Advanced Studies in Contemporary Mathematics
28 (2018), No. 1, pp. 27 - 39

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2018.28.1.27

A GLOBAL BIFURCATION FOR NONLINEAR ELLIPTIC EQUATIONS INVOLVING NONHOMOGENEOUS OPERATORS OF $p(x)$-LAPLACE TYPE

YUN-HO KIM

Abstract. We are concerned with the following nonlinear problem

$$-\mathrm{div}(\psi(x,\nabla u)) = \mu|u|^{p(x)-2}u + f(\lambda,x,u,\nabla u) \quad \text{in } \Omega$$

subject to Dirichlet boundary condition, provided that μ is not an eigenvalue of the $p(x)$-Laplacian. The aim of this paper is to study the structure of the set of weak solutions of nonlinear equations of $p(x)$-Laplace type, by applying a bifurcation result for nonlinear operator equations.

2010 Mathematics Subject Classification. 35B32; 35D30; 35J60; 35P30; 37K50; 47J10.

Keywords and phrases. Bifurcation; $p(x)$-Laplacian; Variable exponent Lebesgue-Sobolev spaces; Weak solutions; Eigenvalue.

1. Introduction

Bifurcation theory for nonlinear eigenvalue problems was originated by Krasnoselskii [20]. Using the topological approach of Krasnoselskii, Rabinowitz [24] investigated that the bifurcation occurring in the Krasnoselskii theorem is actually a global phenomenon. Based on the paper [24], many researchers have studied bifurcation problems about p-Laplacian and generalized operators; see [2, 5, 6, 11, 22, 26, 27]. They mainly used Ljusternik-Schnirelman theory to obtain eigenvalues of nonlinear operators and yielded global bifurcation from the first eigenvalue of the p-Laplacian. While most of them considered global branches in this way, under suitable conditions, Väth [28] suggested the inventive approach to establish the existence of a global branch of solutions for the p-Laplacian with Dirichlet boundary condition by applying nonlinear spectral theory for homogeneous operators. Recently Kim and Väth [17] established the existence of an unbounded branch of solutions for equations involving nonhomogeneous operators of p-Laplace type (see [15] for generalizations to unbounded domains with weighted functions).

In this paper, we are concerned with the existence of an unbounded branch of the set of solutions of nonlinear elliptic equations of $p(x)$-Laplace type subject to Dirichlet boundary condition

$$\text{(B)} \qquad \begin{cases} -\mathrm{div}(\psi(x,\nabla u)) = \mu|u|^{p(x)-2}u + f(\lambda,x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ is not an eigenvalue of

$$\text{(E)} \qquad \begin{cases} -\Delta_{p(x)}u = \mu|u|^{p(x)-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\psi(x,\cdot):\mathbb{R}^N\to\mathbb{R}^N$ is not necessarily positively homogeneous or odd, $p:\overline{\Omega}\to(1,\infty)$ is continuous and $f:\mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ satisfies a Carathéodory condition. Here the operator $-\Delta_{p(x)}u := -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian. Recently, the study about the differential equations and variational problems involving $p(x)$-growth conditions has been extensively investigated because they explain a various physical phenomena, for example, elastic mechanics, electro-rheological fluid dynamics and image processing, *etc.* We refer the readers to [1, 4, 23, 25, 30] and the references therein. However the bifurcation result on nonlinear elliptic equations with variable exponents is rare with the exception of the papers [13, 16, 18]. It is well known that the positivity of the isolated principal eigenvalue for the p-Laplacian plays a key role in obtaining the bifurcation result for elliptic equations involving p-Laplacian. But unlike the p-Laplacian, the infimum of all eigenvalues for the $p(x)$-Laplacian might be zero (see [9]). Based on the work of Väth [28], the global behavior of solutions for degenerated elliptic equations involving the $p(x)$-Laplacian has been considered in [18]; see [13] for Neumann problems with variable exponents. Concerning nonlinear elliptic equations involving nonhomogeneous operators of the p-Laplace type, the key ideas for obtaining the global bifurcation results are to observe the asymptotic behavior of integral operator corresponding to the nonhomogeneous principal part at infinity and to apply an abstract bifurcation result for nonlinear operator equations in terms of the Leray-Schauder degree, in order to include nonodd case of the principal part. The authors in [15, 17] gave the following condition for the divergence part ψ:

(H) There exist positive constants c_1, c_2 such that

$$\langle\psi(x,u)-\psi(x,v),u-v\rangle \geq \begin{cases} c_1\min\{1,(|u|+|v|)^{p-2}\}|u-v|^2 & \text{if } 1<p<2 \text{ and } (u,v)\neq(0,0)\\ c_2|u-v|^p & \text{if } 2\leq p\leq\infty\end{cases}$$

holds for almost all $x\in\mathbb{R}^N$ and for all $u,v\in\mathbb{R}^N$.

Hypothesis (H) means that $\psi(x,\cdot)$ is uniformly monotone on bounded sets and coercive. In particular, this condition plays a decisive role in obtaining the topological properties that the integral operator related to the principal part ψ is bounded homeomorphism and an inverse map of a convex combination of two integral operators which are specified later is continuous. The aim of this paper is that, as considering these topological properties from the strict convexity of $\psi(x,\cdot)$ which is a weaker condition than (H), we obtain the existence of an unbounded branch of the set of solutions for nonlinear elliptic equations of $p(x)$-Laplace type.

