap L_i$.

In [4], 6 different operations **multiplication** were introduced. We will discuss them in the next Section, essentially extending their number.

3.3 Operation "structural subtraction"

$$A \ominus B = [K_1 - L_1, K_2 - L_2, ..., K_n - L_n, \{c_{p_{1,q_1}, p_{2,q_2}, ..., p_{n,q_n}}\}],$$

where

$$c_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}} = a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}$$

for each i, $p_{i,q_i} = k_{i,s_i} \in K_i - L_i$.

3.4 Operation "multiplication with a constant"

$$\alpha.A = [K_1, K_2, \dots, K_n, \{\alpha a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}\}],$$

where α is a (real) constant.

3.5 Operation "projection"

Let $L_i \subseteq K_i$. Then,

$$pr_{L_1,L_2,\dots,L_n} A = [L_1, L_2, \dots, L_n, \{b_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}}\}],$$

where

$$b_{l_{1,t_1},l_{2,t_2},\dots,l_{n,t_n}} = a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}$$

for each i, $l_{i,t_i} = k_{i,s_i} \in L_i$.

3.6 Operation "reduction" over an 3D-IM

First, we introduce operation $(\bot, \dots, \bot, k_{i,s_i}, \bot, \dots, \bot)$-reduction of a given n-DEIM A by

$$A_{(\bot,\dots,\bot,k_{i,s_i},\bot,\dots,\bot)} = [K_1, \dots, K_{i-1}, K_i - \{k_{i,s_i}\}, K_{i+1}, \dots, K_n, \{c_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}\}]$$

where

$$c_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}} = a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}$$

and $p_{j,q_j} = k_{j,s_j} \in K_j$ for $1 \le j \le n$ and $j \ne i$; and $p_{i,r} = k_{i,r} \in K_i$ for $1 \le r \le m_i$ and $r \ne s_i$.

Second, we define

$$\begin{aligned} A_{(k_{1,p_1},k_{2,p_2},\dots,k_{n,p_n})} &= (\dots(A_{(k_{1,p_1},\bot,\dots,\bot)})_{(\bot,k_{2,p_2},\bot,\dots,\bot)}\dots)_{(\bot,\dots,\bot,k_{n,p_n})} \\ &= [K_1 - \{k_{1,p_1}\}, K_2 - \{k_{2,p_2}\}, \dots, K_n - \{k_{n,p_n}\}, \{c_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}}\}], \end{aligned}$$

where

$$c_{p_{1,q_1},p_{2,q_2},\dots,p_{n,q_n}} = a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}$$

and for each i: $p_{i,q_i} = k_{i,s_i} \in K_i - \{k_{i,p_i}\}$ and $1 \le p_{i,q_i} \le m_i$.

Third, let for each i: $L_i = \{l_{i,q_1}, l_{i,q_2}, \dots, l_{i,q_{r_i}}\} \subseteq K_i$, where $1 \le r_i \le m_i$. Now, we define the following operations:

$$A_{(\bot,\dots,\bot,L_i,\bot,\dots,\bot)} = (\dots((A_{(\bot,\dots,\bot,l_{i,q_1},\bot,\dots,\bot)})_{(\bot,\dots,\bot,l_{i,q_2},\bot,\dots,\bot)})\dots)_{(\bot,\dots,\bot,l_{i,q_{r_i}},\bot,\dots,\bot)},$$

$$A_{(L_1,L_2,\dots,L_n)} = (\dots((A_{(L_1,\bot,\dots,\bot)})_{(\bot,L_2,\bot,\dots,\bot)})\dots)_{(\bot,\dots,\bot,L_n)}.$$

3.7 Operation "substitution"

Let the n-DEIM A be given.

