Advanced Studies in Contemporary Mathematics
28 (2018), No. 1, pp. 83 – 92

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.1.83

ON A CLASS OF HYPERBOLIC EQUATION WITH AN INTEGRAL TWO-SPACE-VARIABLES CONDITION WITH THE BESSEL OPERATOR

OUSSAEIF TAKI-EDDINE AND BOUZIANI ABDELFATAH

Abstract. In this paper, we study a mixed problem with an integral two-space-variables condition for hyperbolic equation with the bessel operator. The existence and uniqueness of the solution in functional weighted Sobolev space are proved. The proof is based on a priori estimate "energy inequality" and the density of the range of the operator generated by the problem considered.

1. Introduction

The first who drew attention to these problems with an integral one-space-variables condition is Cannon [**8**], which gold of the study of heat conduction in a bar heated thin, has demonstrated by using the potential method, and the importance of the problems with integral conditions has been pointed out by Samarskii [**17**], we remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics. Always using the potential method, established in Kamynin [**14**] the existence and uniqueness of the solution of a similar problems with a more general representation.

Subsequently, more works related to these problems with an integral one-space-variables have been published, among them we cite the work of Benouar-Yurchuk [**1**], Cannon - Van Der Hoek [**10**], [**11**], Cannon - Esteva - Van Der Hoek [**9**], Ionkin [**12**], Jumarhon - McKee [**13**], Kartynnik [**18**], Lin [**19**], Shi [**15**] Yurchuk [**16**]. In these works, it was discussed problems Mixed related to one-dimensional parabolic equations of second order combining a local condition and an integral condition. Also, mention the articles of Bouziani [**2** – **4**] and Bouziani - Benouar [**5**] – [**7**], in which the authors have studied mixed problems with integral conditions for some partial differential equations, specially hyperbolic equation with integral condition has been investigated in Bouziani [**20**].

The present paper is devoted to the study of a problem with a boundary integral two-space-variables condition for second-order hyperbolic equation with the Bessel operator.

1991 *Mathematics Subject Classification.* 35B45, 35D35, 35B30, 35L20, 35L15.

Key words and phrases. Hyperbolic equation; Energy inequality; A priori estimate; Integral boundary two-space-variables condition.

2. Setting of the problem

In the rectangular domain $\Omega = (0,1) \times (0,T)$, with $T < \infty$, we consider the equation:

$$\mathcal{L}u = u_{tt} - \frac{1}{x}u_x - u_{xx} = f(x,t), \tag{2.1}$$

with the initial data

$$\ell_1 u = u(x,0) = \varphi(x), \quad x \in (0,1), \tag{2.2}$$

$$\ell_2 u = u_t(x,0) = \psi(x), \quad x \in (0,1), \tag{2.3}$$

Neumann boundary condition

$$u_x(1,t) = 0, \tag{2.4}$$

and the integral condition

$$\int_0^\alpha xu(x,t)\,dx + \int_\beta^1 xu(x,t)\,dx = 0,\ \beta > \alpha > 0,\ \alpha + \beta = 1, \qquad t \in (0,T), \tag{2.5}$$

where φ and ψ are known functions.

We shall assume that the function ϕ, ψ and f satisfies a compatibility conditions with (2.5), i.e.,

$$\int_0^\alpha x\varphi(x,t)\,dx + \int_\beta^1 x\varphi(x,t)\,dx = 0, \tag{2.6}$$

$$\int_0^\alpha x\psi(x,t)\,dx + \int_\beta^1 x\psi(x,t)\,dx = 0, \tag{2.7}$$

and

$$\int_0^\alpha xf(x,t)\,dx + \int_\beta^1 xf(x,t)\,dx = 0. \tag{2.8}$$

The presence of integral terms in boundary conditions can, in general, greatly complicate the application of standard functional or numerical techniques, specially the integral two-space-variables condition. Then to avoid this difficulty, we introduce a technique for transfer this problem to another classically less complicated and does not contain integral conditions. For that, we establish the following lemma.

