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SOME IDENTITIES OF DERANGEMENT NUMBERS ARISING FROM DIFFERENTIAL EQUATIONS

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Abstract. Recently, several authors have studied derangement numbers, which are related to second kind of stirling numbers and bell numbers (see [5]). In this paper, we derive the differential equation arising from the generating function of derangement numbers and we give some identities of derangement numbers which are derived from our differential equations.

1. Introduction

As is well known that the Derangement numbers are defined by the generating function to be

$$\frac{1}{1-t}e^{-t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}. \tag{1.1}$$

In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. In other words, derangement is a permutation that has no fixed points. The derangement numbers are $d_0 = 0$, $d_1 = 1$, $d_2 = 2$, $d_3 = 9$, $d_4 = 44$, $d_5 = 265$, $\cdots$

As is well known that the Arrangement numbers are defined by the generating function to be

$$\frac{1}{1-t}e^{t} = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}. \tag{1.2}$$

The number of arrangement of any subset of n distinct objects is the number of one-to-one sequences that can be formed from any subset of n distinct objects. The arrangement numbers are $a_0 = 1$, $a_1 = 2$, $a_2 = 5$, $a_3 = 16$, $a_4 = 65$, $\cdots$.

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From (1.1), we have

$$\begin{aligned}\frac{1}{1-t}e^{-t} &= \frac{1}{1-t}e^{t}e^{-2t}\\ &= \Big(\sum_{l=0}^{\infty} a_l \frac{t^l}{l!}\Big)\Big(\sum_{m=0}^{\infty}(-2)^m\frac{t^m}{m!}\Big)\\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}a_l(-2)^{n-l}\Big)\frac{t^n}{n!}.\end{aligned} \tag{1.3}$$

From (1.1) and (1.3), we get

$$d_n = \sum_{l=0}^{n}\binom{n}{l}a_l(-2)^{n-l}. \tag{1.4}$$

Kim-Kim-Kwon firstly considered the higher-order derangement numbers which are defined by the generating function to be

$$\Big(\frac{1}{1-t}\Big)^k e^{-t} = \sum_{n=0}^{\infty} d_n^{(k)}\frac{t^n}{n!}, \quad (k\in\mathbb{N}), \quad (see[6]). \tag{1.5}$$

From (1.1) and (1.5), we get

$$\begin{aligned}\Big(\frac{1}{1-t}\Big)^k e^{-t} &= \Big(\frac{1}{1-t}\Big)^k e^{-kt}e^{(k-1)t}\\ &= \underbrace{\Big(\frac{1}{1-t}e^{-t}\Big)\times\Big(\frac{1}{1-t}e^{-t}\Big)\times\cdots\times\Big(\frac{1}{1-t}e^{-t}\Big)}_{k-times}\times e^{(k-1)t}\\ &= \Big(\sum_{l_1=0}^{\infty} d_{l_1}\frac{t^{l_1}}{l_1!}\Big)\Big(\sum_{l_2=0}^{\infty} d_{l_2}\frac{t^{l_2}}{l_2!}\Big)\times\cdots\times\Big(\sum_{l_k=0}^{\infty} d_{l_k}\frac{t^{l_k}}{l_k!}\Big)\times e^{(k-1)t}\\ &= \Big(\sum_{l=0}^{\infty}\Big(\sum_{l_1+l_2+\cdots+l_k=l}\binom{l}{l_1,l_2,\cdots,l_k}d_{l_1}d_{l_2}\cdots d_{l_k}\Big)\frac{t^l}{l!}\Big)\\ &\times\Big(\sum_{m=0}^{\infty}(k-1)^m\frac{t^m}{m!}\Big)\\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\sum_{l_1+l_2+\cdots+l_k=l}\binom{l}{l_1,l_2,\cdots,l_k}\binom{n}{l}d_{l_1}d_{l_2}\cdots d_{l_k}\\ &\times(k-1)^{n-m}\Big)\frac{t^n}{n!}.\end{aligned} \tag{1.6}$$

Recently, several authors have studied derangement numbers, which are related to second kind of stirling numbers and bell numbers (see [5]). In this paper, we study the differential equations which are derived from the generating function of derangement numbers. In addition, we give some identities of derangement numbers arising from differential equations.

