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FOURIER SERIES OF SUMS OF PRODUCTS OF HIGHER-ORDER GENOCCHI FUNCTIONS

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Abstract. In this paper, we study three types of functions given by sums of products of higher-order Genocchi functions and derive their Fourier series expansions. Moreover, we express each of them in terms of Bernoulli functions from which the corresponding polynomial identities follow immediately.

1. Introduction

The Genocchi polynomials $G_m^{(r)}(x)$ of order $r(\geq 0)$ are given by

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} G_m^{(r)}(x)\frac{t^m}{m!}, \quad \text{(see [3, 5, 7, 11, 15, 18, 19])}. \tag{1.1}$$

When $x = 0$, $G_m^{(r)} = G_m^{(r)}(0)$ are called Genocchi numbers of order r. In particular, for $r = 1$, $G_m(x) = G_m^{(1)}(x)$, and $G_m = G_m^{(1)}$ are called Genocchi polynomials and numbers, respectively.

From (1.1), it is immediate to show that

$$\begin{aligned} &G_m^{(r)}(x) = 0, \text{ for } 0 \leq m \leq r-1, \ G_r^{(r)}(x) = r!, \\ &\frac{d}{dx}G_m^{(r)}(x) = mG_{m-1}^{(r)}(x), \ (m \geq 1), \\ &G_m^{(r)}(x+1) = 2mG_{m-1}^{(r-1)}(x) - G_m^{(r)}(x), \ (r, m \geq 1). \end{aligned} \tag{1.2}$$

Further, the Genocchi polynomials $G_m^{(r)}(x)$ of order r and the Euler polynomials $E_m^{(r)}(x)$ of order r are related by

$$E_m^{(r)}(x) = \frac{m!}{(m+r)!}G_{m+r}^{(r)}(x), \ (m \geq 0), \tag{1.3}$$

where $E_m^{(r)}(x)$ are defined by

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{m=0}^{\infty} E_m^{(r)}(x)\frac{t^m}{m!}, \quad \text{(see [12, 13])}. \tag{1.4}$$

In view of (1.2), we have

$$G_m^{(r)}(1) = 2mG_{m-1}^{(r-1)} - G_m^{(r)}, \ (r, m \geq 1), \tag{1.5}$$

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$$\int_0^1 G_m^{(r)}(x)dx = \frac{1}{m+1}\left(G_{m+1}^{(r)}(1) - G_{m+1}^{(r)}\right) = 2\left(G_m^{(r-1)} - \frac{1}{m+1}G_{m+1}^{(r)}\right). \tag{1.6}$$

For any real number x, the fractional part of x is denoted by

$$< x >= x - [x] \in [0, 1). \tag{1.7}$$

As is well known, the Bernoulli polynomials $B_m(x)$ are defined by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}, \quad \text{(see [2, 7, 10, 17, 20])}. \tag{1.8}$$

We will make use of the following facts about the Fourier series expansion of the Bernoulli function $B_m(< x >)$:

(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \tag{1.9}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.10}$$

Throughout this paper, we will assume that r and s are fixed positive integers. Let

$$\alpha_m(x) = \sum_{0 \leq k \leq m} G_k^{(r)}(x)G_{m-k}^{(s)}(x), \tag{1.11}$$

$$\beta_m(x) = \sum_{0 \leq k \leq m} \frac{1}{k!(m-k)!}G_k^{(r)}(x)G_{m-k}^{(s)}(x), \tag{1.12}$$

$$\gamma_m(x) = \sum_{1 \leq k \leq m-1} \frac{1}{k(m-k)}G_k^{(r)}(x)G_{m-k}^{(s)}(x). \tag{1.13}$$

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1, 16, 21]).

From (1.2), we immediately note the following:

$$\alpha_m(x) = \beta_m(x) = 0, \text{ for } 0 \leq m < r + s, \tag{1.14}$$

$$\gamma_m(x) = 0, \text{ for } 2 \leq m < r + s, \tag{1.15}$$

$$\alpha_m(x) = \sum_{r \leq k \leq m-s} G_k^{(r)}(x)G_{m-k}^{(s)}(x), \ (m \geq r + s), \tag{1.16}$$

$$\beta_m(x) = \sum_{r \leq k \leq m-s} \frac{1}{k!(m-k)!}G_k^{(r)}(x)G_{m-k}^{(s)}(x), \ (m \geq r + s), \tag{1.17}$$

$$\gamma_m(x) = \sum_{r \le k \le m-s} \frac{1}{k(m-k)} G_k^{(r)}(x) G_{m-k}^{(s)}(x), \ (m \ge r+s), \tag{1.18}$$

$$\alpha_{r+s}(x) = r!s!, \ \beta_{r+s}(x) = 1, \ \gamma_{r+s}(x) = (r-1)!(s-1)!. \tag{1.19}$$