Lemma 1. *Problem* $(2.1) - (2.5)$ *is equivalent to the following problem* (PR) :

$$(PR) \quad \begin{cases} \mathcal{L}u = u_{tt} - \frac{1}{x}u_x - u_{xx} = f(x,t), \\ \ell_1 u = u(x,0) = \varphi(x), \\ \ell_2 u = u_t(x,0) = \psi(x), \\ \alpha u_x(\alpha,t) = \beta u_x(\beta,t), \\ u_x(1,t) = 0. \end{cases} \tag{2.9}$$

Proof. Let $u(x,t)$ be a solution of $(2.1) - (2.5)$, we prove that

$$\alpha u_x(\alpha,t) = \beta u_x(\beta,t).$$

So, multiplying equation (2.1) by x and we integrating with respect to x over $(0,\alpha)$ and $(\beta,1)$, and taking into account of (2.5) and (2.8), we obtain

$$x\frac{\partial u}{\partial x}\Big|_{x=0}^{x=\alpha} dx + x\frac{\partial u}{\partial x}\Big|_{x=\beta}^{x=1} dx = 0,$$

this implies

$$\alpha u_x(\alpha,t)+u_x(1,t)-\beta u_x(\beta,t)=0.$$

Then, from (2.4), we obtain

$$\alpha u_x(\alpha,t)=\beta u_x(\beta,t).$$

Let now $u(x,t)$ be a solution of (PR), we are bound to prove that

$$\int_0^\alpha xu(x,t)\,dx+\int_\beta^1 xu(x,t)\,dx=0.$$

So, multiplying equation (2.1) by x and integrating with respect to x over $(0,\alpha)$ and $(\beta,1)$, and taking into account of

$$\alpha u_x(\alpha,t)=\beta u_x(\beta,t),\ u_x(1,t)=0.$$

we obtain

$$\frac{\partial^2}{\partial t^2}\int_0^\alpha xu(x,t)\,dx=\int_0^\alpha xf(x,t)dx$$

and

$$\frac{\partial^2}{\partial t^2}\int_\beta^1 xu(x,t)\,dx=\int_\beta^1 xf(x,t)dx,$$

combining the two preceding equations, and from (2.8) we get

$$\int_0^\alpha xu(x,t)\,dx+\int_\beta^1 xu(x,t)\,dx=0.$$

□

3. A priori estimate

The method used here is one of the most efficient functional analysis methods in solving partial differential equations with integral conditions, the so-called a priori estimate method or the energy-integral method. This method is essentially based on the construction of multiplicators for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the stated problem. But here we use the energy inequality method for the equivalent problem (PR) given in Lemma 1, so to investigated the posed problem, we introduce the needed function spaces.

We now introduce appropriate function spaces. Let $L^2(\Omega)$ be the usual space of square integrable functions and let $L^2_\varrho(\Omega)$ be the weighted L^2-space with finite norm

$$\|u\|^2_{L^2_\varrho(\Omega)}=\int_\Omega \varrho u^2dx$$

and with associated inner product

$$(u,v)_{L^2_\varrho(\Omega)}=\int_\Omega \varrho uvdx,$$

with $\varrho=\sqrt{x}$.

Problem (2.1) – (2.5) can be viewed as the problem of solving the operator equation

$$Lu=\mathcal{F}, \tag{3.1}$$

where $\mathcal{F} = (f, u_0, u_1)$ and L is the operator given by

$$Lu = (\mathcal{L}u, \ell_1 u, \ell_2 u). \tag{3.2}$$

We consider L as an unbounded operator with the domain $D(L)$ consisting of all functions u belonging to $L^2_{\sqrt{x}}(\Omega)$ for which u_t, u_x, u_{xx}, $u_{tt} \in L^2_{\sqrt{x}}(\Omega)$ and u satisfying conditions (2.9d) and (2.9e). Let B be the Banach space obtained by the closure of $D(L)$ in the norm

$$\|u\|_B^2 = \int_\Omega x^3 u_{tt}^2 dxdt + \int_\Omega x^3 u_{xx}^2 dxdt + \int_0^1 x\left(u_x(x,\tau)\right)^2 dx + \sup_{0\le\tau\le T} \int_0^1 x\left(u_t(x,\tau)\right)^2 dx,$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, \varphi, \psi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega f^2 dxdt + \int_0^1 x\psi^2 dx + \int_0^1 x\varphi_x^2 dx \tag{3.3}$$

is finite. The associated inner product is

$$(\mathcal{F}, \mathcal{F}')_F = (f, f')_{L^2(\Omega)} + \left(x\psi, \psi'\right)_{L^2(0,1)} + (x\varphi_x, \varphi'_x)_{L^2(0,1)}. \tag{3.4}$$

Theorem 1. *For any function $u \in B$ we have the inequality*

$$\|u\|_B \le c\,\|Lu\|_F, \tag{3.5}$$

where c is a positive constant independent of u.