First, local substitution over A is defined for the pairs of indices $(l_{1,p_1} k_{1,p_1})$ and/or (l_{2,p_2}, k_{2,p_2}) and/or ... and/or (l_{n,p_n}, k_{n,p_n}), respectively for arbitrary indices p_i $(1 \leq p_i \leq m_i)$, by

$$\left[\frac{l_{1,p_1}}{k_{1,p_1}}; \perp; ...; \perp\right] A = \left[(K_1 - \{k_{1,p_1}\}) \cup \{l_{1,p_1}\}, K_2, ..., K_n, \{a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}\}\right]$$

$$\left[\perp; \frac{l_{2,p_2}}{k_{2,p_2}}; \perp; ...; \perp\right] A = \left[K_1, (K_2 - \{k_{2,p_2}\}) \cup \{l_{2,p_2}\}, K_3, ..., K_n, \{a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}\}\right]$$

$$\cdots$$

$$\left[\perp; ...; \perp; \frac{l_{n,p_n}}{k_{n,p_n}}\right] A = \left[K_1, K_2, ..., K_{n-1}, (K_n - \{k_{n,p_n}\}) \cup \{l_{n,p_n}\}, \{a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}\}\right].$$

Second,

$$\left[\frac{l_{1,p_1}}{k_{1,p_1}}; \frac{l_{2,p_2}}{k_{2,p_2}}; ...; \frac{l_{n,p_n}}{k_{n,p_n}}\right] A = \left[\frac{l_1}{k_1}; \perp; ...; \perp\right] \left[\perp; \frac{l_2}{k_2}; \perp; ...; \perp\right] ... \left[\perp; \perp; ...; \perp; \frac{l_n}{k_n}\right] A$$

$$= [(K_1 - \{k_{1,p_1}\}) \cup \{l_{1,p_1}\}, (K_2 - \{k_{2,p_2}\}) \cup \{l_{2,p_2}\}, ..., (K_n - \{k_{n,p_n}\}) \cup \{l_{n,p_n}\}, \{a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}\}\Big].$$

Let the sets of indices $L_1, L_2, ..., L_n$ be given, where $card(L_i) = card(K_i)$.

Third, for them we define sequentially:

$$\left[\perp; ...; \perp; \frac{L_i}{K_i}; \perp; ...; \perp\right] A = \left[\perp; ...; \perp; \frac{l_{i,1}}{k_{i,1}} \frac{l_{i,2}}{k_{i,2}} ... \frac{l_{i,m_i}}{k_{i,m_i}}; \perp; \perp\right] A,$$

$$\left[\frac{L_1}{K_1}; \frac{L_2}{K_2}; ...; \frac{L_n}{K_n}\right] A = \left[\frac{L_1}{K_1}; \perp; ...; \perp\right] \left[\perp; \frac{L_2}{K_2}; \perp; ...; \perp\right] ... \left[\perp; ...; \perp; \frac{L_n}{K_n}\right] A.$$

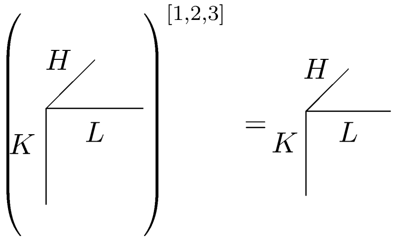
4 Operation "multiplication"

In this section, initially we discuss the case $n = 3$ and after this, the general case.

First, following [4], we start with operation "transposition".

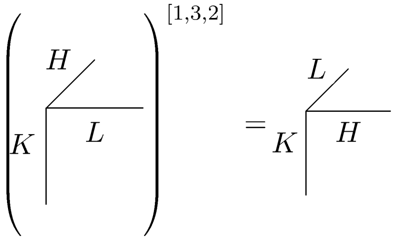
As we saw in [4], there are 2 (= 2!) EIMs, related to this operation: the standard EIM and its transposed EIM. For 3D-EIMs, there are 6 (=3!) cases: the standard 3DEIM and five different transposed 3DEIMs. The geometrical and analytical forms of the individual transposed 3DEIMs are the following.

$[1,2,3]$**-transposition (identity)**



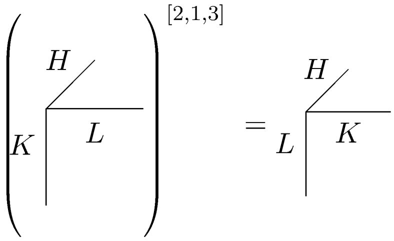
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,2,3]} = [K, L, H, \{a_{k_i,l_j,h_g}\}];$$

$[1,3,2]$**-transposition**



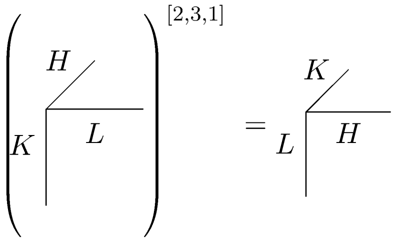
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[1,3,2]} = [K, H, L, \{a_{k_i,h_g,l_j}\}];$$