Taking (1.11) - (1.19) into account, in this paper we will study the following three types of sums of products of higher-order Genocchi functions and find their Fourier series expansions. Moreover, we will express each of them in terms of Bernoulli functions from which the corresponding polynomial identities will easily follow.

(a) $\alpha_m(< x >) = \sum_{0 \le k \le m} G_k^{(r)}(< x >) G_{m-k}^{(s)}(< x >), \ (m > r+s),$
(b) $\beta_m(< x >) = \sum_{0 \le k \le m} \frac{1}{k!(m-k)!} G_k^{(r)}(< x >) G_{m-k}^{(s)}(< x >), \ (m > r+s),$
(c) $\gamma_m(< x >) = \sum_{1 \le k \le m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) G_{m-k}^{(s)}(< x >), \ (m > r+s).$

Before closing this section we can not go without saying that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle)$we can derive the famous Faber-Pandharipande-Zagier identity (see [6]) and the Miki's identity(see [17]). Hence our problem here is a natural extension of the previous works which lead to a simple proof for the important Faber-Pandharipande-Zagier and Miki's identities. For the details, we ask the reader to refer to [14]. Some related recent works can be found in [4, 8, 9, 14].

2. The function $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{0 \le k \le m} G_k^{(r)}(x) G_{m-k}^{(s)}(x), \ (m > r+s)$. Then we will consider the function

$$\alpha_m(< x >) = \sum_{0 \le k \le m} G_k^{(r)}(< x >) G_{m-k}^{(s)}(< x >), \ (m > r+s), \tag{2.1}$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m,r,s)} e^{2\pi i n x}, \tag{2.2}$$

where

$$A_n^{(m)} = A_n^{(m,r,s)} = \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx. \tag{2.3}$$

Before proceeding further, we observe the following.

$$\begin{aligned}\alpha'_m(x) &= \sum_{0\le k\le m}\left(kG^{(r)}_{k-1}(x)G^{(s)}_{m-k}(x)+(m-k)G^{(r)}_{k}(x)G^{(s)}_{m-k-1}(x)\right)\\ &= \sum_{1\le k\le m} kG^{(r)}_{k-1}(x)G^{(s)}_{m-k}(x)+\sum_{0\le k\le m-1}(m-k)G^{(r)}_{k}(x)G^{(s)}_{m-k-1}(x)\\ &= \sum_{0\le k\le m-1} (k+1)G^{(r)}_{k}(x)G^{(s)}_{m-k-1}(x)+\sum_{0\le k\le m-1}(m-k)G^{(r)}_{k}(x)G^{(s)}_{m-k-1}(x)\\ &= (m+1)\sum_{0\le k\le m-1} G^{(r)}_{k}(x)G^{(s)}_{m-1-k}(x)\\ &= (m+1)\alpha_{m-1}(x).\end{aligned} \tag{2.4}$$

This implies that

$$\frac{d}{dx}\left(\frac{\alpha_{m+1}(x)}{m+2}\right)=\alpha_m(x), \tag{2.5}$$

and

$$\int_0^1 \alpha_m(x)dx=\frac{1}{m+2}\left(\alpha_{m+1}(1)-\alpha_{m+1}(0)\right). \tag{2.6}$$

For $m > r+s$, we let

$$\begin{aligned}\Delta_m=\Delta_m(r,s) &= \alpha_m(1)-\alpha_m(0)\\ &= \sum_{0\le k\le m}\left(G^{(r)}_{k}(1)G^{(s)}_{m-k}(1)-G^{(r)}_{k}G^{(s)}_{m-k}\right)\\ &= \sum_{0\le k\le m}\left((2kG^{(r-1)}_{k-1}-G^{(r)}_{k})(2(m-k)G^{(s-1)}_{m-k-1}-G^{(s)}_{m-k})-G^{(r)}_{k}G^{(s)}_{m-k}\right)\\ &= 2\sum_{0\le k\le m}\Big(2k(m-k)G^{(r-1)}_{k-1}G^{(s-1)}_{m-k-1}-kG^{(r-1)}_{k-1}G^{(s)}_{m-k}\\ &\qquad\qquad -(m-k)G^{(r)}_{k}G^{(s-1)}_{m-k-1}\Big).\end{aligned} \tag{2.7}$$

We note here that

$$\alpha_m(0)=\alpha_m(1)\Longleftrightarrow \Delta_m=0, \tag{2.8}$$

and

$$\int_0^1 \alpha_m(x)dx=\frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

We are now going to determine the Fourier coefficients $A^{(m)}_n$.