This paper is organized as follows: In Section 2, we state some basic results on the variable exponent Lebesgue-Sobolev spaces. In Section 3, some properties of the corresponding integral operators are presented. We will prove the main result on a global bifurcation for the problem (B) in Section 4.

2. Preliminaries

In this section, we state some elementary properties for the variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces can be taken from [4, 10, 19].

To make a self-contained paper, we first recall some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\Omega)$.

Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1 \right\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h_+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h_- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \left\{ u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

The dual space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. The variable exponent Lebesgue spaces are a special case of Orlicz-Musielak spaces treated by Musielak in [23].

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

where the norm is

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}. \tag{2.1}$$

Lemma 2.1. ([10, 19]) *The space $L^{p(\cdot)}(\Omega)$ is a separable and uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right) \|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Lemma 2.2. ([10]) *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx, \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

Then

(1) $\rho(u) > 1$ $(= 1;\ < 1)$ *if and only if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$ $(= 1;\ < 1)$, *respectively;*
(2) *If* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, *then* $\|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}$;
(3) *If* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, *then* $\|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}$.

Lemma 2.3. ([7]) *Let $q \in L^\infty(\Omega)$ be such that $1 \le p(x)q(x) \le \infty$ for almost all $x \in \Omega$. If $u \in L^{q(\cdot)}(\Omega)$ with $u \neq 0$, then*

(1) *If $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} > 1$, then $\|u\|^{q-}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \| \, |u|^{q(x)} \, \|_{L^{p(\cdot)}(\Omega)} \le \|u\|^{q+}_{L^{p(\cdot)q(\cdot)}(\Omega)}$;*

(2) *If $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} < 1$, then $\|u\|^{q+}_{L^{p(\cdot)q(\cdot)}(\Omega)} \le \| \, |u|^{q(x)} \, \|_{L^{p(\cdot)}(\Omega)} \le \|u\|^{q-}_{L^{p(\cdot)q(\cdot)}(\Omega)}$.*

Lemma 2.4. ([3, 4]) *Let $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$ is globally log-Hölder continuous on Ω. Then, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the $p(\cdot)$-Poincaré inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds, where a positive constant C depends on p and Ω.

Lemma 2.5. ([8]) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$. If $q \in L^\infty(\Omega)$ with $q_- > 1$ satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } \ p(x) < N \\ \infty & \text{if } \ p(x) \ge N \end{cases} \quad \text{for all } x \in \Omega,$$

then we have a continuous embedding

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$.

3. Properties of the integral operators

In this section, we give some properties of the integral operators corresponding to the problem (B), by applying the basic properties of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ which were given in the previous section.

Here and in the sequel, we assume that $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$ is globally log-Hölder continuous on Ω (see Definition 4.11 and Remark 4.1.5 of [4]). Let $X := W_0^{1,p(\cdot)}(\Omega)$ with the norm

$$\|u\|_X = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

which is equivalent to the norm (2.1) due to Lemma 2.4.

We assume that $\psi : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $\Psi_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $\Psi_0 = \Psi_0(x,v)$, that is, $\psi(x,v) = \frac{d}{dv}\Psi_0(x,v)$. Suppose that ψ and Ψ_0 satisfy the following assumptions:

(H1) The equalities

$$\Psi_0(x,\mathbf{0}) = 0 \quad \text{and} \quad \Psi_0(x,v) = \Psi_0(x,-v)$$

hold for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(H2) There are a function $a \in L^{p'(\cdot)}(\Omega)$ and a nonnegative constant b such that

$$|\psi(x,v)| \le a(x) + b|v|^{p(x)-1}$$

holds for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(H3) There exists a constant d such that

$$d|v|^{p(x)} \le \psi(x,v)\cdot v \quad \text{and} \quad d|v|^{p(x)} \le p_+\Psi_0(x,v)$$

hold for all $x \in \Omega$ and $v \in \mathbb{R}^N$.

(H4) $\Psi_0(x,\cdot)$ is strictly convex for almost all $x \in \Omega$.

Let $\langle\cdot,\cdot\rangle$ denote the usual of X and its dual X^*. Let us define the functional $\Psi : X \to \mathbb{R}$ by

$$\Psi(u) = \int_\Omega \Psi_0(x,\nabla u)dx.$$

Under assumptions (H1)–(H3), it follows from [14, Lemma 3.2] that the functional Ψ is well defined on X, $\Psi \in C^1(X,\mathbb{R})$ and its Fréchet derivative is given by

$$\langle\Psi'(u),v\rangle = \int_\Omega \psi(x,\nabla u)\cdot\nabla v\,dx. \tag{3.1}$$

The main idea in obtaining our bifurcation result is to study the asymptotic behavior of the integral operator Ψ' and then to deduce a spectral result for operators that are not necessarily homogeneous. To do this, we consider a function $\psi_{p(\cdot)} : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\psi_{p(x)}(x,v) := |v|^{p(x)-2}v$$

and an operator $\Psi'_{p(\cdot)} : X \to X^*$ defined by

$$\langle\Psi'_{p(\cdot)}(u),v\rangle := \int_\Omega \psi_{p(x)}(x,\nabla u(x))\cdot\nabla v(x)\,dx$$

for all $v \in \mathbb{R}^N$.