$[2,1,3]$**-transposition**



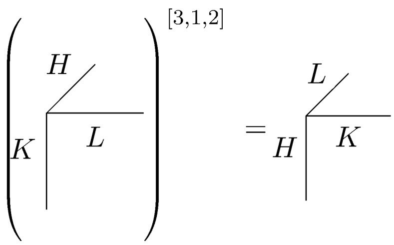
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,1,3]} = [L, K, H, \{a_{l_j,k_i,h_g}\}];$$

$[2,3,1]$**-transposition**



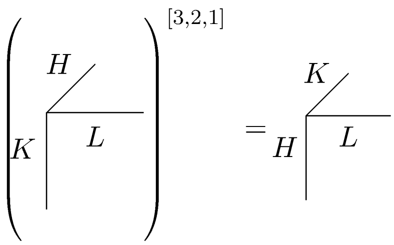
$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[2,3,1]} = [L, H, K, \{a_{l_j,h_g,k_i}\}];$$

$[3, 1, 2]$**-transposition**



$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,1,2]} = [H, K, L, \{a_{h_g,k_i,l_j}\}];$$

$[3, 2, 1]$**-transposition**



$$[K, L, H, \{a_{k_i,l_j,h_g}\}]^{[3,2,1]} = [H, L, K, \{a_{h_g,l_j,k_i}\}].$$

Let $[x, y, z]$ be a permutation of the triple $[1, 2, 3]$.

In [4], there are six different operations "multiplication".

Let the 3-DEIMs

$$A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$$

and

$$B = [P, Q, R, \{b_{p_r,q_s,e_d}\}]$$

be given.

Operation $[x, y, z]$-multiplication is defined by:

$$A \odot_{(\circ,*)}^{[x,y,z]} B = A \odot_{(\circ,*)} B^{[x,y,z]},$$

where the first multiplication operation $\odot_{(\circ,*)}^{[x,y,z]}$ is:

$$A \odot_{(\circ,*)} B = [K \cup (P - L), Q \cup (L - P), H \cup R, \{c_{t_u,v_w,x_y}\}],$$

where

$$c_{t_u,v_w,x_y}$$

$$= \begin{cases} a_{k_i,l_j,h_g}, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \text{ and } x_y = h_g \in H \\ & \text{or } t_u = k_i \in K - P - Q \\ & \text{and } v_w = l_j \in L \text{ and } x_y = h_g \in H \\ \\ b_{p_r,q_s,e_d}, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L - K \text{ and } x_y = e_d \in R \\ & \text{or } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L - K \text{ and } x_y = e_d \in R \\ \\ \underset{l_j = p_r \in L \cap P}{\circ} a_{k_i,l_j,h_g} * b_{p_r,q_s,e_d}, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ & \text{and } x_y = h_g = e_d \in H \cap R \\ \\ \bot, & \text{otherwise} \end{cases}$$

Now, we introduce two other operations "multiplication" and a new form of the above operation for the case, when A and B are 3-DEIM in the forms from Section 3 for $n = 3$.

4.1 First operation "multiplication"

$$A \odot_{(\circ,*,1)} B = [K_1, L_3, \{c_{p_{1,q_1},p_{2,q_2}}\}],$$

where

$$c_{p_{1,q_1},p_{2,q_2}} =$$

$$= \begin{cases} \underset{\substack{k_{2,s_2} = l_{1,t_1} \in K_2 \cap L_1 \\ k_{3,s_3} = l_{2,t_2} \in K_3 \cap L_2}}{\circ} (a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}} * b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}), & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \\ & \text{and } p_{2,q_2} = l_{3,t_3} \in L_3 \\ \\ \bot, & \text{otherwise} \end{cases}$$

Therefore, this operation generates a 2-DEIM, i.e., decreases the number of dimensions.

Number 1 in the record $\odot_{(\circ,*,1)}$ means that the first component of the first 3-DEIM and the third (i.e., the frist in the opposite order) component of the second 3-DEIM are used as the two components of the new 3-DEIM. These two components are interpreted as the first and last components of the new 3-DEIM.