Case 1 : $n \neq 0$.

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}[\alpha_m(x)e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1 (\frac{d}{dx}\alpha_m(x))e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m. \end{aligned} \tag{2.10}$$

Thus we have shown that

$$A_n^{(m)} = \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m. \tag{2.11}$$

Noting that $A_n^{(r+s)} = r!s! \int_0^1 e^{-2\pi i n x} dx = 0$, and by induction on m, from (2.11) we can easily deduce that

$$A_n^{(m)} = -\frac{1}{m+2} \sum_{1 \leq j \leq m-(r+s)} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1}. \tag{2.12}$$

Case 2 : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.13}$$

$\alpha_m(< x >), (m > r + s)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers $m > r + s$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers $m > r + s$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for some integer $m > r + s$. Then $\alpha_m(0) = \alpha_m(1)$. Thus $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} \alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{1 \leq j \leq m-(r+s)} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{1 \leq j \leq m-(r+s)} \binom{m+2}{j} \Delta_{m-j+1} \\ &\qquad \times \Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \Big) \\ &= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{2 \leq j \leq m-(r+s)} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) \\ &\quad + \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.14}$$

We are now ready to state our first result.

Theorem 2.1. *For each positive integer $l > r+s$, we let*

$$\Delta_l = 2\sum_{0\le k\le l}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right).$$

Assume that $\Delta_m = 0$, for some integer $m > r+s$. Then we have the following.

(a) $\sum_{0\le k\le m} G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>)$ has the Fourier series expansion

$$\sum_{0\le k\le m} G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>)$$
$$=\frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{1\le j\le m-(r+s)}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}, \tag{2.15}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$(b)\ \sum_{0\le k\le m} G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>) = \frac{1}{m+2}\sum_{j=0,j\neq 1}^{m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>), \tag{2.16}$$

for all x in $\mathbb{R}$.

Assume next that $\Delta_m \neq 0$, for an integer $m > r+s$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m, \tag{2.17}$$

for $x \in \mathbb{Z}$.

We are now going to state our second result.

Theorem 2.2. *For each positive integer $l > r+s$, we let*

$$\Delta_l = 2\sum_{0\le k\le l}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right).$$

Assume that $\Delta_m \neq 0$, for some integer $m > r+s$. Then we have the following.

$$(a)\ \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{1\le j\le m-(r+s)}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{0\le k\le m} G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>), & \text{for } x\notin\mathbb{Z},\\ \sum_{0\le k\le m} G_k^{(r)}G_{m-k}^{(s)} + \frac{1}{2}\Delta_m, & \text{for } x\in\mathbb{Z}.\end{cases} \tag{2.18}$$

$$(b)\ \frac{1}{m+2}\sum_{0\le j\le m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) = \sum_{0\le k\le m} G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>), \tag{2.19}$$

for $x \notin \mathbb{Z}$;

$$\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)=\sum_{0\leq k\leq m}G_k^{(r)}G_{m-k}^{(s)}+\frac{1}{2}\Delta_m,\ for\ x\in\mathbb{Z}. \tag{2.20}$$

Corollary 2.3. *For each positive integer $l>r+s$, we let*

$$\Delta_l=2\sum_{0\leq k\leq l}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)}-kG_{k-1}^{(r-1)}G_{l-k}^{(s)}-(l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right).$$

Then we have the following polynomial identity.

$$\sum_{0\leq k\leq m}G_k^{(r)}(x)G_{m-k}^{(s)}(x)=\frac{1}{m+2}\sum_{0\leq j\leq m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}B_j(x),\ (m>r+s). \tag{2.21}$$