Proof. Multiplying the equation (2.1) by the following Mu :

$$Mu = \begin{cases} xu_t + x^3u_{tt} - x^3u_{xx}, & 0 \le x \le \alpha, \\ xu_t + x^3u_{tt} - 2x^3u_{xx}, & \alpha \le x \le \beta, \\ xu_t + x^3u_{tt} - x^3u_{xx}, & \beta \le x \le 1, \end{cases}$$

and integrating over Ω^τ, where $\Omega^\tau = (0,1) \times (0,\tau)$, we have

$$\begin{aligned} \int_\Omega \mathcal{L}u \cdot Mudxdt \;\; = \;\; & \int_{\Omega_\alpha} \left(u_{tt} - \frac{1}{x}u_x - u_{xx}\right)\left(xu_t + x^3u_{tt} - x^3u_{xx}\right) dxdt \\ & + \int_{\Omega_{\alpha,\beta}} \left(u_{tt} - \frac{1}{x}u_x - u_{xx}\right)\left(xu_t + x^3u_{tt} - 2x^3u_{xx}\right) dxdt \\ & + \int_{\Omega_\beta} \left(u_{tt} - \frac{1}{x}u_x - u_{xx}\right)\left(xu_t + x^3u_{tt} - x^3u_{xx}\right) dxdt. \end{aligned}$$

Consequently,

$$\begin{cases} \frac{1}{2}\int_0^\alpha xu_t^2(x,\tau)\,dx - \int_{\Omega_\alpha} (xu_x)_x u_t dxdt + \int_{\Omega_\alpha} x^3u_{tt}^2 dxdt + \int_{\Omega_\alpha} x^3u_{xx}^2 dxdt + \int_{\Omega_\alpha} x^2u_xu_{xx} dxdt \\ \quad = \frac{1}{2}\int_0^\alpha x\psi^2(x)\,dx + \int_{\Omega_\alpha} x^3u_{xx}u_{tt} dxdt + \int_{\Omega_\alpha} x^2u_xu_{tt} dxdt + \int_{\Omega_\alpha} \left(xu_t + x^3u_{tt} - x^3u_{xx}\right) f dxdt, \qquad \Omega \\ \frac{1}{2}\int_\alpha^\beta xu_t^2(x,\tau)\,dx - \int_{\Omega_{\alpha,\beta}} (xu_x)_x u_t dxdt + \int_{\Omega_{\alpha,\beta}} x^3u_{tt}^2 dxdt + 2\int_{\Omega_{\alpha,\beta}} x^3u_{xx}^2 dxdt + 2\int_{\Omega_{\alpha,\beta}} x^2u_xu_{xx} dxdt \\ = \frac{1}{2}\int_\alpha^\beta x\psi^2(x)\,dx + 2\int_{\Omega_{\alpha,\beta}} x^3u_{xx}u_{tt} dxdt + \int_{\Omega_{\alpha,\beta}} x^2u_xu_{tt} dxdt + \int_{\Omega_{\alpha,\beta}} \left(xu_t + x^3u_{tt} - 2x^3u_{xx}\right) f dxdt, \\ \frac{1}{2}\int_\beta^1 xu_t^2(x,\tau)\,dx - \int_{\Omega_\beta} (xu_x)_x u_t dxdt + \int_{\Omega_\beta} x^3u_{tt}^2 dxdt + \int_{\Omega_\beta} x^3u_{xx}^2 dxdt + \int_{\Omega_\beta} x^2u_xu_{xx} dxdt \\ \quad = \frac{1}{2}\int_\beta^1 x\psi^2(x)\,dx + \int_{\Omega_\beta} x^3u_{xx}u_{tt} dxdt + \int_{\Omega_\beta} x^2u_xu_{tt} dxdt + \int_{\Omega_\beta} \left(xu_t + x^3u_{tt} - x^3u_{xx}\right) f dxdt, \qquad \Omega_{,} \end{cases}$$