2. Some identities of Derangement numbers arising from differential equation

Let

$$F = F(t) = \frac{1}{1-t}e^{-t}. \tag{2.1}$$

Taking the derivatives of (2.1) with respect to t, we have

$$\begin{aligned} F^{(1)} = \frac{d}{dt}F(t) &= \frac{-e^{-t}(1-t)+e^{-t}}{(1-t)^2} \\ &= -\frac{e^{-t}}{1-t} + \left(\frac{e^{-t}}{1-t}\right)^2 e^t \\ &= -F + e^t F^2. \end{aligned} \tag{2.2}$$

From (2.2), we have

$$\begin{aligned} F^{(2)} &= -F^{(1)} + 2e^t F F^{(1)} + e^t F^2 \\ &= F^{(1)}(-1+2e^t F) + e^t F^2 \\ &= (-F + e^t F^2)(-1+2e^t F) + e^t F^2 \\ &= F - 2e^t F^2 + 2e^{2t} F^3. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we note that

$$\begin{aligned} F^{(3)} &= F^{(1)} - 4e^t F F^{(1)} - 2e^t F^2 + 6e^{2t} F^2 F^{(1)} + 4e^{2t} F^3 \\ &= F^{(1)}(1 - 4e^t F + 6e^{2t} F^2) - 2e^t F^2 + 4e^{2t} F^3 \\ &= (-F + e^t F^2)(1 - e^t 4F + 6e^{2t} F^2) - 2e^t F^2 + 4e^{2t} F^3 \\ &= -F + 3e^t F^2 - 6e^{2t} F^3 + 6e^{3t} F^4. \end{aligned} \tag{2.4}$$

Continuing this process, we get

$$F^{(N)} = \sum_{k=0}^{N} (-1)^{N-k} c_k(N) e^{kt} F^{k+1}. \tag{2.5}$$

Let us take the derivative on the both sides of (2.5) with respect to t, Then we have

$$F^{(N+1)} = \sum_{k=0}^{N} (-1)^{N-k} c_k(N) e^{kt} \big(kF^{k+1} + (k+1) F^k F^{(1)} \big). \tag{2.6}$$

Applying the identity (2.2) to (2.6), we get

$$\begin{aligned} F^{(N+1)} &= \sum_{k=0}^{N} (-1)^{N-k} k c_k(N) e^{kt} F^{k+1} + \sum_{k=0}^{N} (-1)^{N-k} (k+1) c_k(N) e^{kt} \\ &\times F^k F^{(1)} \\ &= \sum_{k=0}^{N} (-1)^{N-k} k c_k(N) e^{kt} k F^{k+1} + \sum_{k=0}^{N} (-1)^{N-k} (k+1) c_k(N) e^{kt} \\ &\times F^k (-F + F^2 e^t) \\ &= \sum_{k=0}^{N} (-1)^{N-k+1} c_k(N) e^{kt} F^{k+1} + \sum_{k=0}^{N} (-1)^{N-k} (k+1) c_k(N) \\ &\times e^{(k+1)t} F^{k+2} \\ &= \sum_{k=0}^{N} (-1)^{N-k+1} c_k(N) e^{kt} F^{k+1} + \sum_{k=1}^{N+1} (-1)^{N-k+1} k c_{k-1}(N) \\ &\times e^{kt} F^{k+1} \\ &= (-1)^{N+1} c_0(N) F + (N+1) c_N(N) e^{(N+1)t} F^{N+2} \\ &+ \sum_{k=1}^{N} (-1)^{N-k+1} \Big(c_k(N) + k c_{k-1}(N) \Big) e^{kt} F^{k+1}. \end{aligned} \tag{2.7}$$

By replacing N by $N+1$ in (2.5), we get

$$\begin{aligned} F^{(N+1)} &= \sum_{k=0}^{N+1} (-1)^{N-k+1} c_k(N+1) e^{kt} F^{k+1} \\ &= (-1)^{N+1} c_0(N+1) F + c_{N+1}(N+1) e^{(N+1)t} F^{N+2} \\ &+ \sum_{k=1}^{N} (-1)^{N-k+1} c_k(N+1) e^{kt} F^{k+1}. \end{aligned} \tag{2.8}$$

Comparing the coefficients on the both sides of (2.7) and (2.8), we have

$$c_0(N+1) = c_0(N), \quad c_{N+1}(N+1) = (N+1) c_N(N), \tag{2.9}$$

and

$$c_k(N+1) = c_k(N) + kc_{k-1}(N), \tag{2.10}$$

where $1 \leq k \leq N$.