3. The function $\beta_m(<x>)$

Let $\beta_m(x)=\sum_{0\leq k\leq m}\frac{1}{k!(m-k)!}G_k^{(r)}(x)G_{m-k}^{(s)}(x),\ (m>r+s)$. Then we will consider the function

$$\beta_m(<x>)=\sum_{0\leq k\leq m}\frac{1}{k!(m-k)!}G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>),\ (m>r+s),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m,r,s)}e^{2\pi inx}, \tag{3.1}$$

where

$$B_n^{(m)}=B_n^{(m,r,s)}=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx. \tag{3.2}$$

To continue further, we need to observe the following.

$$\begin{aligned}\beta_m'(x)&=\sum_{0\leq k\leq m}\left\{\frac{k}{k!(m-k)!}G_{k-1}^{(r)}(x)G_{m-k}^{(s)}(x)+\frac{(m-k)}{k!(m-k)!}G_k^{(r)}(x)G_{m-k-1}^{(s)}(x)\right\}\\&=\sum_{1\leq k\leq m}\frac{1}{(k-1)!(m-k)!}G_{k-1}^{(r)}(x)G_{m-k}^{(s)}(x)\\&+\sum_{0\leq k\leq m-1}\frac{1}{k!(m-k-1)!}G_k^{(r)}(x)G_{m-k-1}^{(s)}(x)\\&=\sum_{0\leq k\leq m-1}\frac{1}{k!(m-1-k)!}G_k^{(r)}(x)G_{m-1-k}^{(s)}(x)\\&+\sum_{0\leq k\leq m-1}\frac{1}{k!(m-1-k)!}G_k^{(r)}(x)G_{m-1-k}^{(s)}(x)\\&=2\beta_{m-1}(x).\end{aligned} \tag{3.3}$$

From this, we have

$$\frac{d}{dx}\left(\frac{\beta_{m+1}(x)}{2}\right)=\beta_m(x), \tag{3.4}$$

and

$$\int_0^1 \beta_m(x)dx=\frac{1}{2}\Big(\beta_{m+1}(1)-\beta_{m+1}(0)\Big). \tag{3.5}$$

For $m>r+s$, we let

$$\begin{aligned}\Omega_m&=\Omega_m(r,s)=\beta_m(1)-\beta_m(0)\\&=\sum_{0\le k\le m}\frac{1}{k!(m-k)!}(G_k^{(r)}(1)G_{m-k}^{(s)}(1)-G_k^{(r)}G_{m-k}^{(s)})\\&=\sum_{0\le k\le m}\frac{1}{k!(m-k)!}\Big((2kG_{k-1}^{(r-1)}-G_k^{(r)})(2(m-k)G_{m-k-1}^{(s-1)}-G_{m-k}^{(s)})-G_k^{(r)}G_{m-k}^{(s)}\Big)\\&=\sum_{0\le k\le m}\frac{2}{k!(m-k)!}\Big(2k(m-k)G_{k-1}^{(r-1)}G_{m-k-1}^{(s-1)}-kG_{k-1}^{(r-1)}G_{m-k}^{(s)}\\&\qquad-(m-k)G_k^{(r)}G_{m-k-1}^{(s-1)}\Big).\end{aligned} \tag{3.6}$$

Now, we note here that

$$\beta_m(0)=\beta_m(1)\iff\Omega_m=0, \tag{3.7}$$

and

$$\int_0^1\beta_m(x)dx=\frac{1}{2}\Omega_{m+1}. \tag{3.8}$$

Next, we are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1 : $n\neq 0$

$$\begin{aligned}B_n^{(m)}&=\int_0^1\beta_m(x)e^{-2\pi inx}dx\\&=-\frac{1}{2\pi in}\Big[\beta_m(x)e^{-2\pi inx}\Big]_0^1+\frac{1}{2\pi in}\int_0^1(\frac{d}{dx}\beta_m(x))e^{-2\pi inx}dx\\&=-\frac{1}{2\pi in}\Big(\beta_m(1)-\beta_m(0)\Big)+\frac{2}{2\pi in}\int_0^1\beta_{m-1}(x)e^{-2\pi inx}dx\\&=\frac{2}{2\pi in}B_n^{(m-1)}-\frac{1}{2\pi in}\Omega_m,\end{aligned} \tag{3.9}$$

