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VARIATIONS ON THE CUSA-HUYGENS INEQUALITY

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ABSTRACT. In this paper, the author refines the classical inequalities of the trigonometric functions, such as Jordan's inequality, Cusa-Huygens inequality and Kober's inequality.

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1. Introduction

The study of the classical inequalities of the trigonometric functions such as Adamović-Mitrinović inequality, Cusa-Huygens inequality, Jordan inequality, Redheffer inequality, Becker-Stark inequality, Wilker inequality, Huygens inequality, and Kober inequality has got big attention of the numerous authors. Since last ten years, the huge number of papers on the refinement and the generalization of these inequalities have appeared, e.g. see [2, 3, 9, 10, 12, 13, 14, 17, 18, 19, 21] and the references therein. Motivated by these studies, this paper deals with the variations of Cusa-Huygens inequality, and our results refine the some existing results in the literature.

In literature, the following inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < |x| < \frac{\pi}{2}, \tag{1.1}$$

are known as Adamović-Mitrinović inequality [11, p.238] and Cusa-Huygens [19] inequality, respectively. For the refinement of (1.1), e.g. see [9, 12, 14, 18, 19, 21] and the bibliography of these papers. Most of the refinements of (1.1) are involving very complicated upper and lower bound of $\sin(x)/x$. In the following theorem we refine (1.1) by giving the upper and lower bound of $\sin(x)/x$ in terms of simple functions, and these functions are also independent of the exponent.

1.2. **Theorem.** *For $x \in (0, \pi)$, we have*

$$\frac{1+\cos(x)}{2-\alpha x^2} < \frac{\sin(x)}{x} < \frac{1+\cos(x)}{2-\beta x^2},$$

with the best possible constants $\alpha = 1/6 \approx 0.166667$ and $\beta = 2/\pi^2 \approx 0.202642$.

1.3. **Lemma.** *For $x \in (0, \pi/2)$, the function*

$$f_1(x) = \frac{x^2(\sin x)^2}{x^2 - (\sin x)^2}$$

is strictly decreasing.

Proof. Let

$$t(x) = 1/f_1(x) = \frac{1}{(\sin x)^2} - \frac{1}{x^2}.$$

We will prove that, $t(x)$ is strictly increasing, which will imply the assertion from Lemma 1.3. One has

$$x^3(\sin x)^3 \cdot t'(x) = (\sin x)^3 - (x^3) \cdot \cos(x) > 0,$$

by the Mitrinović-Adamović inequality $\sin(x)/x > \cos(x)^{1/3}$. This gives $t'(x) > 0$, so the result follows. □

1.4. **Theorem.** *For $x \in (0, \pi/2)$, the following inequalities*

$$\frac{(\cos x)^2 + b - 1}{b} < \frac{(\sin x)^2}{x^2} < \frac{(\cos x)^2 + a - 1}{a} \tag{1.5}$$

hold, with the best positive constants $a = 3$ and $b = \pi^2/(\pi^2 - 4)$.

Proof. In what follows, we shall denote by $f(u+)$ the limit from the right at $x = u$ of the function $f(x)$. Similarly, $f(v-)$ will be the limit from the left at $y = v$. The right side of inequality (1.5) may be written as $a > f_1(x)$, where $f_1(x)$ is defined in Lemma 1.3. Since $f_1(\pi/2-) < f_1(x) < f_1(0+)$, the inequalities of Theorem 1.4 are consequences, as $a = f_1(0+)$ (simple computations), and $b = f_1(\pi/2-)$. □

1.6. **Lemma.** *For $x > 0$, the function*

$$f_2(x) = \frac{x\cosh(x) - x}{\sinh(x) - x}$$

is strictly increasing.