By integration by parts, we obtain

$$(3.6)\qquad -\int_{\Omega_\alpha}(xu_x)_x\,u_t dxdt = -\frac{\alpha}{2}\int_0^\tau u_x(\alpha,t)\,u_t(\alpha,t)\,dt+\frac{1}{2}\int_0^\alpha x\left(u_x(x,\tau)\right)^2dx-\frac{1}{2}\int_0^\alpha x\varphi_x^2dx.$$

$$(3.7)\qquad \int_{\Omega_\alpha}x^2u_xu_{xx}dxdt=\frac{\alpha^2}{2}\int_0^\tau\left(u_x(\alpha,t)\right)^2dt-\int_{\Omega_\alpha}xu_x^2dxdt,$$

$$(3.8)\qquad -\int_{\Omega_{\alpha,\beta}}(xu_x)_x\,u_t dxdt=-\frac{\beta}{2}\int_0^\tau u_x(\beta,t)\,u_t(\beta,t)\,dt+\frac{\alpha}{2}\int_0^\tau u_x(\alpha,t)\,u_t(\alpha,t)\,dt$$
$$+\frac{1}{2}\int_\alpha^\beta x\left(u_x(x,\tau)\right)^2dx-\frac{1}{2}\int_\alpha^\beta x\varphi_x^2dx,$$

$$(3.9)\qquad 2\int_{\Omega_{\alpha,\beta}}x^2u_xu_{xx}dxdt = \beta^2\int_0^\tau\left(u_x(\beta,t)\right)^2dt-\alpha^2\int_0^\tau\left(u_x(\alpha,t)\right)^2dt-2\int_{\Omega_{\alpha,\beta}}xu_x^2dxdt,$$

$$(3.10)\qquad -\int_{\Omega_\beta}(xu_x)_x\,u_t dxdt=\frac{\beta}{2}\int_0^\tau u_x(\beta,t)\,u_t(\beta,t)\,dt$$
$$+\frac{1}{2}\int_\beta^1 x\left(u_x(x,\tau)\right)^2dx-\frac{1}{2}\int_\beta^1 x\varphi_x^2dx,$$

$$(3.11)\qquad \int_{\Omega_\beta}x^2u_xu_{xx}dxdt=-\frac{\beta^2}{2}\int_0^\tau\left(u_x(\beta,t)\right)^2dt-\int_{\Omega_\beta}x\left(u_x(x,\tau)\right)^2dxdt.$$

By using the Cauchy inequality with ε, and substituting (3.)-(3.) into (3.), we get
(3.12)