Thus we have shown that

$$B_n^{(m)}=\frac{2}{2\pi in}B_n^{(m-1)}-\frac{1}{2\pi in}\Omega_m. \tag{3.10}$$

From (3.10) and noting that $B_n^{(r+s)}=\int_0^1 e^{-2\pi inx}dx=0$, by induction on m, we can easily show that

$$B_n^{(m)}=-\sum_{1\le j\le m-(r+s)}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}. \tag{3.11}$$

Case 2 : $n = 0$

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}. \tag{3.12}$$

$\beta_m(< x >)$, $(m > r+s)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those integers $m > r+s$ with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m > r+s$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for some integer $m > r+s$. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{1\le j\le m-(r+s)}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{1\le j\le m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\Big(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi n x}}{(2\pi i n)^j}\Big) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{2\le j\le m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) \\
&\qquad + \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.13}$$

We are now ready to state our first result.

Theorem 3.1. *For each integer $l > r+s$, we put*

$$\Omega_l = \sum_{0\le k\le l}\frac{2}{k!(l-k)!}\big(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\big). \tag{3.14}$$

Assume that $\Omega_m = 0$, for some integer $m > r+s$. Then we have the following.

(a) $\sum_{0\le k\le m}\frac{1}{k!(m-k)!}G_k^{(r)}(< x >)G_{m-k}^{(s)}(< x >)$ has the Fourier series expansion

$$\begin{aligned}
&\sum_{0\le k\le m}\frac{1}{k!(m-k)!}G_k^{(r)}(< x >)G_{m-k}^{(s)}(< x >) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\Big(-\sum_{1\le j\le m-(r+s)}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big)e^{2\pi i n x},
\end{aligned} \tag{3.15}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$\begin{aligned}
(b)\ &\sum_{0\le k\le m}\frac{1}{k!(m-k)!}G_k^{(r)}(< x >)G_{m-k}^{(s)}(< x >) \\
&= \sum_{j=0, j\neq 1}^{m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >),
\end{aligned} \tag{3.16}$$

for all $x \in \mathbb{R}$.

Assume next that $\Omega_m \neq 0$, for some integer $m > r+s$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Then the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0)+\beta_m(1)) = \beta_m(0) + \frac{1}{2}\Omega_m, \tag{3.17}$$

for $x \in \mathbb{Z}$.

Next, we are ready to state our second result.

Theorem 3.2. *For each integer $l > r+s$, we put*

$$\Omega_l = \sum_{0\le k\le l} \frac{2}{k!(l-k)!}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right). \tag{3.18}$$

Assume that $\Omega_m \neq 0$, for an integer $m > r+s$. Then we have the following.

$$(a)\ \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty}\left(-\sum_{1\le j\le m-(r+s)} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right)e^{2\pi i n x}$$
$$= \begin{cases} \sum_{0\le k\le m} \frac{1}{k!(m-k)!}G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>), & \text{for } x\notin\mathbb{Z}, \\ \sum_{0\le k\le m} \frac{1}{k!(m-k)!}G_k^{(r)}G_{m-k}^{(s)} + \frac{1}{2}\Omega_m, & \text{for } x\in\mathbb{Z}. \end{cases} \tag{3.19}$$

$$(b)\ \sum_{0\le j\le m-(r+s)} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)$$
$$= \sum_{0\le k\le m}\frac{1}{k!(m-k)!}G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>), \quad \text{for } x\notin\mathbb{Z};$$
$$\sum_{j=0, j\neq 1}^{m-(r+s)} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)$$
$$= \sum_{0\le k\le m}\frac{1}{k!(m-k)!}G_k^{(r)}G_{m-k}^{(s)} + \frac{1}{2}\Omega_m, \quad \text{for } x\in\mathbb{Z}. \tag{3.20}$$

Corollary 3.3. *For each integer $l > r+s$, we let*

$$\Omega_l = \sum_{0\le k\le l} \frac{2}{k!(l-k)!}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right). \tag{3.21}$$

Then we have the following polynomial identity.

$$\sum_{0\le k\le m}\frac{1}{k!(m-k)!}G_k^{(r)}(x)G_{m-k}^{(s)}(x) = \sum_{0\le j\le m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(x), \ (m > r+s). \tag{3.22}$$

4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{1\leq k\leq m-1} \frac{1}{k(m-k)} G_k^{(r)}(x) G_{m-k}^{(s)}(x)$, $(m > r+s)$. Then we will consider the function

$$\gamma_m(< x >) = \sum_{1\leq k\leq m-1} \frac{1}{k(m-k)} G_k^{(r)}(< x >) G_{m-k}^{(s)}(< x >),\ (m > r+s),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m,r,s)} e^{2\pi i n x}, \tag{4.1}$$

where

$$C_n^{(m)} = C_n^{(m,r,s)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.2}$$