The property (2) of the following assertions can be found in [21]. But for reader's convenience, we give the proof because it is slightly different from that of [21].

Lemma 3.1. *Assume that* (H1)–(H4) *hold. Then we have*

(1) $\Psi' : X \to X^*$ *is bounded, continuous, strictly monotone, coercive and hemicontinuous on* X.

(2) Ψ *is convex and weakly lower semicontinuous on* X. *Moreover,* Ψ' *is a mapping of type* (S_+) *i.e. if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty}\langle\Psi'(u_n)-\Psi'(u),u_n-u\rangle \le 0$, *then* $u_n \to u$ *in* X *as* $n\to\infty$.

(3) Ψ' *is a bounded homeomorphism onto* X^*.

Proof. (1) From the same reasoning made in the proof of Lemma 3.4 in [14], we can show that Ψ' is bounded and continuous on X. The coercivity and strict monotonicity of Ψ' are achieved by (H3) and (H4) respectively. It only remains to prove that Ψ' is hemicontinuous on X. Since $\psi(x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost all $x\in\Omega$,

$$\psi(x,\nabla(u_1+tu_2)) \to \psi(x,\nabla u_1) \quad \text{as} \quad t\to 0$$

for almost all $x\in\Omega$ and for all $u_1,u_2\in X$. So

$$\big\langle\Psi'(u_1+tu_2),v\big\rangle = \int_\Omega \langle\psi(x,\nabla(u_1+tu_2),v\rangle\;dx$$

$$\to \int_\Omega \langle \psi(x, \nabla u_1), v\rangle \, dx = \big\langle \Psi'(u_1), v\big\rangle$$

as $t \to 0$ for all $v \in X$. Hence Ψ' is hemicontinuous on X.

(2) Let $\{u_n\}$ be any sequence in X which converges to u in X and

$$\limsup_{n\to\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle \le 0. \tag{3.2}$$

Since Ψ' is strictly monotone on X, it follows from the relation (3.2) that

$$\begin{aligned}\lim_{n\to\infty} &\int_\Omega (\psi(x, \nabla u_n) - \psi(x, \nabla u)) \cdot (\nabla u_n - \nabla u) \, dx \\ &= \lim_{n\to\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u\rangle = 0.\end{aligned}$$

Since $\| (\psi(x, \nabla u_n) - \psi(x, \nabla u)) \cdot (\nabla u_n - \nabla u)\|_{L^1(\Omega)} \to 0$ as $n \to \infty$, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ satisfying

$$\lim_{n\to\infty} (\psi(x, \nabla u_{n_k}(x)) - \psi(x, \nabla u(x))) \cdot (\nabla u_{n_k(x)} - \nabla u(x)) = 0 \tag{3.3}$$

for almost all $x \in \Omega$. This implies that there is a constant C satisfying

$$\begin{aligned}\psi(x, \nabla u_{n_k}(x)) \cdot \nabla u_{n_k}(x) \le\ & C + |\psi(x, \nabla u_{n_k}(x))|\, |\nabla u(x)| \\ &+ |\psi(x, \nabla u(x))|\, |\nabla u_{n_k}(x)| + |\psi(x, \nabla u(x))|\, |\nabla u(x)|\end{aligned}$$

for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$. By (H2) and (H3) we have

$$\begin{aligned}d\, |\nabla u_{n_k}(x)|^{p(x)} &\le \psi(x, \nabla u_{n_k}(x)) \cdot \nabla u_{n_k}(x) \\ &\le C + |\psi(x, \nabla u_{n_k}(x))|\, |\nabla u(x)| \\ &\quad + |\psi(x, \nabla u(x))|\, |\nabla u_{n_k}(x)| + |\psi(x, \nabla u(x))|\, |\nabla u(x)| \qquad (3.4)\\ &\le C + \Big(a(x) + b\, |\nabla u_{n_k}(x)|^{p(x)-1}\Big)\, |\nabla u(x)| \qquad (3.5)\\ &\quad + |\psi(x, \nabla u(x))|\, |\nabla u_{n_k}(x)| + |\psi(x, \nabla u(x))|\, |\nabla u(x)|\end{aligned}$$

for almost all $x \in \Omega$. By Young's inequality, we have

$$b\, |\nabla u_{n_k}(x)|^{p(x)-1}\, |\nabla u(x)| \le \frac{d}{3}\, |\nabla u_{n_k}(x)|^{p(x)} + \left(\frac{d}{3b^{p'(x)}}\right)^{1-p(x)} |\nabla u(x)|^{p(x)}$$

and

$$|\psi(x, \nabla u(x))|\, |\nabla u_{n_k}(x)| \le \left(\frac{d}{3}\right)^{\frac{1}{1-p(x)}} |\psi(x, \nabla u(x))|^{p'(x)} + \frac{d}{3}\, |\nabla u_{n_k}(x)|^{p(x)}$$