$$\begin{cases} -\frac{\alpha}{2}\int_0^{\tau} u_x(\alpha,t)\,u_t(\alpha,t)\,dt+\frac{\alpha^2}{2}\int_0^{\tau}(u_x(\alpha,t))^2\,dt+\frac{1}{8}\int_{\Omega_\alpha}x^3u_{tt}^2dxdt+\frac{1}{4}\int_{\Omega_\alpha}x^3u_{xx}^2dxdt \\ +\frac{1}{2}\int_0^{\alpha}xu_x^2(x,\tau)\,dx+\frac{1}{2}\int_0^{\alpha}xu_t^2(x,\tau)\,dx \\ \le\frac{7}{2}\int_{\Omega_\alpha}f^2dxdt+\frac{1}{2}\int_0^{\alpha}x\psi^2dx+\frac{1}{2}\int_0^{\alpha}x\varphi_x^2dx+2\int_{\Omega_\alpha}xu_x^2dxdt+\frac{1}{2}\int_{\Omega_\alpha}xu_t^2dxdt, \qquad \Omega_\alpha, \\ -\frac{\beta}{2}\int_0^{\tau}u_x(\beta,t)\,u_t(\beta,t)\,dt+\frac{\alpha}{2}\int_0^{\tau}u_x(\alpha,t)\,u_t(\alpha,t)\,dt+\beta^2\int_0^{\tau}(u_x(\beta,t))^2\,dt-\alpha^2\int_0^{\tau}(u_x(\alpha,t))^2\,dt \\ +\frac{1}{2}\int_{\alpha}^{\beta}xu_x^2(x,\tau)\,dx+\frac{1}{2}\int_{\alpha}^{\beta}xu_t^2(x,\tau)\,dx+\frac{1}{8}\int_{\Omega_{\alpha,\beta}}x^3u_{tt}^2dxdt+\frac{11}{8}\int_{\Omega_{\alpha,\beta}}x^3u_{xx}^2dxdt \\ \le\frac{23}{2}\int_{\Omega_{\alpha,\beta}}f^2dxdt+\frac{1}{2}\int_{\alpha}^{\beta}x\psi^2(x)\,dx+\frac{1}{2}\int_{\alpha}^{\beta}x\varphi_x^2dx+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}xu_t^2dxdt+3\int_{\Omega_{\alpha,\beta}}xu_x^2dxdt, \qquad \Omega_{\alpha,\beta} \\ +\frac{\beta}{2}\int_0^{\tau}u_x(\beta,t)\,u_t(\beta,t)\,dt-\frac{\beta^2}{2}\int_0^{\tau}(u_x(\beta,t))^2\,dt+\frac{1}{8}\int_{\Omega_\beta}x^3u_{tt}^2dxdt+\frac{1}{4}\int_{\Omega_\beta}x^3u_{xx}^2dxdt \\ +\frac{1}{2}\int_{\beta}^{1}xu_x^2(x,\tau)\,dx+\frac{1}{2}\int_{\beta}^{1}xu_t^2(x,\tau)\,dx \\ \le\frac{7}{2}\int_{\Omega_\beta}f^2dxdt+\frac{1}{2}\int_{\beta}^{1}x\psi^2(x)\,dx+\frac{1}{2}\int_{\beta}^{1}x\varphi_x^2dx+2\int_{\Omega_\beta}xu_x^2dxdt+\frac{1}{2}\int_{\Omega_\beta}xu_t^2dxdt, \qquad \Omega_\beta. \end{cases}$$

Now, combining inequalities (3.), (3.) and (3.), and by using the conditions in (PR) ,we obtain

$$\text{(3.13)}\quad \frac{1}{8}\int_{\Omega}x^3u_{tt}^2dxdt+\frac{1}{4}\int_{\Omega}x^3u_{xx}^2dxdt+\frac{1}{2}\int_0^1xu_x^2(x,\tau)\,dx+\frac{1}{2}\int_0^1xu_t^2(x,\tau)\,dx$$
$$\le \frac{23}{2}\int_{\Omega}f^2dxdt+\frac{1}{2}\int_0^1x\psi^2dx+\frac{1}{2}\int_0^1x\varphi_x^2dx$$
$$+3\int_{\Omega}xu_x^2dxdt+\frac{1}{2}\int_{\Omega}xu_t^2dxdt.$$

Then, we obtain

$$\text{(3.14)}\quad \int_{\Omega}x^3u_{tt}^2dxdt+\int_{\Omega}x^3u_{xx}^2dxdt+\left(\int_0^1xu_x^2(x,\tau)\,dx+\int_0^1xu_t^2(x,\tau)\,dx\right)$$
$$\le 92\left(\int_{\Omega}f^2dxdt+\int_0^1x\psi^2(x)\,dx+\int_0^1x\varphi_x^2dx\right)$$
$$+24\int_0^{\tau}\left(\int_0^1xu_x^2dxdt+\int_0^1xu_t^2dx\right)dt,$$

Using Lemma 1 in [21] , we have

$$\text{(3.15)}\quad \int_{\Omega}x^3u_{tt}^2dxdt+\int_{\Omega}x^3u_{xx}^2dxdt+\int_0^1xu_x^2(x,\tau)\,dx+\int_0^1xu_t^2(x,\tau)\,dx$$
$$\le k\left(\int_{\Omega}f^2dxdt+\int_0^1x\psi^2(x)\,dx+\int_0^1x\varphi_x^2dx\right),$$

where

$$k=92\exp(24T).$$

The right-hand side of (3.21) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, we obtain the desired inequality, where $c = (k)^{\frac{1}{2}}$. □

Corollary 1. *A solution of the problem* (2.1) – (2.5) *is unique if it exists, and depends continuously on* $\mathcal{F} \in F$.