To proceed further, we need to observe the following.

$$\begin{aligned}
\gamma_m'(x) &= \sum_{1\leq k\leq m-1} \left(\frac{1}{m-k} G_{k-1}^{(r)}(x) G_{m-k}^{(s)}(x) + \frac{1}{k} G_k^{(r)}(x) G_{m-k-1}^{(s)}(x) \right) \\
&= \sum_{0\leq k\leq m-2} \frac{1}{m-k-1} G_k^{(r)}(x) G_{m-k-1}^{(s)}(x) + \sum_{1\leq k\leq m-1} \frac{1}{k} G_k^{(r)}(x) G_{m-k-1}^{(s)}(x) \\
&= \sum_{1\leq k\leq m-2} \left(\frac{1}{m-k-1} + \frac{1}{k} \right) G_k^{(r)}(x) G_{m-k-1}^{(s)}(x) \\
&+ \frac{1}{m-1} G_0^{(r)}(x) G_{m-1}^{(s)}(x) + \frac{1}{m-1} G_{m-1}^{(r)}(x) G_0^{(s)}(x) \\
&= (m-1) \sum_{1\leq k\leq m-2} \frac{1}{k(m-1-k)} G_k^{(r)}(x) G_{m-1-k}^{(s)}(x) \\
&= (m-1)\gamma_{m-1}(x),
\end{aligned} \tag{4.3}$$

where we note that $G_0^{(r)}(x) = 0 = G_0^{(s)}(x)$.

From (4.3), we have

$$\frac{d}{dx}\left(\frac{1}{m} \gamma_{m+1}(x) \right) = \gamma_m(x), \tag{4.4}$$

and

$$\int_0^1 \gamma_m(x) dx = \frac{1}{m}\Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) \Big). \tag{4.5}$$

For $m \geq 2$, we let

$$\begin{aligned}\Lambda_m &= \gamma_m(1) - \gamma_m(0)\\ &= \sum_{1\leq k\leq m-1} \frac{1}{k(m-k)}\Big(G_k^{(r)}(1)G_{m-k}^{(s)}(1) - G_k^{(r)}G_{m-k}^{(s)}\Big)\\ &= \sum_{1\leq k\leq m-1} \frac{1}{k(m-k)}\Big((2kG_{k-1}^{(r-1)} - G_k^{(r)})(2(m-k)G_{m-k-1}^{(s-1)} - G_{m-k}^{(s)}) - G_k^{(r)}G_{m-k}^{(s)}\Big)\\ &= \sum_{1\leq k\leq m-1} \frac{2}{k(m-k)}\Big(2k(m-k)G_{k-1}^{(r-1)}G_{m-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{m-k}^{(s)}\\ &\qquad - (m-k)G_k^{(r)}G_{m-k-1}^{(s-1)}\Big).\end{aligned} \tag{4.6}$$

From (1.15) and (1.19), we observe that

$$\Lambda_m = 0,\ (2 \leq m \leq r+s). \tag{4.7}$$

Also, it is obvious that

$$\gamma_m(0) = \gamma_m(1) \ \Leftrightarrow\ \Lambda_m = 0, \tag{4.8}$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\Lambda_{m+1}. \tag{4.9}$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.

Case 1 : $n \neq 0$

$$\begin{aligned}C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 (\frac{d}{dx}\gamma_m(x))e^{-2\pi inx}dx\\ &= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) + \frac{1}{2\pi in}\int_0^1 \{(m-1)\gamma_{m-1}(x)e^{-2\pi inx}dx\\ &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m,\end{aligned} \tag{4.10}$$

from which by induction on m, we obtain

$$\begin{aligned}C_n^{(m)} &= -\sum_{1\leq j\leq m-(r+s)} \frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1}\\ &= -\frac{1}{m}\sum_{1\leq j\leq m-(r+s)} \frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}.\end{aligned} \tag{4.11}$$