Proof. One has

$$(\sinh(x) - x)^2 \cdot (f_2)'(x) = g(x) = 2(\sinh x)^2 x \sinh(x) x^2 \cosh(x).$$

We shall prove that $g(x) > 0$. For this aim, use the series expansions of the occurring functions, namely

$$\sinh(x) = x/1! + x^3/3! + x^5/5! + \cdots, \quad \cosh x = 1 + x^2/2! + x^4/4! + \ldots,$$

$$(\sinh x)^2 = 2x^2/2! + 8x^4/4! + 32x^6/6! + \cdots,$$

which may be easily obtained by the Maclaurin expansion of the functions. We will obtain

$$g(x) = x^6(2^6/6! - 1/4! - 1/5!) + x^8(2^8/8! - 1/6! - 1/7!) + \ldots > 0,$$

as generally

$$2^{2k}/(2k)! > 1/(2k-2)! + 1/(2k-1)!$$

for all $k \geq 3$. This is equivalent to $2^{2k} > 4k^2$, or $4^{k-1} > k^2$, which follows by mathematical induction. □

1.7. **Theorem.** *For* $x \in (0,1)$, *one has*

$$\frac{\cosh(x)+b-1}{b} < \frac{\sinh(x)}{x} < \frac{\cosh(x)+a-1}{a}, \tag{1.8}$$

with best possible constants $a = 3$ *and* $b = (\cosh(1)-1)/(\sinh(1)-1) = 3.0998\ldots$.

Proof. The right side of inequality 1.8 may be rewritten as $a < f_2(x)$. By Lemma 1.6, $f_2(x) > f_2(0+) = 3$ (simple computations, which we omit here), so the right side of (1.8) holds true for any $x > 0$. The left side of (1.8) follows by $f_2(x) < f_2(1-) = (\cosh(1)-1)/(\sinh(1)-1) = 3.0998\ldots$. □

1.9. **Remark.** The right side inequality of (1.8) for $a = 3$ is due to Neuman and Sándor [14]. We will prove that the inequality $g(x) > 0$ of Lemma 1.6 offers the following counterpart to this inequality:

$$\frac{\sinh(x)}{x} > \frac{1+\sqrt{1+8\cosh(x)}}{4} \quad \text{for} \quad x > 0. \tag{1.10}$$

Indeed, $g(x) > 0$ may be written also as $2t^2 - t - \cosh(x) > 0$, where $t = \sinh(x)/x$. Solving this quadratic equation, relation (1.10) follows.

1.11. **Lemma.** *The function*

$$f_3(x) = \frac{x\cosh(x)}{3\sinh(x)-2x}$$

is strictly increasing for $x > 0$.

Proof. After elementary computations one gets

$$(3\sinh(x)-2x)^2 f_3'(x) = h(x) = 3\sinh(x)\cosh(x) - 2x^2\sinh(x) - 3x.$$

One has

$$h'(x) = 3(\sinh x)^2 - 2x\sinh(x) - x^2\cosh(x) = 2x^2(3t^2 - 2t - \cosh(x)),$$

with $t = \sinh(x)/x$. The fact that $3t^2 - 2t - \cosh x > 0$ is equivalent to $t > (1 + \sqrt{1+3\cosh(x)})/3$. We will show that this follows by (1.10). Indeed, put $\cosh(x) = u$. The following inequality is valid:

$$\frac{1+\sqrt{1+8u}}{4} > \frac{1+\sqrt{1+3u}}{3}.$$

This follows by elementary computations, reducing to $u = \cosh(x) > 1$. Now, as $h'(x) > 0$, we get $h(x) > h(0) = 0$, so the function $f_3(x)$ is strictly increasing for $x > 0$. □

1.12. **Theorem.** *For* $x \in (0,1)$, *we have*

$$\frac{\cosh(x)+2b}{3b} < \frac{\sinh(x)}{x} < \frac{\cosh(x)+2a}{3a}, \tag{1.13}$$

with the best positive constants and $a = 1$ *and* $b = \cosh(1)/(3\sinh(1) - 2) = 1.01146\ldots$.

Proof. The right side of (1.13) may be rewritten as $a < f_3(x)$. By lemma 1.11, $f_3(x)$ is strictly increasing, $f_3(x) < f_3(1-) = b$. Inequality (1.13) follows. □

1.14. **Remark.** The right side of (1.13) holds true with best constant $a = 1$ for any $x > 0$.

1.15. **Lemma.** *The function*

$$f_4(x) = \frac{x\cos(x) + 2x}{\sin(x)}$$

is strictly increasing on $(0, \pi/2)$.