From (2.2) and (2.5), we get

$$\begin{aligned} F^{(1)} &= \sum_{k=0}^{1} (-1)^{1-k} c_k(1) e^{kt} F^{k+1} \\ &= -c_0(1)F + c_1(1)e^t F^2 \\ &= -F + e^t F^2. \end{aligned} \tag{2.11}$$

By (2.11), we get

$$c_0(1) = 1, \quad c_1(1) = 1. \tag{2.12}$$

Thus, by (2.9) and (2.12), we have

$$c_0(N+1) = c_0(N) = c_0(N-1) = \cdots = c_0(1) = 1, \tag{2.13}$$

and

$$\begin{aligned} c_{N+1}(N+1) &= (N+1)c_N(N) = (N+1)Nc_{N-1}(N-1) = \cdots \\ &= (N+1)N\cdots 2c_1(1) = (N+1)N\cdots 2\cdot 1 = (N+1)!. \end{aligned} \tag{2.14}$$

From (2.10), we have

$$\begin{aligned} c_k(N+1) &= c_k(N) + kc_{k-1}(N) \\ &= c_k(N-1) + kc_{k-1}(N-1) + kc_{k-1}(N) \\ &= \cdots \\ &= c_k(k) + kc_{k-1}(k) + \cdots + kc_{k-1}(N). \end{aligned} \tag{2.15}$$

By (2.13) and (2.15), we get

$$\begin{aligned} c_k(N+1) &= c_k(k) + kc_{k-1}(k) + \cdots + kc_{k-1}(N) \\ &= kc_{k-1}(k-1) + kc_{k-1}(k) + \cdots + kc_{k-1}(N) \\ &= \sum_{i_1=0}^{N-k+1} kc_{k-1}(k-1+i_1) \\ &= \sum_{i_1=0}^{N-k+1} k \sum_{i_2=0}^{i_1} (k-1)c_{k-2}(k-2+i_2) \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} k(k-1)c_{k-2}(k-2+i_2) \\ &= \cdots \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} k(k-1)\cdots 1 c_0(i_k). \end{aligned} \tag{2.16}$$

By (2.13) and (2.16), we get

$$\begin{aligned} c_k(N+1) &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} k(k-1)\cdots 1 \\ &= \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} k! \\ &= k! \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} 1 \\ &= k! \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} (i_{k-1}+1). \end{aligned} \tag{2.17}$$

Theorem 2.1. *Let $N \in \mathbb{N} \cup \{0\}$.*
Then the following differential equations

$$F^{(N)} = \sum_{k=0}^{N} (-1)^{N-k} c_k(N) e^{kt} F^{k+1},$$

have a solution for $F(t) = \frac{1}{1-t}e^{-t}$, *where*

$$c_0(N) = 1, \quad c_N(N) = N!,$$
$$c_k(N) = k! \sum_{i_1=0}^{N-k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} (i_{k-1}+1),$$

for $1 \le k \le N-1$.

By (1.1), we have

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\ &= \left(\frac{d}{dt}\right)^N \Big(\sum_{n=0}^{\infty} d_n \frac{t^n}{n!}\Big) \\ &= \sum_{n=0}^{\infty} d_{n+N} \frac{t^n}{n!}. \end{aligned} \tag{2.18}$$

Thus by (1.5), (2.5) and (2.18), we get

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n+N} \frac{t^n}{n!} &= \sum_{k=0}^{N} (-1)^{N-k} c_k(N) e^{kt} F^{k+1} \\ &= \sum_{k=0}^{N} (-1)^{N-k} c_k(N) e^{kt} \Big(\frac{1}{1-t} e^{-t}\Big)^{k+1} \\ &= \sum_{k=0}^{N} (-1)^{N-k} c_k(N) \Big(\frac{1}{1-t}\Big)^{k+1} e^{-t} \\ &= \sum_{k=0}^{N} (-1)^{N-k} c_k(N) \sum_{n=0}^{\infty} d_n^{(k+1)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{N} (-1)^{N-k} c_k(N) d_n^{(k+1)}\Big) \frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