4. Existence of solution

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in B$ and for arbitrary $\mathcal{F} = \mathcal{F} = (f, \varphi, \psi) \in F$.

Theorem 2. *Suppose the conditions of Theorem* 1 *are satisfied. Then the problem* (2.1) – (2.3) *admits a unique solution* $u = L^{-1}\mathcal{F}$.

Proof. First we prove that $R(L)$ is dense in F for the special case where $D(L) \equiv B$ is reduced to $D_0(L)$, where $D_0(L) = \{u,\ u \in D(L) : \ell_1 u = 0,\ \ell_2 u = 0\}$. □

Proposition 1. *Let the conditions of theorem 2 be satisfied. If, for* $\omega \in L^2_{\sqrt{x}}(\Omega)$ *and for all* $u \in D_0(L)$, *we have*

$$\int_\Omega \mathcal{L}u \cdot \omega dxdt = 0, \tag{4.1}$$

then ω *vanishes almost everywhere in* Ω.

Proof. The scalar product of F is defined by

$$(Lu, \omega)_F = \int_\Omega \mathcal{L}u \cdot \omega dxdt + \int_0^1 x \left(\frac{\partial \ell_1 u}{\partial x}\right)\left(\frac{\partial \omega_1}{\partial x}\right) dx + \int_0^1 x (\ell_2 u)(\omega_2)\, dx, \tag{4.2}$$

the equality (4.1) can be written as follows:

$$\int_\Omega u_{tt} \omega dxdt = \int_\Omega \left(\frac{1}{x} u_x + u_{xx}\right) \omega dxdt. \tag{4.3}$$

If we put

$$u(x,t) = \begin{cases} 0 & t = 0, \\ -\Im_t^\star z = -\int_t^T z(x,\tau)\, d\tau & 0 < t \le T, \end{cases} \tag{4.4}$$

where $z, \frac{\partial z}{\partial x}, \frac{\partial \Im_t^\star z}{\partial x} \in L^2_{\sqrt{x}}(\Omega)$. Then, relation (4.4) and the assumptions on the function z imply that u is in $D_0(L)$.

$$\int_\Omega \frac{\partial z}{\partial t} \omega dxdt = -\int_\Omega \frac{1}{x}\frac{\partial}{\partial x}\left(x \frac{\partial \Im_t^\star z}{\partial x}\right) \omega dxdt. \tag{4.5}$$

In terms of the given function ω, and from the equality (4.5) we give the function ω in terms of z as follows:

$$\omega = \begin{cases} \left(x - \frac{x^2}{2}\right) \Im_t^\star \frac{\partial z}{\partial x}, & 0 \le x \le \alpha, \\ x \Im_t^\star \frac{\partial z}{\partial x}, & \alpha \le x \le \beta, \\ \left(x - \frac{x^2}{2}\right) \Im_t^\star \frac{\partial z}{\partial x}, & \beta \le x \le 1, \end{cases} \tag{4.6}$$

So, relation (4.6) and the assumptions on the function z imply that ω is in $L^2_{\sqrt{x}}(\Omega)$.

and z satisfies the same conditions of the function u in (PR) :

$$\alpha\frac{\partial z}{\partial x}(\alpha,t)=\beta\frac{\partial z}{\partial x}(\beta,t)\,,\;\frac{\partial z}{\partial x}(1,t)=0, \tag{4.7}$$

Replacing ω in (4.5) by its representation (4.6) and integrating by parts each term of (4.5) and by taking the condition (4.7), we obtain