4.2 Second operation "multiplication"

$$A \odot_{(\circ,*,1,1)} B = [K_1 \cup (L_2 - K_2), K_3 \cup L_1, L_3 \cup (K_2 - L_2), \{c_{p_{1,q_1},p_{2,q_2},p_{3,q_3}}\}],$$

where

$$c_{p_{1,q_1},p_{2,q_2},p_{3,q_3}} =$$

$$= \begin{cases} a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}}, & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \\ & \quad \text{and } p_{2,q_2} = k_{2,s_2} \in K_2 - L_1 - L_2 \\ & \quad \text{and } p_{3,q_3} = k_{3,s_3} \in K_3 \\ & \text{or } p_{1,q_1} = k_{1,s_1} \in K_1 - L_1 - L_2 \\ & \quad \text{and } p_{2,q_2} = k_{2,s_2} \in K_2 \\ & \quad \text{and } p_{3,q_3} = k_{3,s_3} \in K_3 \\ b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}, & \text{if } p_{1,q_1} = l_{1,t_1} \in L_1 \\ & \quad \text{and } p_{2,q_2} = l_{2,t_2} \in L_2 - K_1 - K_2 \\ & \quad \text{and } p_{3,q_3} = l_{3,t_3} \in L_3 \\ & \text{or } p_{1,q_1} = l_{1,t_1} \in L_1 \\ & \quad \text{and } p_{2,q_2} = l_{2,t_2} \in L_2 - K_1 - K_2 \\ & \quad \text{and } p_{3,q_3} = l_{3,t_3} \in L_3 \\ \underset{p_{2,q_2} = l_{1,t_1} \in K_2 \cap L_1}{\circ} (a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}}, & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \;\&\; p_{2,q_2} = l_{2,t_2} \in L_2 \\ \quad * b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}), & \text{and } p_{3,q_3} = k_{3,s_3} = l_{3,t_3} \in K_3 \cap L_3 \\ \bot, & \text{otherwise} \end{cases}$$

This operation can be represented by the older multiplication, because

$$A \odot_{(\circ,*,1,1)} B = [K_1 \cup (L_2 - K_2), K_3 \cup L_1, L_3 \cup (K_2 - L_2), \{c_{p_{1,q_1},p_{2,q_2},p_{3,q_3}}\}]$$

$$= [K_1 \cup (L_2 - K_2), L_3 \cup (K_2 - L_2), K_3 \cup L_1, \{c_{p_{1,q_1},p_{3,q_3},p_{2,q_2}}\}]^{[1,3,2]}$$

$$= ([K_1, K_2, K_3, \{a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}}\}] \odot_{(\circ,*)} [L_2, L_3, L_1, \{b_{l_{2,t_2},l_{3,t_3},l_{1,t_1}}\}])^{[1,3,2]}$$

$$= ([K_1, K_2, K_3, \{a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}}\}] \odot_{(\circ,*)} [L_1, L_2, L_3, \{b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}\}]^{[3,1,2]})^{[1,3,2]}$$

$$= ([K_1, K_2, K_3, \{a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}}\}] \odot_{(\circ,*)}^{[3,1,2]} [L_1, L_2, L_3, \{b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}\}])^{[1,3,2]}.$$

Therefore, this operation generates a 3-DEIM, i.e., preserves the number of dimensions.

The particular case when index sets $K_2 = L_2$, is very interesting. In it we obtain

$$A \odot_{(\circ,*,1,1)} B = [K_1, K_3 \cup L_1, L_3, \{c_{p_{1,q_1},p_{2,q_2},p_{3,q_3}}\}],$$

where

$$c_{p_{1,q_1},p_{2,q_2},p_{3,q_3}} =$$

$$= \begin{cases} \underset{p_{2,q_2}=l_{1,t_1}\in K_2\cap L_1}{\circ} (a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}}, & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \ \& \ p_{2,q_2} = l_{2,t_2} \in L_2 \\ \quad *b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}), & \text{and } p_{3,q_3} = k_{3,s_3} = l_{3,t_3} \in K_3 \cap L_3 \\ \perp, & \text{otherwise} \end{cases}$$

Exactly this form of the second operation corresponds to the record $\odot_{(\circ,*,1,1)}$. Here, the first 1 has the above sense, while the second 1 means that one component from the first and of the second 3-DEIM (the third of the first 3-DEIM and the first of the second 3-DEIM) will be used as a second component of the resulting 3-DEIM.