for almost all $x \in \Omega$. These together with the relation (3.5) imply that

$$\begin{aligned}\frac{d}{3}\, |\nabla u_{n_k}(x)|^{p(x)} \le\ & M + a_0(x)\, |\nabla u(x)| + \left(\frac{d}{3b^{p'(x)}}\right)^{1-p(x)} |\nabla u(x)|^{p(x)} \\ &+ \left(\frac{d}{3}\right)^{\frac{1}{1-p(x)}} |\psi(x, \nabla u(x))|^{p'(x)} + |\psi(x, \nabla u(x))|\, |\nabla u(x)|\end{aligned}$$

for almost all $x \in \Omega$. This implies that $\{\nabla u_{n_k}(x)\}$ is bounded in $\mathbb{R}^N$ for almost all $x \in \Omega$. Without loss of generality we suppose that $\nabla u_{n_k}(x) \to \xi(x)$ as $k \to \infty$ for almost all $x \in \Omega$. It follows from (3.3) that

$$\begin{aligned}0 &= \lim_{k\to\infty} (\psi(x, \nabla u_{n_k}(x)) - \psi(x, \nabla u(x))) \cdot (\nabla u_{n_k}(x) - \nabla u(x)) \\ &= (\psi(x, \xi(x)) - \psi(x, \nabla u(x))) \cdot (\xi(x) - \nabla u(x))\end{aligned}$$

for almost all $x \in \Omega$. Since ψ is strictly monotone, $\xi(x) = \nabla u(x)$, i.e., $\nabla u_{n_k}(x) \to \nabla u(x)$ as $k \to \infty$ for almost all $x \in \Omega$. Since these arguments hold for any subsequence of the sequence $\{u_n\}$, we get that $\nabla u_n(x) \to \nabla u(x)$ as $n \to \infty$ for almost all $x \in \Omega$. Taking (3.2) into account, we infer

$$\lim_{n\to\infty} \int_\Omega \psi(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx = 0.$$

Owing to the convexity of Ψ, one has

$$\Psi(u) + \int_\Omega \psi(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx \geq \Psi(u_n),$$

so that $\Psi(u) \geq \limsup_{n\to\infty} \Psi(u_n)$. Since Ψ is strictly convex, we ensure that Ψ is weakly lower semicontinuous on X and thus $\Psi(u) \leq \liminf_{n\to\infty} \Psi(u_n)$. In conclusion,

$$\Psi(u) = \lim_{n\to\infty} \Psi(u_n). \tag{3.6}$$

Consider the sequence $\{g_n\}$ in $L^1(\Omega)$ defined by

$$g_n(x) = \frac{1}{2}\big(\Psi_0(x, \nabla u_n) + \Psi_0(x, \nabla u)\big) - \Psi_0\big(x, \frac{1}{2}(\nabla u_n - \nabla u)\big).$$

Then $g_n(x) \to \Psi_0(x, \nabla u)$ as $n \to \infty$ for almost all $x \in \Omega$ and $g_n \geq 0$ by the convexity and continuity of Ψ_0. Using Fatou lemma and (3.6), we have

$$\Psi(u) \leq \liminf_{n\to\infty} \int_\Omega g_n(x)\, dx = \Psi(u) - \limsup_{n\to\infty} \int_\Omega \Psi_0\big(x, \frac{1}{2}(\nabla u_n - \nabla u)\big)\, dx.$$

Hence,

$$\limsup_{n\to\infty} \int_\Omega \Psi_0\big(x, \frac{1}{2}(\nabla u_n - \nabla u)\big)\, dx \leq 0,$$

that is

$$\lim_{n\to\infty} \int_\Omega \Psi_0\big(x, \frac{1}{2}(\nabla u_n - \nabla u)\big)\, dx = 0.$$

It follows from (H3) that $\lim_{n\to\infty} \|\nabla u_n - \nabla u\|_{L^{p(\cdot)}(\Omega)} = 0$. Since $\|u_n - u\|_X \leq c\|\nabla u_n - \nabla u\|_{L^{p(\cdot)}(\Omega)}$ for some constant c, we conclude $\lim_{n\to\infty} \|u_n - u\|_X = 0$.

(3) Since Ψ' satisfies all conditions of the Browder-Minty theorem in [29], the inverse operator $(\Psi')^{-1} : X^* \to X$ exists and is bounded on X^*. To show that $(\Psi')^{-1}$ is continuous on X^*, consider a sequence $\{h_n\}$ in X^* which converges to h in X^* for each $h \in X^*$. Set $u_n = (\Psi')^{-1}(h_n)$ and $u = (\Psi')^{-1}(h)$. Then $\Psi'(u_n) = h_n$ and $\Psi'(u) = h$. Since $\{u_n\}$ is bounded in X, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \rightharpoonup u_0$ in X as $k \to \infty$ for some $u_0 \in X$. Since $h_{n_k} \to h$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \langle \Psi'(u_{n_k}) - \Psi'(u_0), u_{n_k} - u_0\rangle = \lim_{k\to\infty} \langle h_{n_k}, u_{n_k} - u_0\rangle = 0.$$

Since Ψ' is a mapping of type (S_+), $u_{n_k} \to u_0$ in X as $k \to \infty$. Since Ψ' is strictly monotone on X, it follows from the relation

$$\Psi'(u_0) = \lim_{k\to\infty} \Psi'(u_{n_k}) = \lim_{k\to\infty} h_{n_k} = h = \Psi'(u)$$

that $u_0 = u$. Without loss of generality we may replace up to a subsequence, still denoted by u_n and thus $u_n \to u_0$ in X as $n \to \infty$, namely, $(\Psi')^{-1}$ is continuous at each $h \in X^*$. Hence Ψ' is bounded homeomorphism onto X^*. The proof is complete. □

Now we consider an operator $\Psi'_t := t\Psi' + (1-t)\Psi'_{p(\cdot)}$ defined as a convex combination of Ψ' and $\Psi'_{p(\cdot)}$ for $t \in [0,1]$.