$$\int_{\Omega^\tau}\frac{\partial z}{\partial t}\omega dxdt=\begin{cases} -\int_0^\tau\left(\frac{\alpha}{2}-\frac{\alpha^2}{4}\right)\left(\Im_t^\star\frac{\partial z(\alpha,t)}{\partial x}\right)^2dt-\frac{1}{2}\int_{\Omega_\alpha^\tau}\left(\Im_t^\star\frac{\partial z}{\partial x}\right)^2dxdt & \Omega_\alpha^\tau=(0,\alpha)\times(0,\tau)\\ -\frac{\beta}{2}\int_0^\tau\left(\Im_t^\star\frac{\partial z(\alpha,t)}{\partial x}\right)^2dt+\frac{\alpha}{2}\int_0^\tau\left(\Im_t^\star\frac{\partial z(\alpha,t)}{\partial x}\right)^2dt & \\ -\frac{1}{2}\int_{\Omega_{\alpha,\beta}^\tau}\left(\Im_t^\star\frac{\partial z}{\partial x}\right)^2dxdt & \Omega_{\alpha,\beta}^\tau=(\alpha,\beta)\times(0,\tau)\\ \int_0^\tau\left(\frac{\beta}{2}-\frac{\beta^2}{4}\right)\left(\Im_t^\star\frac{\partial z(\beta,t)}{\partial x}\right)^2dxdt-\frac{1}{2}\int_{\Omega_\beta^\tau}\left(\Im_t^\star\frac{\partial z}{\partial x}\right)^2dxdt & \Omega_\beta^\tau=(\beta,1)\times(0,\tau)\end{cases}$$

So, we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\alpha^\tau}(1-x)\,z^2dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}^\tau}z^2dxdt+\frac{1}{2}\int_{\Omega_\beta^\tau}(1-x)\,z^2dxdt\\ =\;&\frac{\alpha^2}{4}\int_0^\tau\left(\Im_t^\star\frac{\partial z\,(\alpha,t)}{\partial x}\right)^2dt-\frac{\beta^2}{4}\int_0^\tau\left(\Im_t^\star\frac{\partial z\,(\alpha,t)}{\partial x}\right)^2dt\\ &-\frac{1}{2}\int_{\Omega^\tau}\left(\Im_t^\star\frac{\partial z}{\partial x}\right)^2dxdt,\end{aligned}$$

by taking into account the boundary conditions (4.6) of the function z yields

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\alpha^\tau}(1-x)\,z^2dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}^\tau}z^2dxdt+\frac{1}{2}\int_{\Omega_\beta^\tau}(1-x)\,z^2dxdt\\ =\;&-\frac{1}{2}\int_{\Omega^\tau}\left(\Im_t^\star\frac{\partial z}{\partial x}\right)^2dxdt\\ \le\;&0.\end{aligned}$$

Then, we obtain

$$\frac{1}{2}\left(\int_{\Omega_\alpha^\tau}(1-x)\,z^2dxdt+\int_{\Omega_{\alpha,\beta}^\tau}z^2dxdt+\int_{\Omega_\beta^\tau}(1-x)\,z^2dxdt\right)\le 0.$$

and thus $z=0$ in Ω, then $\omega=0$ in Ω. This proves Proposition 2. □

We return to the proof of Theorem 2. We have already noted that it is sufficient to prove that the set $R(L)$ dense in F.

Suppose that, for some $W=(\omega,\omega_1,\omega_2)\in R(L)^\perp$ and for all $u\in D(L)\equiv B$, it holds

$$(Lu,\omega)_F=\int_\Omega\mathcal{L}u.\omega dxdt+\int_0^1x\left(\frac{\partial\ell_1u}{\partial x}\right)\left(\frac{\partial\omega_1}{\partial x}\right)dx+\int_0^1x\,(\ell_2u)\,(\omega_2)\,dx. \tag{4.8}$$

Then we must prove that $W=0$. Putting $u\in D_0(L)$ in (4.8), we have

$$\int_\Omega\mathcal{L}u.\omega dxdt=0,\quad u\in D_0(L).$$

Hence Proposition 1 implies that $\omega=0$. Thus (4.8) takes the form

$$\int_0^1x\left(\frac{\partial\ell_1u}{\partial x}\right)\left(\frac{\partial\omega_1}{\partial x}\right)dx+\int_0^1x\,(\ell_2u)\,(\omega_2)\,dx=0,\quad u\in D(L). \tag{4.9}$$

Since the range of the trace operators ℓ_1 and ℓ_2 are every where dense in the Hilbert space F with the norm

$$\left(\int_0^1 \left[x\left(\frac{\partial \ell_1 u}{\partial x}\right)^2 dx + \int_0^1 x\,(\ell_2 u)^2\,dx\right]\right)^{\frac{1}{2}},$$

the equality (4.9) implies that $\omega_1 = 0$ and $\omega_2 = 0$. Hence $W = 0$ implies $\left(\overline{R(L)} = F\right)$. Therefore, the proof of Theorem 2 is complete.