4.3 Third operation "multiplication"

$$A \odot_{(\circ,*,2)} B = [K_1, K_2, L_3, L_4, \{c_{p_{1,q_1},p_{2,q_2},p_{3,q_3},p_{4,q_4}}\}].$$

where

$$c_{p_{1,q_1},p_{2,q_2},p_{3,q_3},p_{4,q_4}} =$$

$$\begin{cases} \underset{k_{3,s_3}=l_{1,t_1}\in K_3\cap L_1}{\circ} (a_{k_{1,s_1},k_{2,s_2},k_{3,s_3}} * b_{l_{1,t_1},l_{2,t_2},l_{3,t_3}}), & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \\ & \text{and } p_{2,q_2} = k_{2,s_2} \in K_2 \\ & \text{and } p_{3,q_3} = l_{2,t_2} \in L_2 \\ & \text{and } p_{4,q_4} = l_{3,t_3} \in L_3 \\ \perp, & \text{otherwise} \end{cases}$$

Therefore, this operation generates a 4-DEIM, i.e., increases the number of dimensions.

Number 2 in the record $\odot_{(\circ,*,2)}$ means that the first two components of the first 3-DEIM and the second two (i.e., the frist two in the opposite order) components of the second 3-DEIM are used as the four components of the new already 4-DEIM.

Following the sense of the introduced notation, we can define the general case of the operation multiplication.

4.4 General case of the operation "multiplication"

Let A and B be the two n-DEIM from Section 3. Then, we define one of the possible general cases for operation multiplication. It includes the first, third and the particular case of the second operations introduced above. By the moment, an **Open Problem** is what is the general form for the second multiplication.

Let α and β be two natural numbers for which $1 \le \alpha \le n-1$, $0 \le \beta \le n-2$ and $\alpha + \beta \le n-1$. Then, we obtain the following $(2\alpha+\beta)$-DEIM

$$A \odot_{(\circ,*,\alpha,\beta)} B = [K_1, ..., K_\alpha, K_{n-\beta+1} \cup L_1, ..., K_n \cup L_\beta, L_{n-\alpha+1}, ..., L_n, \{c_{p_{1,q_1}, p_{2,q_2}, ..., p_{2\alpha+\beta, q_{2\alpha+\beta}}}\}],$$

where

$$c_{p_{1,q_1}, p_{2,q_2}, ..., p_{2\alpha+\beta, q_{2\alpha+\beta}}}$$

$$= \begin{cases} \underset{\substack{k_{\alpha+1,s_{\alpha+1}} = l_{\beta+1,t_{\beta+1}} \in K_{\alpha+1} \cap L_{\beta+1} \\ \dots \\ k_{n-\beta,s_{n-\beta}} = l_{n-\alpha,t_{n-\alpha}} \in K_{n-\beta} \cap L_{n-\alpha}}}{\circ} (a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} * b_{l_{1,t_1}, l_{2,t_2}, ..., l_{n,t_n}}), & \text{if } p_{1,q_1} = k_{1,s_1} \in K_1 \\ & \dots \\ & p_{\alpha,q_\alpha} = k_{\alpha,s_\alpha} \in K_\alpha \\ & p_{\alpha+1,q_{\alpha+1}} = k_{n-\beta+1,s_{n-\beta+1}} \\ & = l_{1,t_1} \in K_{n-\beta+1} \cap L_1 \\ & \dots \\ & p_{\alpha+\beta,q_{\alpha+\beta}} = k_{n,s_n} \\ & = l_{\beta,t_\beta} \in K_n \cap L_\beta \\ & p_{\alpha+\beta+1,q_{\alpha+\beta+1}} \\ & = l_{n-\alpha+1,t_{n-\alpha+1}} \in L_{n-\alpha+1} \\ & \dots \\ & p_{2\alpha+\beta,q_{2\alpha+\beta}} = l_{n,t_n} \in L_n \\ \perp, & \text{otherwise} \end{cases}$$

5 Relations over n-DEIMs

Let the two n-DEIMs A and B be given. We will introduce the following definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion"*.
The strict relation "inclusion about dimension" is

$$A \subset_d B \text{ iff } ((\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \ \& \ (\exists j : 1 \le j \le n)(K_j \ne L_j))$$
$$\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} = b_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}).$$