Lemma 3.2. *Assume that* (H1)–(H4) *hold. Then* $\Psi'_t : X \to X^*$ *is a mapping of type* (S_+) *for* $t \in [0,1]$.

Proof. We know that the operator $\Psi'_{p(\cdot)}$ have all properties in Lemma 3.1 since $\psi_{p(\cdot)}$ satisfies conditions (H1)–(H4). Hence $\Psi'_{p(\cdot)}$ is a mapping of type (S_+) and so it is clear for $t = 0$. Let $\{u_n\}$ be any sequence in X such that $u_u \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle \Psi'_t(u_n), u_n - u\rangle \le 0$ for $t \in (0,1]$. Since $\Psi'_{p(\cdot)}$ is strictly monotone on X, for $t \in (0,1]$

$$\begin{aligned}
&\limsup_{n\to\infty}\langle t\Psi'(u_n), u_n - u\rangle \\
&\le \limsup_{n\to\infty}\langle t\Psi'(u_n), u_n - u\rangle + \liminf_{n\to\infty}\langle (1-t)\Psi'_{p(\cdot)}(u_n), u_n - u\rangle \\
&\le \limsup_{n\to\infty}\langle t\Psi'(u_n) + (1-t)\Psi'_{p(\cdot)}(u_n), u_n - u\rangle \\
&= \limsup_{n\to\infty}\langle \Psi'_t(u_n), u_n - u\rangle \\
&\le 0.
\end{aligned}$$

Since Ψ' is a mapping of type (S_+), we obtain that $u_n \to u$ in X as $n \to \infty$. Thus the operator Ψ'_t is also a mapping of type (S_+) for $t \in [0,1]$. □

Theorem 3.3. *Assume that* (H1)–(H4) *hold. Then the operator* $\Psi'_t : X \to X^*$ *is bounded homeomorphism. Moreover, the map* $h : [0,1] \times X^* \to X$, $(t, f) \mapsto (\Psi'_t)^{-1}(f)$ *is continuous on* $[0,1] \times X^*$.

Proof. For all $t \in [0,1]$ and for all $u, v \in X$, we have

$$\begin{aligned}
&\langle \Psi'_t(u) - \Psi'_t(v), u - v\rangle \\
&\ge \min\{\langle \Psi'(u) - \Psi'(v), u - v\rangle, \langle \Psi'_{p(\cdot)}(u) - \Psi'_{p(\cdot)}(v), u - v\rangle\}.
\end{aligned}$$

This implies that Ψ'_t is strictly monotone and coercive on X because Ψ' and $\Psi'_{p(\cdot)}$ are strictly monotone and coercive on X. With the aid of Lemma 3.2, it is clear that Ψ'_t is bounded homeomorphism onto X^*. It remains to show that h is continuous on $[0,1] \times X^*$. To do this, let $\{(t_n, f_n)\}$ be any sequence in $[0,1] \times X^*$ such that $t_n \to t$ in $[0,1]$ and $f_n \to f$ in X^* as $n \to \infty$. By the same argument as in Lemma 3.1, the inverse operator $(\Psi'_t)^{-1} : X^* \to X$ exists and is bounded on X^*. Set $u_n = (\Psi'_{t_n})^{-1}(f_n)$ and $u = (\Psi'_t)^{-1}(f)$. Then $\Psi'_{t_n}(u_n) = f_n$ and $\Psi'_t(u) = f$. Since $\{u_n\}$ is bounded in X, there is a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \rightharpoonup u_0$ in X as $k \to \infty$ for some $u_0 \in X$. Since $\Psi'_{t_{n_k}}$ is a mapping of type (S_+) and $f_{n_k} \to f$ in X^* as $k \to \infty$,

$$\lim_{k\to\infty}\langle \Psi'_{t_{n_k}}(u_{n_k}) - \Psi'_{t_{n_k}}(u), u_{n_k} - u\rangle = \lim_{k\to\infty}\langle f_{n_k}, u_{n_k} - u\rangle = 0$$

and thus $u_{n_k} \to u_0$ in X as $k \to \infty$. Hence we have

$$\Psi'_t(u_0) = \lim_{k\to\infty}\Psi'_{t_{n_k}}(u_{n_k}) = \lim_{k\to\infty} h_{n_k} = h = \Psi'_t(u).$$

Since $\Psi'_{t_{n_k}}$ is strictly monotone on X, we obtain that $u_0 = u$. Without loss of generality we may replace u_{n_k} by u_n and thus

$$u_n \to u \text{ and equivalently } h(t_n, f_n) \to h(t, f) \text{ as } n \to \infty.$$

Therefore h is continuous on $[0,1] \times X^*$. □

To discuss the asymptotic behavior of Ψ', we require the following hypothesis:

(H5) For each $\varepsilon > 0$ there is a function $\ell \in L^{p(\cdot)}(\Omega)$ such that for almost all $x \in \Omega$ the following holds:
$$\frac{\left|\psi(x,v) - \psi_{p(x)}(x,v)\right|}{|v|^{p_- -1}} \leq \varepsilon$$
for all $v \in \mathbb{R}^N$ with $|v| > |\ell(x)|$.