References

[1] N.E. Benouar and N.I. Yurchuk, Mixed problem with an integral condition for parabolic equations with the Bessel operator, Differentsial'nye Uravneniya, 27 (1991), 2094-2098.

[2] A. Bouziani, Mixed problem for certain nonclassical equations with a small parameter, Bulletin de la Classe des Sciences, Académie Royale de Belgique, 5 (1994), 389-400.

[3] A. Bouziani, Mixed problem with integral conditions for a certain parabolic equation, J. of Appl. Math. and Stoch. Anal. 9 (1996), 323-330.

[4] A. Bouziani, Mixed problem with nonlocal condition for certain pluriparabolic equations, Hiroshima Math. J. 27 (1997), 373-390.

[5] A. Bouziani and N.E. Benouar, Problème mixte avec conditions intégrales pour une classe d'équations paraboliques, Comptes Rendus de l'Académie des Sciences, Paris t.321, Série I, (1995), 1177-1182.

[6] A. Bouziani and N.-E. Benouar, Mixed problem with integral conditions for a third order parabolic equation, Kobe J. Math. 15 (1998), no. 1, 47–58.

[7] A. Bouziani and N.E. Benouar, Problèmes aux limites pour use classe d'équations de type non classique pour use structure pluri-dimensionnelle, Bull. of the Polish Acad. of Sciences-Mathematics 43,(1995), 317-328

[8] J. R. Cannon, The solution of the heat equation subject to the specification of energy, Quart. Appl. Math. 21 (1963), 155–160.

[9] J. R. Cannon, S. Pérez Esteva, and J. van der Hoek, A Galerkin procedure for the diffusion equation subject to the specification of mass, SIAM J., Numer. Anal. 24 (1987), no. 3, 499–515.

[10] J.R. Cannon and J. Van der Hoek, The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, Boll. Uni. Math. Ital. Suppl. 1 (1981), 253-282.

[11] J.R. Cannon and J. Van der Hoek, An implicit finite difference scheme for the diffusion of mass in a portion of the domain, Numer. Solutions of PDEs (ed. by J. Noye), North-Holland, Amsterdam (1982), 527 539.

[12] N.I. Ionkin, Solution of boundary value problem in heat conduction theory with nonlocal boundary conditions, Differentsial'nye Uravneniya, 13 (1977), 294-304.

[13] B. Jumarhon, and S. McKee, On the heat equation with nonlinear and nonlocal boundary conditions, J. Math. Anal. Appl. 190 (1995), 806-820.

[14] N.I. Kamynin, , A boundary value problem in the theory of the heat conduction with non-classical boundary condition, Th. Vychisl. Mat. Mat. Fiz. 43 (1964),1006-1024.

[15] P. Shi, Weak solution to an evolution problem with a nonlocal constraint, SIAM. J. Math. Anal. 24 (1993), 46-58.

[16] N.I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, Differentsial'nye Uravneniya 22 (1986), 2117-2126.

[17] A.A. Samarskii, Some problems in differential equations theory, Differents. Uravn. 16 (1980), 1925-1935

[18] A. V Kartynnik, Three points boundary value problem with an integral space variables conditions for second order parabolic equations,Differentsial'nye Uravneniya., 26, N'9, 1568-1575, 1990.

[19] Y. Lin, Parabolic partial differential equation subject to nonlocal boundary conditions, Ph.D. thesis, Washington State University, Pullman, WA, 1988.

[20] Solution forte d'un problème mixte avec une condition non locale pour une classe d'équations hyperboliques, Acad. Roy. Belg. Bull. Cl. Sci. (6) 8 (1997), 53–70 (French).

Current address: Bp 103 El aouinet, Tebessa, 12005, Algeria.
E-mail address: taki_maths@live.fr

Current address: Department of Mathematics and Informatics., Larbi Ben M'Hidi University, Oum El Bouaghi, 04000, Algeria.
E-mail address: af_bouziani@yahoo.com