Case 2 : $n = 0$

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\Lambda_{m+1}. \tag{4.12}$$

$\gamma_m(< x >)$ is piecewise C^∞. Furthermore, $\gamma_m(< x >)$ is continuous for those integers $m > r+s$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$, for some integer $m > r+s$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$
\begin{aligned}
\gamma_m(< x >) &= \frac{1}{m}\Lambda_{m+1} - \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\Big(\sum_{1\leq j\leq m-(r+s)}\frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1}\Big)e^{2\pi i n x}\\
&= \frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{1\leq j\leq m-(r+s)}\binom{m}{j}\Lambda_{m-j+1}\\
&\times\Big(-j!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\Big)\\
&= \frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{2\leq j\leq m-(r+s)}\binom{m}{j}\Lambda_{m-j+1}B_j(< x >)\\
&\qquad + \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}
\tag{4.13}
$$

We are now ready to state our first result.

Theorem 4.1. *For each integer $l > r+s$, we let*

$$
\Lambda_l = \sum_{1\leq j\leq l-1}\frac{2}{k(l-k)}\big(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\big),
\tag{4.14}
$$

Assume that $\Lambda_m = 0$, for some integer $m > r+s$. Then we have the following.

(a) $\sum_{1\leq j\leq m-1}\frac{1}{k(m-k)}G_k^{(r)}(< x >)G_{m-k}^{(s)}(< x >)$ has the Fourier series expansion

$$
\begin{aligned}
&\sum_{1\leq j\leq m-1}\frac{1}{k(m-k)}G_k^{(r)}(< x >)G_{m-k}^{(s)}(< x >)\\
&= \frac{1}{m}\Lambda_{m+1} - \frac{1}{m}\sum_{n=-\infty, n\neq 0}^{\infty}\Big\{\sum_{1\leq j\leq m-(r+s)}\frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1}\Big\}e^{2\pi i n x},
\end{aligned}
\tag{4.15}
$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

$$
\begin{aligned}
(b)\quad &\sum_{1\leq k\leq m-1}\frac{1}{k(m-k)}G_k^{(r)}(< x >)G_{m-k}^{(s)}(< x >)\\
&= \frac{1}{m}\sum_{j=0, j\neq 1}^{m-(r+s)}\binom{m}{j}\Lambda_{m-j+1}B_k(< x >),
\end{aligned}
\tag{4.16}
$$

for all $x \in \mathbb{R}$.

Assume next that $\Lambda_m \neq 0$, for some integer $m > r+s$. Then $\gamma_m(0) \neq \gamma_m(1)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0)+\gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m, \tag{4.17}$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

Theorem 4.2. *For each integer $l > r+s$, we let*

$$\Lambda_l = \sum_{1\le k\le l-1} \frac{2}{k(l-k)}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right), \tag{4.18}$$

Assume that $\Lambda_m \neq 0$, for some integer $m > r+s$.
Then we have the following.

$$\begin{aligned}(a)\ &\frac{1}{m}\Lambda_{m+1} - \frac{1}{m}\sum_{n=-\infty,n\neq 0}^{\infty}\left\{\sum_{1\le j\le m-(r+s)}\frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1}\right\}e^{2\pi i n x}\\ &= \begin{cases}\sum_{1\le k\le m-1}\frac{1}{k(m-k)}G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>), & \text{for } x\notin\mathbb{Z},\\ \sum_{1\le k\le m-1}\frac{1}{k(m-k)}G_k^{(r)}G_{m-k}^{(s)} + \frac{1}{2}\Lambda_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned} \tag{4.19}$$

$$\begin{aligned}(b)\ &\frac{1}{m}\sum_{0\le j\le m-(r+s)}\binom{m}{j}\Lambda_{m-j+1}B_k(<x>)\\ &= \sum_{1\le k\le m-1}\frac{1}{k(m-k)}G_k^{(r)}(<x>)G_{m-k}^{(s)}(<x>),\ \text{for } x\notin\mathbb{Z};\\ &\frac{1}{m}\sum_{j=0,j\neq 1}^{m-(r+s)}\binom{m}{j}\Lambda_{m-j+1}B_k(<x>)\\ &= \sum_{1\le k\le m-1}\frac{1}{k(m-k)}G_k^{(r)}G_{m-k}^{(s)} + \frac{1}{2}\Lambda_m,\ \text{for } x\in\mathbb{Z}.\end{aligned} \tag{4.20}$$