Now we can give that the operators Ψ' and $\Psi'_{p(\cdot)}$ are asymptotic at infinity in the following sense.

Lemma 3.4. *Assume that* (H1)–(H5) *are fulfilled. Then we have*
$$\lim_{\|u\|_X \to \infty} \frac{\|\Psi'(u) - \Psi'_{p(\cdot)}(u)\|_{X^*}}{\|u\|_X^{p_- -1}} = 0.$$

Proof. The proof is absolutely the same as that of Proposition 3.4 in [16]. □

Next we deal with the properties for the superposition operator induced by the function f in (B). In particular, we give the compactness of this operator and the behavior of this operator at infinity, respectively. The ideas of the proof about these properties are completely the same as in [18]. We assume that the variable exponents are subject to the following restrictions
$$\begin{cases} p^*(x) := \frac{Np(x)}{N-p(x)},\ q(x) \in \left(\frac{Np(x)}{Np(x)-N+p(x)}, \infty\right) & \text{if } N > p(x), \\ p^*(x), q(x) \in (1,\infty) \text{ arbitrary} & \text{if } N \leq p(x) \end{cases}$$
for almost all $x \in \Omega$. Assume that

(F1) $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\lambda, \cdot, u, v)$ is measurable for all $(\lambda, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and $f(\cdot, x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$.

(F2) For each bounded interval $I \subset \mathbb{R}$, there are a function $a_I \in L^{q(\cdot)}(\Omega)$ and a nonnegative constant b_I such that
$$|f(\lambda, x, u, v)| \leq a_I(x) + b_I\left(|u|^{\frac{p^*(x)}{q(x)}} + |v|^{\frac{p(x)}{q(x)}}\right)$$
for almost all $x \in \Omega$ and all $(\lambda, u, v) \in I \times \mathbb{R} \times \mathbb{R}^N$.

(F3) There exist a function $a \in L^{p'(\cdot)}(\Omega)$ and a locally bounded function $b : [0,\infty) \to \mathbb{R}$ with $\lim_{r\to\infty} b(r)/r = 0$ such that
$$|f(0,x,u,v)| \leq a(x) + \left[b(|u| + |v|)\right]^{p_- -1}$$
for almost all $x \in \Omega$ and all $(u,v) \in \mathbb{R} \times \mathbb{R}^N$.

Under assumptions (F1) and (F2), we can define an operator $F : \mathbb{R} \times X \to X^*$ by
$$\langle F(\lambda, u), v\rangle = \int_\Omega f(\lambda, x, u(x), \nabla u(x)) v(x)\, dx \tag{3.7}$$

and an operator $G : X \to X^*$ by

$$\langle G(u), v\rangle = \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)\,dx \tag{3.8}$$

for all $v \in X$.

The following consequence is given in [16].

Lemma 3.5. *If* (F1)–(F3) *hold, then the operators* $F : \mathbb{R} \times X \to X^*$ *and* $G : X \to X^*$ *are continuous and compact. Moreover, the operator* $F(0,\cdot) : X \to X^*$ *satisfies the following property:*

$$\lim_{\|u\|_X\to\infty} \frac{\|F(0,u)\|_{X^*}}{\|u\|_X^{p_- -1}} = 0.$$

Recall that a real number μ is called an *eigenvalue of* (E) if the equation

$$\Psi'_{p(\cdot)}(u) = \mu G(u)$$

has a solution u_0 in X that is different from the origin.

Now we consider the following result in the sense of nonlinear spectral theory; see [18]. If $p(x)$ is a constant, we can obtain the following assertion by using the Furi-Martelli-Vignoli spectrum; see Theorem 4 of [28] or Lemma 27 of [12].

Lemma 3.6. *If μ is not an eigenvalue of (E), then we have*

$$\liminf_{\|u\|_X\to\infty} \frac{\|\Psi'_{p(\cdot)}(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- -1}} > 0. \tag{3.9}$$

With the aids of Lemmas 3.4 and 3.6, we give the following spectral result for nonhomogeneous operators which will be used to get our main theorem. For the case of a constant function $p(x) \equiv p$, the analogy of this assertion can be found in [15, 17].

Lemma 3.7. *Suppose that conditions* (H1)–(H3), (H5), *and* (F1)–(F3) *are satisfied. If μ is not an eigenvalue of* (E), *we have*

$$\liminf_{\|u\|_X\to\infty} \min_{t\in[0,1]} \frac{\|\Psi'_t(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- -1}} > 0,$$

where Ψ'_t is a convex combination of Ψ' and $\Psi'_{p(\cdot)}$.