Proof. Elementary computations offer the relation

$$(\sin x)^2 \cdot f_4'(x) = g(x) = \sin(x)\cos(x) - x + 2\sin(x) - 2x\cos(x).$$

It is immediate that

$$g'(x) = 2\sin(x)(x - \sin(x)) > 0,$$

so we get $g(x) > g(0) = 0$, giving the result. □

1.16. **Theorem.** *The following inequalities*

$$\frac{\cos(x) + 2}{b} < \frac{\sin(x)}{x} < \frac{\cos(x) + 2}{a}, \tag{1.17}$$

holds for $x \in (0, \pi/2)$*, where* $a = 3$ *and* $b = \pi$ *are the possible best constants.*

Proof. The right side of (1.17) may be written as $a < f_4(x)$, while the left side as $b > f_4(x)$. Relations (1.17) are consequences of Lemma 1.15, as $a = f_4(0+)$, $b = f_4(\pi/2-)$. □

1.18. **Remark.** The right side of (1.17) for $a = 3$ is the Cusa-Huygens inequality. As in the proof of Lemma 1.15, the inequality $x - \sin(x) > 0$ is valid for any $x \in (0, \pi)$, this shows that the Cusa-Huygens inequality holds true also in $(0, \pi)$.

1.19. **Lemma.** *The function*

$$f_5(x) = \frac{3\sin(x) - x\cos(x)}{x}$$

is strictly decreasing in $(0, \pi)$.

Proof. One has

$$x^2 \cdot f_5'(x) = k(x) = 3x\cos(x) - 3\sin(x) + x^2\sin(x).$$

As $k'(x) = x(x\cos(x) - \sin(x))$, and remarking that $x\cos(x) - \sin(x) < 0$ for $x \in (0, \pi/2)$ by the known inequality $\tan(x) > x$, and as $\cos(x) < 0, \sin(x) > 0$ for $x \in (\pi/2, \pi)$, so we get $k'(x) < 0$ for any $x \in (0, \pi)$. □

1.20. **Theorem.** *For $x \in (0, \pi/2)$, the best positive constants a and b such that*

$$\frac{\cos(x) + b}{3} < \frac{\sin(x)}{x} < \frac{\cos(x) + a}{3} \tag{1.21}$$

are $a = 2$ and $b = 6/\pi$. The best constants a and b in the same double inequality for $x \in (0, \pi)$ are $a = 2$ and $b = 1$.

Proof. The right side of (1.21) may be written as $a < f_5(x)$ and all are consequences of Lemma 1.19, as $f_5(0) = f_5(0+) = 2$ and $f_5(\pi/2-) = 6/\pi$, $\quad f_5(\pi-) = 1$. □

1.22. **Remark.** Thus, one has the inequalities

$$\frac{\cos(x) + 6\pi}{3} < \frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}, \quad \text{for} \quad x \in (0, \pi/2),$$

$$\frac{\cos(x) + 1}{3} < \frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}, \quad \text{for} \quad x \in (0, \pi).$$

1.23. **Lemma.** *The function*

$$f_6(x) = \frac{3\sin(x) - 2x}{x\cos(x)}$$

is strictly decreasing in $(0, \pi/2)$.

Proof. One has the numerator of

$$f_6'(x) = 3x - 3\sin(x)\cos(x) - 2x^2\sin(x) = r(x).$$

Now,

$$r'(x) = 2x^2(3p^2 - 2p - \cos(x)),$$

where $p = \sin(x)/x$. Now, the quadratic inequality $3p^2 - 2p - \cos(x) < 0$ is equivalent to $p < [1 + \sqrt{1 + 3\cos(x)}]/3$. We show this is true, by the Cusa-Huygens inequality $p < (u + 2)/3$ where $u = \cos(x)$. It is sufficient to show that $(u + 2)/3 < [1 + \sqrt{1 + 3u}]/3$, which after elementary computations becomes $u^2 < u$, or (as x is in $(0, pi/2)$), $u < 1$, which is valid. □

1.24. **Theorem.** *The best positive constant a such that*

$$\frac{\sin(x)}{x} < \frac{a\cos(x) + 2}{3}, \quad \text{for} \quad x \in (0, \pi/2) \tag{1.25}$$

is $a = 1$.