The non-strict relation "inclusion about dimension" is

$$A \subseteq_d B \text{ iff } ((\forall i)(K_i \subseteq L_i)\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} = b_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}).$$

The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i)\&(\forall i)(\forall s_i : 1 \le s_i \le m_i)(a_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}} < b_{k_{1,s_1}, k_{2,s_2}, ..., k_{n,s_n}}).$$

The non-strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (\forall i)(K_i = L_i)\&(\forall i)(\forall s_i : 1 \le s_i \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} \le b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The strict relation "inclusion" is

$$A \subset_d B \text{ iff } ((\forall i)(K_i \subset L_i) \vee (\forall i)(K_i \subseteq L_i \ \& \ (\exists j : 1 \le j \le n)(K_j \ne L_j))$$
$$\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} < b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

The non-strict relation "inclusion" is

$$A \subseteq_d B \text{ iff } ((\forall i)(K_i \subseteq L_i)\&(\forall i)(\forall j : 1 \le j \le m_i)(a_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}} \le b_{k_{1,s_1},k_{2,s_2},\dots,k_{n,s_n}}).$$

6 Conclusion

The results from present paper extend the idea for index matrices. As we have shown in Section 4, in the case of n-DEIMs new types of operations (in this case – operation "multiplication") arise. In future, we will introduce some new operations without analogues in matrix theory, as well as in IM theory.

From the definition of n-DEIMs it is clear, that they are an essential extension of the concept of a tensor, too. At the moment, an **Open Problem** is whether some of the IM-operations can be transferred to tensors.

Following ideas from [8, 10], we see that the n-DEIMs give a suitable model of the OLAP-cube. But as it is clear, there are a lot of new operations that may be defined over the n-DEIMs. An **Open Problem** is whether all of these new operations can be translated in the terms of OLAP-cube.

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References

[1] Atanassov K., Conditions in Generalized nets, Proc. of the XIII Spring Conf. of the Union of Bulg. Math., Sunny Beach, April 1984, 219-226.

[2] Atanassov K., Generalized index matrices, Comptes rendus de l'Academie Bulgare des Sciences, Vol.40, 1987, No.11, 15-18.

[3] Atanassov, K., On index matrices. Part 5: Three dimensional index matrices. Advanced Studies in Contemporary Mathematics, Vol. 24, 2014, No. 4, 423-432.

[4] Atanassov, K., Index Matrices: Towards an Augmented Matrix Calculus, Springer, Cham, 2014.

[5] Atanassov, K. Extended index matrices. Intelligent systems2014 (P. Angelov, K. Atanassov, L. Doukovska, M. Hadjiski, V. Jotsov, J. Kacprzyk, N. Kasabov, S. Sotirov, E. Szmidt, S. Zadrozny, Eds.), Springer, Cham, 2015, 305-317.

[6] Atanassov, K. Intuitionistic Fuzzy Logics, Springer, Cham, 2017.

[7] Atanassov K., E. Szmidt, J. Kasprzyk, V. Bureva. Two Examples for the Use of 3-dimensional Intuitionistic Fuzzy Index Matrices, Notes of Intuitionistic Fuzzy Sets, Vol. 20, N 2, 2014, 52-59.

[8] Bureva, V., V. Traneva, E. Sotirova, K. Atanassov. Index matrices and OLAP-cube. Part 2: An presentation of the OLAP-analysis by index matrices. Advanced Studies in Contemporary Mathematics, 27, 4, 2017, 647-672

[9] Traneva V. Internal operations over 3-dimensional extended index matrices, Proceedings of the Jangjeon Mathematical Society, Vol. 18 (4), 2015, 547-569.

[10] Traneva, V., V. Bureva, E. Sotirova, K. Atanassov. Index matrices and OLAP-cube. Part 1: Application of the index matrices to presentation of operations in OLAP-cube. Advanced Studies in Contemporary Mathematics, 27, 2, 2017, 253-278.

[11] Traneva, V., E. Sotirova, V. Bureva, K. Atanassov. Aggregation operations over 3-dimensional extended index matrices. Advanced Studies in Contemporary Mathematics, Vol. 25, 2015, No. 3, 407-416.