Proof. Applying Lemma 3.6, we deduce that

$$\alpha := \liminf_{\|u\|_X\to\infty} \frac{\|\Psi'_{p(\cdot)}(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p_- -1}} > 0. \tag{3.10}$$

Let ε be an arbitrary positive number. By Lemma 3.4 and the relation (3.10), we choose a positive constant R such that $\|u\|_X \geq R$ implies

$$\|\Psi'_{p(\cdot)}(u) - \mu G(u)\|_{X^*} > (\alpha - \varepsilon)\|u\|_X^{p_- -1}$$

and

$$\|\Psi'_{p(\cdot)}(u) - \Psi'(u)\|_{X^*} < \frac{\alpha}{2}\|u\|_X^{p_- -1}.$$

For all $u \in X$ with $\|u\|_X \geq R$, we get

$$\begin{aligned}\min_{t\in[0,1]} \|\Psi'_t(u) - \mu G(u)\|_{X^*} &= \|\Psi'_{p(\cdot)}(u) - \mu G(u)\|_{X^*} - \max_{t\in[0,1]} \|\Psi'_{p(\cdot)}(u) - \Psi'_t(u)\|_{X^*} \\ &= \|\Psi'_{p(\cdot)}(u) - \mu G(u)\|_{X^*} - \|\Psi'(u) - \Psi'_{p(\cdot)}(u)\|_{X^*} \\ &> \left(\frac{\alpha}{2} - \varepsilon\right) \|u\|_X^{p_- - 1}.\end{aligned}$$

Since ε was arbitrary, the conclusion is achieved. This completes the proof. □

4. Main Result

In this section, we give the existence of an unbounded branch of solutions for our problem (B).

Definition 4.1. *A weak solution of* (B) *is a pair* (λ, u) *in* $\mathbb{R} \times X$ *such that*

$$\Psi'(u) - \mu G(u) = F(\lambda, u) \quad \text{in } X^*,$$

where Ψ', F, *and* G *are defined by* (3.1), (3.7) *and* (3.8), *respectively.*

The following result is taken from Theorem 2.2 of [17] (see also [28]), as a key tool in obtaining our bifurcation result.

Lemma 4.2. *Let* X *be a Banach space and* Y *a normed space. Suppose that* $\Psi' : X \to Y$ *is a homeomorphism and* $G : X \to Y$ *is a continuous and compact operator such that the Leray-Schauder degree in* X *satisfies*

$$deg_X \left(I_X - ((\Psi')^{-1} \circ (-G)), B_r, 0\right) \neq 0$$

for all sufficiently large $r > 0$, *where* I_X *is the identity operator on* X *and* B_r *is the open ball in* X *centered at* 0 *of radius* r, *respectively. Let* $F : \mathbb{R} \times X \to Y$ *be a continuous and compact operator. If the set*

$$S := \bigcup_{t\in[0,1]} \{u \in X : \Psi'(u) + G(u) = tF(0, u)\}$$

is bounded, then the solution set

$$\{(\lambda, u) \in \mathbb{R} \times X : \Psi'(u) + G(u) = F(\lambda, u)\}$$

contains an unbounded connected set $C \subseteq \left(\mathbb{R}\backslash\{0\}\right) \times X$ *such that its closure intersects* $\{0\} \times X$.

Based on the above lemma, we now can prove the main result on bifurcation result for problem (B) with the helps of nonlinear spectral theory and degree theory. The proof is similar to that of Theorem 3.2. in [15]; see also [17].

Theorem 4.3. *Suppose that conditions* (H1)–(H5) *and* (F1)–(F3) *are satisfied. If* μ *is not an eigenvalue of* (E), *then there is an unbounded connected set* $C \subseteq (\mathbb{R} \setminus \{0\}) \times X$ *such that every point* (λ, u) *in* C *is a weak solution of problem* (B) *and* $\overline{C}$ *intersects* $\{0\} \times X$.

Proof. Apply Lemma 4.2 with $X = W_0^{1,p(\cdot)}(\Omega)$ and $Y = X^*$. From Lemmas 3.1 and 3.5 we obtain that $\Psi' : X \to X^*$ is a homeomorphism and the operators F and G are continuous and compact. Since μ is not an eigenvalue of (E), Lemmas 3.5 and 3.7 imply that for some $\beta > 0$, there is a positive constant $R > 1$ such that

$$\|\Psi'(u) - \mu G(u)\|_{X^*} > \beta \|u\|_X^{p_- -1} > \|F(0,u)\|_{X^*} \geq \|tF(0,u)\|_{X^*}$$

for all $u \in X$ with $\|u\|_X \geq R$ and for all $t \in [0,1]$. Therefore, the set

$$S = \bigcup_{t\in[0,1]} \{u \in X : \Psi'(u) - \mu G(u) = tF(0,u)\}$$

is bounded. To apply Lemma 4.2, it remains to prove that

$$\deg_X(I_X - (\Psi')^{-1} \circ (\mu G), B_r) \neq 0 \tag{4.1}$$

holds for sufficiently large $r > R$. Consider the homotopy $H : [0,1] \times X \to X$ defined by