Proof. Inequality (1.25) may be rewritten as $a > f_6(x)$. By lemma 1.23, one has $f_6(x) < f_6(0+) = 1$, so the theorem is proved. □

1.26. **Remark.** Remark that there is a single x_0 in $(0, \pi/2)$ such that $\sin(x) = (2/3)x$, and that $f_6(x) > 0$ for $x \in (0, x_0)$ and $f_6(x) < 0$ for $x \in (x_0, \pi/2)$. Therefore, (1.25) is trivial for $x \in (x_0, \pi/2)$, and it is relevant only for $x \in (0, x_0)$. There is no positive b such that $\sin(x)/x > (b\cos(x) + 2)/3$ for $x \in (0, \pi/2)$, but there is $b \geq 0$ with this property for $x \in (0, x_0)$, namely $b = 0$.

1.27. **Lemma.** *The function*

$$f_7(x) = \cos(x)x\sin(x) + 2 - 3\cos(x)$$

is strictly increasing in $x \in (0, \pi)$.

Proof. We get

$$f_7'(x) = -x\sin(x) + 2 - 2\cos(x) = m(x).$$

One has $m'(x) = \sin(x) - x\cos(x) > 0$ for $x \in (0, \pi/2)$, by the inequality $\tan(x) > x$, and in $[\pi/2, \pi)$ by $\sin(x) > 0$, $\cos(x) < 0$. Therefore, $g(x) > g(0) = 0$, giving $f_7'(x) > 0$. □

1.28. **Theorem.** *For* $x \in (0, \pi/2)$, *the best non-negative constants* a *and* b *such that*

$$\frac{\sin(x) + b}{x} < \frac{\cos(x) + 2}{3} < \frac{\sin(x) + a}{x} \tag{1.29}$$

are $b = 0$ *and* $a = (\pi - 3)/2$. *For* $x \in (0, \pi)$ *the best non-negative constants* a *and* b *such that*

$$\frac{\sin(x) + b}{x} < \frac{\cos(x) + 2}{3} < \frac{\sin(x) + a}{x} \tag{1.30}$$

are $b = 0$ *and* $a = \pi/3$.

Proof. The first inequality of (1.29) may be written as $b < f_7(x)/3 = F(x)$, where $f_7(x)$ is the function of Lemma 1.27. The inequalities (1.29) and (1.30) are consequences of the fact that this function is strictly increasing and that $F(0+) = 0$, $F(\pi/2-) = (\pi - 3)/3$ and $F(\pi-) = \pi/3$. □

1.31. **Lemma.** *The function*

$$f_8(x) = \frac{3\sin(x) - x\cos(x) - 2x}{\cos(x) + 2}$$

is strictly decreasing for $x \in (0, \pi)$.

Proof. One has

$$(\cos(x) + 2)^2 f_8'(x) = (\sin x)^2 + 2\cos(x) - 2 = n(x).$$

As

$$n'(x) = 2\sin(x)(\cos(x) - 1) < 0$$

for $x \in (0, \pi)$, we get $n(x) < n(0) = 0$, implying $f_8'(x) < 0$. □

1.32. **Theorem.** *For* $x \in (0, \pi/2)$, *the best constants* a *and* b *such that*

$$\frac{(x + b)(\cos(x) + 2)}{3} < \sin(x) < \frac{(x + a)(\cos(x) + 2)}{3} \tag{1.33}$$

are $a = 0$ *and* $b = (3 - \pi)/2$.

Proof. The second inequality of (1.33) may be written as $a > f_8(x)$, where this function appears in Lemma 1.31. As $f_8(0+) = 0 = a$, $f_8(\pi/2-) = (3 - \pi)/2 = b$, the result follows. □

The first inequality of (1.33) with best constant may be written also as

$$\frac{\cos(x)+2}{3} < \frac{\sin(x)}{x+(3-\pi)/2}$$

for $x \in ((\pi-3)/2, \pi/2)$.

1.34. **Lemma.** *The function*

$$f_9(x) = \frac{3x\cos(x)+12x-x^3}{\sin(x)}$$

is strictly decreasing on $(0,\pi)$.