Corollary 4.3. *For each integer $l > r+s$, we let*

$$\Lambda_l = \sum_{1\le k\le l-1} \frac{2}{k(l-k)}\left(2k(l-k)G_{k-1}^{(r-1)}G_{l-k-1}^{(s-1)} - kG_{k-1}^{(r-1)}G_{l-k}^{(s)} - (l-k)G_k^{(r)}G_{l-k-1}^{(s-1)}\right). \tag{4.21}$$

Then we have the following polynomial identity.

$$\sum_{1\le k\le m-1}\frac{1}{k(m-k)}G_k^{(r)}(x)G_{m-k}^{(s)}(x) = \frac{1}{m}\sum_{0\le j\le m-(r+s)}\binom{m}{j}\Lambda_{m-j+1}B_k(<x>),\ (m > r+s). \tag{4.22}$$

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, **1970**.

[2] R. P. Agarwal, D. S. Kim, T. Kim and J. Kwon, *Sums of finite products of Bernoulli functions*, Adv. Diff. Equ., **2017**, 2017:237.

[3] S. Araci, M. Acikgoz, H. Jolany and J. -J. Seo, *A unified generating function of the q-Genocchi polynomials with their interpolation functions*, Proc. Jangjeon Math. Soc., **15** (2012), no. 2, 227-233.

[4] A. Bayad, *Special values of Lerch zeta function and their Fourier expansions*, Adv. Stud. Contemp. Math.(Kyungshang), **21**(2011), no. 1, 1-4.

[5] I. N. Cangul, V. Curt, H. Ozden and Y. Simsek, *On the higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math.(Kyungshang), **19**(2009), no. 1, 39-57.

[6] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139(1)**(2000), 173-199.

[7] S. Gaboury, R. Tremblay and B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc., **17** (2014), no. 1, 115-123.

[8] A. Guven and D. M. Israfilov, *Approximation by means of Fourier trigonometric series in weighted Orlicz spaces*, Adv. Stud. Contemp. Math.(Kyungshang), **19**(2009), no. 2, 283-295.

[9] G.-W. Jang, T. Kim, D. S. Kim and T. Mansour, *Fourier series of functions related to Bernoulli polynomials*, Adv. Stud. Contemp. Math.(Kyungshang), **27**(2017), no. 1, 49-62.

[10] D. S. Kim, T. Kim, H. -I. Kwon, G. -W. Jang and T. Mansour, *A generalization of symmetric property beyond appell polynomials*, Proc. Jangjeon Math. Soc., **20** (2017), no. 2, 145-152.

[11] T. Kim, *On the multiple q-Genocchi and Euler numbers*, Russ. J. Math. Phys., **15** (2008), no. 4, 481-485.

[12] T. Kim, *Euler numbers and polynomials associated with zeta functions*, Abstr. Appl. Anal., **2008**, Art. ID 581582, 11pages.

[13] T. Kim, J. Choi and Y. H. Kim, *A note on the values of Euler zeta functions at positive integers*, Adv. Stud. Contemp. Math.(Kyungshang), **22**(2012), no. 1, 27-34.

[14] T. Kim, D. S. Kim, L.-C. Jang and G.-W. Jang, *Fourier series of sums of products of Bernoulli functions and their applications*, J. Nonlinear Sci. Appl., **10** (2017), 2798-2815.

[15] B. Kurt and Y. Simsek, *On the Hermite based Genocchi polynomials*, Adv. Stud. Contemp. Math.(Kyungshang), **23**(2013), no. 1, 13-17.

[16] J. E. Marsden, *Elementary Classical Analysis*, W. H. Freeman and Company, 1974.

[17] H. Miki, *A relation between Bernoulli numbers*, J. Number Theory, **10**(1978), no. 3, 297-302.

[18] S. -H. Rim, S. J. Lee, E. J. Moon and J. H. Jin, *On the q-Genocchi numbers and polynomials associated with q-zeta function*, Proc. Jangjeon Math. Soc., **12** (2009), no. 3, 261-267.

[19] C. S. Ryoo, *On the structure of the zeros of the twisted (h, q)-Genocchi polynomials*, Proc. Jangjeon Math. Soc., **15** (2012), no. 3, 283-291.

[20] Y. Simsek, *Identities on the Changhee numbers and Apostol-type Daehee polynomials*, Adv. Stud. Contemp. Math.(Kyungshang), **27**(2017), no. 2, 199-212.

[21] D. G. Zill and M. R. Cullen, *Advanced Engineering Mathematics*, Jones and Bartlett Publishers 2006.

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