From (2.19), we get the following theorem

Theorem 2.2. *For* $n, N \in \mathbb{N} \cup \{0\}$, *we have*

$$d_{n+N} = \sum_{k=0}^{N} (-1)^{N-k} c_k(N) d_n^{(k+1)}.$$

3. Inversion formula

Lemma 3.1. *The inversion formula of Theorem1 is given by*
$N!e^{Nt}F^{N+1}=\sum_{k=0}^{N}\binom{N}{k}F^{(k)}$.
Proof By (1.2) we have $e^tF^2=\binom{1}{0}F+\binom{1}{1}F^{(1)}$, which proves the lemma for $N=1$. Assume that $N!e^{Nt}F^{N+1}$ is given by$\sum_{k=0}^{N}\binom{N}{k}F^{(k)}$.Then

$$N!e^{Nt}\big(NF^{N+1}+(N+1)F^NF^{(1)}\big)=\sum_{k=0}^{N}\binom{N}{k}F^{(k+1)}. \tag{3.1}$$

By (2.2) and (3.1), we have

$$N!e^{Nt}\big(NF^{N+1}+(N+1)F^N(-F+e^tF^2)\big)=\sum_{k=0}^{N}\binom{N}{k}F^{(k+1)}. \tag{3.2}$$

From (3.2), we get

$$\begin{aligned}(N+1)!e^{(N+1)t}F^{N+2}&=N!e^{Nt}F^{N+1}+\sum_{k=0}^{N}\binom{N}{k}F^{(k+1)}\\&=\sum_{k=0}^{N}\binom{N}{k}F^{(k)}+\sum_{k=0}^{N}\binom{N}{k}F^{(k+1)}\\&=\sum_{k=0}^{N}\binom{N}{k}F^{(k)}+\sum_{k=1}^{N+1}\binom{N}{k-1}F^{(k)}\\&=\binom{N}{0}F+\binom{N}{N}F^{(N+1)}\\&+\sum_{k=1}^{N}\left(\binom{N}{k}+\binom{N}{k-1}\right)F^{(k)}.\end{aligned} \tag{3.3}$$

Since

$$\binom{N}{0}=1=\binom{N+1}{0},\quad \binom{N}{N}=1=\binom{N+1}{N+1}, \tag{3.4}$$

and

$$\binom{N+1}{k}=\binom{N}{k}+\binom{N}{k-1}. \tag{3.5}$$

By (3.3), (3.4) and (3.5), we get

$$\begin{aligned}(N+1)!e^{(N+1)t}F^{N+2} &= \binom{N+1}{0}F + \binom{N+1}{N+1}F^{(N+1)} \\ &\quad + \sum_{k=1}^{N}\binom{N+1}{k}F^{(k)} \\ &= \sum_{k=0}^{N+1}\binom{N+1}{k}F^{(k)}.\end{aligned} \tag{3.6}$$

We complete the proof.

Therefore by lemma 3.1. we get the following theorem.

Theorem 3.2. *For $N \in \mathbb{N} \cup \{0\}$, Then the following differential equations*

$$N!e^{Nt}F^{N+1} = \sum_{k=0}^{N}\binom{N}{k}F^{(k)}.$$

have a solution $F(t) = \frac{1}{1-t}e^{-t}$.

From left sides of Theorem 3.2, we get

$$\begin{aligned}N!e^{Nt}F^{N+1} &= N!e^{Nt}\Big(\frac{1}{1-t}e^{-t}\Big)^{N+1} \\ &= N!\Big(\frac{1}{1-t}\Big)^{N+1}e^{-t}.\end{aligned} \tag{3.7}$$

From (1.5) and (3.7), we have

$$N!e^{Nt}F^{N+1} = N!\sum_{n=0}^{\infty}d_n^{(N+1)}\frac{t^n}{n!}. \tag{3.8}$$

Thus, by (2.18) and (3.8), we get

Theorem 3.3. *Let $n, N \in \mathbb{N} \cup \{0\}$, we get*

$$d_n^{(N+1)} = \frac{1}{N!}\sum_{k=0}^{N}\binom{N}{k}d_{n+k}.$$

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