Proof. One has

$$\sin(x)^2 f_9'(x) = 3\sin(x)\cos(x) - 3x + 12\sin(x) - 3x^2\sin(x) - 12x\cos(x) + x^3\cos(x) = s(x).$$

Simple computations give

$$s'(x) = \sin(x)(6x - 6\sin(x) - x^3).$$

It is well-known that $\sin(x) > x - x^3/6$ for any $x > 0$ [1], so we get $s'(x) < 0$ for $x \in (0,\pi)$. This implies $s(x) < s(0) = 0$, so the function $f_9(x)$ is strictly decreasing, and the lemma is proved. □

1.35. **Theorem.** *(a) For* $x \in (0, \pi/2)$, *the best positive constants* a *and* b *such that*

$$\frac{3\cos(x)-x^2+12}{a} < \frac{\sin(x)}{x} < \frac{3\cos(x)-x^2+12}{b} \tag{1.36}$$

are $a = 15$ *and* $b = \pi(48-\pi^2)/8 \approx 14.97$.
(b) For the left side of (1.36), *the best constant* a, *when* $x \in (0,\pi)$ *is* $a = 15$

Proof. The left side of (1.36) may be written as $a > f_9(x)$, where $f_9(x)$ is the decreasing function of Lemma 1.34. Since $f_9(0+) = 15$, $f_9(\pi/2-) = \pi(48-\pi^2)/8$, the result follows. The left side inequality holds true for any $x \in (0,\pi)$, so part (b) follows as well. □

1.37. **Remark.** For $x \in (0, 2\pi/3)$, the best constant b in the right side of (1.36) is $b = 2\sqrt{3}(189\pi - 8\pi^3)/81 \approx 14.78$. Indeed, as $\sin(2\pi/3) = \sin(\pi/3) = \sqrt{3}/2$ and $\cos(2\pi/3) = -\cos(\pi/3) = -1/2$, we get after simple computations $f_9(2\pi/3) = b$, where b is the value given above.

1.38. **Lemma.** *The function*

$$f_{10}(x) = \frac{2x + x\cos(x) - 3\sin(x)}{1-\cos(x)}$$

is strictly increasing $x \in (0, \pi/2)$.

Proof. After elementary computations we get

$$\begin{aligned}(1-\cos(x))^2 f'_{10}(x) &= u(x) = 4 - 4\cos(x) - 5\sin(x)^2 + x\sin(x) + 2x\sin(x)\cos(x)\\ &= (1/2)(3 - 8\cos(x) + 2x\sin(x) + 5\cos(2x) + 2x\sin(2x)),\end{aligned}$$

where we have used $\sin(2x) = 2\sin(x)\cos(x)$ and $1 - \cos(2x) = 2(\sin x)^2$. This implies

$$2u'(x) = 10\sin(x) + 2x\cos(x) - 8\sin(2x) + 4x\cos(2x) = x^5 a_5 - x^7 a_7 + \dots,$$

by using the series expansions of the functions

$$\sin(x) = x/1! - x^3/3! + x^5/5! - \dots$$

and

$$\cos(x) = 1 - x^2/2! + x^4/4! - \dots.$$

After some computations, we get

$$a_5 = 10/5! + 2/4! - 8.2^5/3! + 4.2^4/2!,$$

and generally

$$a_{2n+1} = 10/(2n+1)! + 2/(2n)! - 8 \cdot 2^{2n+1}/(2n+1)! + 4 \cdot 2^{2n}/(2n)!.$$

We will prove that

$$x^{2n+1} a_{2n+1} > x^{2n+3} a_{2n+3}.$$

As $x^2 < \pi^2/4 < 5/2$, after some elementary transformations, the inequality becomes:

$$2(2n+2)(2n+3)(4n+12) + 2(2n+2)(2n+3)(2n-3)2^{2n+2}$$

$$> 5(4n+16) + 5(2n-1).2^{2n+4}$$

for $n \geq 2$. This is true for $n = 2$, and for $n \geq 3$ follows from the inequality $(2n+2)(2n+3)(2n-3) > 10(2n-1)$. Therefore we get $u'(x) > 0$, so $u(x) > u(0) = 0$, giving $f'_{10}(x) > 0$. □

1.39. **Theorem.** *(a) For $x \in (0, \pi/2)$, the best positive constants a and b such that*

$$\frac{\sin(x)+a}{x+a} < \frac{\cos(x)+2}{3} < \frac{\sin(x)+b}{x+b} \tag{1.40}$$

are $a = 0$ and $b = \pi - 3$.

Proof. The left side inequality of (1.40) is $f_{10}(x) > a = f_{10}(0+)$, where $f_{10}(x)$ is the function defined in Lemma 1.38. As $b = f_{10}(\pi/2-) = \pi - 3$, the result follows. □