$$H(t,u) = (\Psi'_t)^{-1}(\mu G(u)),$$

where $\Psi'_t = t\Psi' + (1-t)\Psi'_{p(\cdot)}$. Let us take maps h_1 and h_2 where $h_1 : [0,1] \times X \to [0,1] \times X^*$ is defined by $h_1(t,u) := (t, \mu G(u))$ and $h_2 : [0,1] \times X^* \to X$ is defined by $h_2(t,v) := (\Psi'_t)^{-1}(v)$. Since h_1 is compact continuous map and h_2 is continuous map by Theorem 3.3 and Lemma 3.5, the composition of h_2 with h_1, that is, H is compact and continuous. Moreover it follows from Lemma 3.7 that for large sufficiently $r > R$, $H(t,u) \neq u$ for all $(t,u) \in [0,1] \times \partial B_r$. Thus the homotopy invariance property of the Leray-Schauder degree implies that

$$\begin{aligned}\deg_X(I_X - (\Psi')^{-1} \circ (\mu G), B_r, 0) &= \deg_X(I_X - H(1,\cdot), B_r, 0)\\ &= \deg_X(I_X - H(0,\cdot), B_r, 0)\\ &= \deg_X(I_X - (\Psi'_{p(\cdot)})^{-1} \circ (\mu G), B_r, 0).\end{aligned}$$

Since $\Psi'_{p(\cdot)}$ and G are odd, Borsuk's theorem implies that the last degree is odd and so (4.1) holds. This completes the proof. □

References

[1] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (2006) 1383–1406.

[2] M. A. Del Pino and R.F. Manasevich, *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations 92 (1991) 226–251.

[3] L. Diening, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. 268 (2004) 31–43.

[4] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math., vol. 2017, Springer-Verlag, Berlin, 2011.

[5] P. Drábek, *On the global bifurcation for a class of degenerate equations*, Ann. Mat. Pura Appl. 159 (1991) 1–16.

[6] P. Drábek, A. Kufner, and F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, de Gruyter, Berlin, 1997.

[7] D. E. Edmunds and J. Rákosník, *Sobolev embeddings with variable exponent*, Studia Math. 143 (2000) 267–293.

[8] X. Fan, J. Shen, and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl. 262 (2001) 749–760.

[9] X. Fan, Q. Zhang, and D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2005) 306–317.
[10] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001) 424–446.
[11] N. Fukagai, M. Ito, and K. Narukawa, *A bifurcation problem of some nonlinear degenerate elliptic equations*, Adv. Differential Equations 2 (1997) 895–926.
[12] E. Giorgieri, J. Appell and M. Väth, *Nonlinear spectral theory for homogeneous operators*, Nonlinear Funct. Anal. Appl. 7 (2002) 589–618.
[13] B. -H. Hwang, S. D. Lee, and Y. -H. Kim, *Existence of an unbounded branch of the set of solutions for Neumann problems involving the $p(x)$-Laplacian*, Bound. Value Probl. 92 (2014) 1–16.
[14] I. H. Kim and Y. -H. Kim, *Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents*, Maunscripta Math. 147 (2015) 169–191.
[15] I. -S. Kim, *A global bifurcation result for generalized Laplace operators in $\mathbb{R}^N$*, J. Nonlinear Convex Anal. 13 (2012) 667–680.
[16] Y. -H. Kim, *Existence of an unbounded branch of the set of solutions for equations of $p(x)$-Laplace type*, Bound. Value Probl. 27 (2014) 1–20.
[17] Y. -H. Kim and M. Väth, *Global solution branches for equations involving nonhomogeneous operators of p-Laplace type*, Nonlinear Anal. 74 (2011) 1878–1891.
[18] Y. -H. Kim, L. Wang, and C. Zhang, *Global bifurcation for a class of degenerate elliptic equations with variable exponents*, J. Math. Anal. Appl. 371 (2010) 624–637.
[19] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. 41 (1991) 592–618.
[20] M. A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon, New York, 1965.
[21] V. K. Le, *On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces*, Nonlinear Anal. 71 (2009) 3305–3321.
[22] V. K. Le and K. Schmitt, *Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems*, Springer, New York, 1997.
[23] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.
[24] P. H. Rabinowitz, *Some global results for non-linear eigenvalue problems*, J. Funct. Anal. 7 (1971) 487–513.
[25] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math., vol. 1748, Springer, Berlin, 2000.
[26] K. Schmitt and I. Sim, *Bifurcation problems associated with generalized Laplacians*, Adv. Differential Equations 9 (2004) 797–828.
[27] C. A. Stuart, *Some bifurcation theory for k-set contractions*, Proc. London Math. Soc. 27 (1973) 531–550.
[28] M. Väth, *Global bifurcation of the p-Laplacian and related operators*, J. Differential Equations 213 (2005) 389–409.
[29] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B*, Springer, New York, 1990.
[30] V. V. Zhikov, *On the density of smooth functions in Sobolev-Orlicz spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. 226 (2004) 67–81.

Yun-Ho Kim, Department of Mathematics Education, Sangmyung University, Seoul 110-743, Republic of Korea
E-mail address: kyh1213@smu.ac.kr